

Regular local algebras over a Prüfer domain: weak dimension and regular sequences

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Abstract

A not necessarily noetherian local ring O is called regular if every finitely generated ideal $I \triangleleft O$ possesses finite projective dimension. In the article localizations $O = A_q$, $q \in \text{Spec } A$, of a finitely presented, flat algebra A over a Prüfer domain R are investigated with respect to regularity: this property of O is shown to be equivalent to the finiteness of the weak homological dimension $\text{wdim } O$. A formula to compute $\text{wdim } O$ is provided. Furthermore regular sequences within the maximal ideal $M \triangleleft O$ are studied: it is shown that regularity of O implies the existence of a maximal regular sequence of length $\text{wdim } O$. If $\text{height}(q \cap R) \neq \infty$, then this sequence can be chosen such that the radical of the ideal generated by the members of the sequence equals M . As a consequence it is proved that if O is regular, then the (noetherian) factor ring $O/(q \cap R)O$ is Cohen-Macaulay. If $(q \cap R)R_{q \cap R}$ is not finitely generated, then $O/(q \cap R)O$ itself is regular.

Key words: regular local ring, weak dimension, flat dimension, regular sequence, Prüfer domain, coherent ring
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Introduction

In the present article finitely generated, flat algebras A over a Prüfer domain R and their localizations A_q at prime ideals $q \triangleleft A$ are investigated. The finiteness properties and the ideal theory of these algebras have been the subject of various investigations, two of the main results being:

- coherence [Gla1], Ch. 7: every finitely generated ideal $I \triangleleft A$ is finitely presented,
- catenarity [BDF]: the length ℓ of a non-refinable chain of primes $q_0 \subset q_1 \subset \dots \subset q_\ell$ in A depends on q_0 and q_ℓ only, provided that the Krull dimension $\dim R_p$ is finite for all $p \in \text{Spec } R$.

Motivated by questions concerning the geometry of schemes over Prüfer domains in the sequel we study localizations A_q possessing a distinguished property that generalizes the noetherian notion of regularity. A noetherian local ring O is called regular if its global homological dimension $\text{gldim } O$ is finite. This property plays an important role in the theory of noetherian rings and links algebra with geometry. Accordingly attempts were made to generalize it to non-noetherian rings. They led to the following definition first given by J. Bertin [Ber]: the local ring O is called regular if for every *finitely generated* ideal $I \triangleleft O$ the projective dimension $\text{pdim}_O O/I$ of the O -module O/I is finite. Auslander's Lemma $\text{gldim } O = \sup(\text{pdim}_O(O/I) \mid I \triangleleft O)$ shows that Bertin's definition indeed generalizes the noetherian notion of regularity. However a local ring regular in the sense of Bertin needs not have finite global dimension.

In the sequel the words *regular* and *regularity* always refer to Bertin's definition.

The weak homological dimension $\text{wdim } O$ of a local ring O in many cases has shown to be an appropriate tool to check for regularity. Analogously to the global dimension it is defined as the supremum of the flat dimensions $\text{fdim } M$ of all O -modules M , where $\text{fdim } M$ denotes the length of the shortest flat resolution of M . It is well-known that a coherent local ring O of finite weak dimension $\text{wdim } O$ is regular and that the finiteness of the global dimension $\text{gldim } O$ implies that of $\text{wdim } O$. However there exist coherent, regular, local rings possessing an infinite weak dimension, so that in contrast to the noetherian case regularity of a coherent local ring cannot in general be checked using the weak dimension.

Several classes of coherent local rings for which regularity is *equivalent* to the finiteness of the weak dimension have been identified in the past – an overview can be found in [Gla1]. Among them are for example the local rings with finitely generated maximal ideal [TZZ] and localizations of certain group algebras. One of the main results of the present article describes a new class

of local rings of that type:

Theorem 1 *The localization A_q , $q \in \text{Spec } A$, of a finitely generated, flat algebra A over the Prüfer domain R is regular if and only if its weak dimension is finite. For a regular localization A_q the weak dimension satisfies:*

$$\text{wdim } A_q = \begin{cases} \dim(A_q \otimes_R kp) & \text{if } p = 0 \\ \dim(A_q \otimes_R kp) + 1 & \text{if } p \neq 0, \end{cases}$$

where $p := q \cap R$ and kp denotes the field of fractions of R/p .

As the second theme of this article we discuss a method to construct regular sequences within a localization A_q . It is copied from the noetherian case and is shown to work for an arbitrary coherent local ring of finite weak dimension:

Theorem 2 *In a coherent local ring O of finite weak dimension every sequence (t_1, \dots, t_ℓ) composed of elements of the maximal ideal $M \triangleleft O$ such that $t_1 + M^2, \dots, t_\ell + M^2 \in M/M^2$ are O/M -linearly independent is a regular sequence. Its members generate a prime ideal $q \triangleleft O$ with the property*

$$\text{wdim } O/q = \text{wdim } O - \ell;$$

in particular O/q is a coherent regular ring.

It is well-known that under the assumptions made in Theorem 2 the length ℓ of a regular sequence within the maximal ideal satisfies $\ell \leq \text{wdim } O$. The theorem thus yields the inequality

$$\dim M/M^2 \leq \text{wdim } O. \tag{1}$$

In contrast to the noetherian case however lifting a basis of M/M^2 does not in general yield a *maximal* regular sequence. Moreover the length of the longest regular sequence in M needs not be equal to $\text{wdim } O$. Against this background the behavior of the lifting procedure in the case of algebras over Prüfer domains is remarkable:

Theorem 3 *The maximal ideal $M := qA_q$ of a regular localization $O := A_q$ of a finitely generated, flat algebra A over a Prüfer domain R contains a maximal regular sequence (t_1, \dots, t_d) of length $d = \text{wdim } O$.*

Every sequence (t_1, \dots, t_ℓ) of elements of M such that $(t_1 + M^2, \dots, t_\ell + M^2)$ forms an O/M -basis of M/M^2 is regular and has the following properties:

- (1) If M is finitely generated, then (t_1, \dots, t_ℓ) is a maximal regular sequence of length $\ell = \text{wdim } O$ and its members generate M .
- (2) If M is not finitely generated, then for every non-zero $t \in p := M \cap R$ the sequence (t_1, \dots, t_ℓ, t) is maximal regular; its length satisfies $\ell + 1 = \text{wdim } O$. Moreover if pR_p is the radical of the principal ideal tR_p , then M equals the radical of the ideal $\sum_{i=1}^{\ell} Ot_i + Ot$.

Note that as a consequence of Theorem 1 and inequality (1) the dimension of M/M^2 in Theorem 3 is finite. Furthermore notice that the maximal ideal M is finitely generated if the maximal ideal pR_p , $p = M \cap R$, is finitely generated. Finally it should be mentioned that the local ring R_p is a valuation domain. Thus if $\dim R_p$ is finite, then pR_p is the radical of a principal ideal.

The results presented so far can be used to get insight into the geometry of integral separated R -schemes \mathcal{X} of finite type over a Prüfer domain R – in the sequel such schemes are called R -varieties. They are known to be R -flat and to possess coherent structure sheaf $\mathcal{O}_{\mathcal{X}}$. Furthermore all fibres $\mathcal{X} \times_R k_p$, $p \in \text{Spec } R$, are equidimensional schemes of a common dimension $n \in \mathbb{N}$; in particular the generic fibre $X := \mathcal{X} \times_R K$, $K := \text{Frac } R$, is an algebraic variety of dimension n . The case of a noetherian base ring R , that is R -varieties over a Dedekind domain, are studied extensively in arithmetic algebraic geometry. R -varieties over a non-noetherian valuation domain appear in the valuation theory of algebraic function fields: the function field F of a *normal* R -variety \mathcal{X} carries a finite set V of non-archimedean valuations induced by \mathcal{X} . Namely the valuation ring of each of the valuations $v \in V$ is one of the local rings $\mathcal{O}_{\mathcal{X}, \eta}$, where $\eta \in \mathcal{X}$ runs through the generic points of the closed fibre $\overline{X} := \mathcal{X} \times_R kN$, $N \triangleleft R$ the maximal ideal of R . By definition the restriction $v|_K$ is a valuation v_K of K independent of v and possessing R as its valuation domain. Moreover the extension of residue fields of $v|v_K$ equals $k\eta|kN$ and thus has the same transcendence degree as $F|K$. Valuations with the latter property are called constant reductions of $F|K$. They were first studied using a purely valuation-theoretic approach in the case of transcendence degree one by M. Deuring [Deu] and E. Lamprecht [Lam]. For higher transcendence degree P. Roquette [Roq] introduced a geometric approach by (essentially) studying projective R -varieties. His work supplements and generalizes results obtained by G. Shimura [Shi] in the case of a discrete valuation ring R .

In the articles just mentioned a focus lies on clarifying the relation between the divisor theory of the generic fibre X of an R -variety \mathcal{X} and that of its closed fibre \overline{X} . In this task Weil divisors on \mathcal{X} itself enter in. On the non-noetherian scheme \mathcal{X} they must be defined utilizing sheaves instead of taking the usual approach through subschemes of codimension one: a Weil divisor on \mathcal{X} is a coherent fractional $\mathcal{O}_{\mathcal{X}}$ -module \mathcal{J} possessing the property $\mathcal{J} = (\mathcal{O}_{\mathcal{X}} : (\mathcal{O}_{\mathcal{X}} : \mathcal{J})) =: \widehat{\mathcal{J}}$ called reflexivity. The set $D(\mathcal{X})$ of all Weil divisors

on \mathcal{X} becomes an abelian semigroup through the multiplication $\mathcal{I} \circ \mathcal{J} := \widehat{\mathcal{I}\mathcal{J}}$. If \mathcal{X} is normal, then $D(\mathcal{X})$ is a group [Kn1], that in the noetherian case is isomorphic to the group of ordinary Weil divisors. Given a Weil prime divisor $P \subset X$ of the generic fibre of \mathcal{X} a divisor \overline{P} on \overline{X} can be defined essentially by intersecting the Zariski closure \mathcal{P} of P on \mathcal{X} with the subscheme $\overline{X} \subset \mathcal{X}$. The ideal sheaf \mathcal{J} associated to the subscheme \mathcal{P} is an element of $D(\mathcal{X})$ – a non-trivial result in the present non-noetherian context. However to define a divisor on the possibly singular scheme \overline{X} it is desirable that \mathcal{J} be invertible. At that point regularity becomes important.

Every stalk \mathcal{J}_P , $P \in \mathcal{X}$, of a sheaf $\mathcal{J} \in D(\mathcal{X})$ is a fractional $\mathcal{O}_{\mathcal{X},P}$ -ideal with the property $(\mathcal{O}_{\mathcal{X},P} : (\mathcal{O}_{\mathcal{X},P} : \mathcal{J}_P)) = \mathcal{J}_P$. Such fractional ideals are principal if in the local ring $\mathcal{O}_{\mathcal{X},P}$ finitely many elements always possess a greatest common divisor. One of the main results of the theory of finite free resolutions states that a coherent, regular, local ring is of that type – see [Nor]. Thus as in the noetherian case the sheaves $\mathcal{J} \in D(\mathcal{X})$ are locally principal on the regular locus $\text{Reg } \mathcal{X} := \{P \in \mathcal{X} \mid \mathcal{O}_{\mathcal{X},P} \text{ is regular.}\}$ and on a regular R -variety – meaning that $\text{Reg } \mathcal{X} = \mathcal{X}$ – Weil and Cartier divisors coincide.

A description of the regular locus can be achieved using Theorem 1:

The regular locus of an R -variety \mathcal{X} of relative dimension $n \in \mathbb{N}$ can be characterized as $\text{Reg } \mathcal{X} = \{P \in \mathcal{X} \mid \text{wdim } \mathcal{O}_{\mathcal{X},P} \leq n + 1\}$ and satisfies

$$\bigcup_{p \in \text{Spec } R} \text{Reg } (\mathcal{X} \times_R kp) \subseteq \text{Reg } \mathcal{X}.$$

A regular algebraic variety X over the field K is said to have good reduction at the valuation ring $R \subset K$ if there exists an R -variety \mathcal{X} such that $X = \mathcal{X} \times_R K$ and the closed fibre $\mathcal{X} \times_R k$ is regular. If R is noetherian, then the scheme \mathcal{X} itself is regular. Vice versa it is known that the closed fibre $\mathcal{X} \times_R k$ of a regular variety over a discrete valuation domain R is Cohen-Macaulay. As the main geometric result of the present article we prove that these facts remain valid over a general Prüfer domain R – with an interesting non-noetherian twist though:

Theorem 4 *The fibres $\mathcal{X} \times_R kp$ of a regular R -variety \mathcal{X} over the Prüfer domain R are Cohen-Macaulay. If the prime ideal pR_p is not finitely generated, then the fibre $\mathcal{X} \times_R kp$ is a regular kp -scheme.*

Vice versa: an R -variety \mathcal{X} with closed structure morphism $\mathcal{X} \rightarrow \text{Spec } R$ and such that all fibres $\mathcal{X} \times_R kp$ over maximal ideals $p \in \text{Spec } R$ are regular is itself regular.

As an end of this introduction some facts about normal *curves* over a non-noetherian valuation domain R should be mentioned: first of all the relation-

ship between valued function fields of transcendence degree one and proper normal R -curves was clarified by B. Green, M. Matignon and F. Pop [GMP3], [G] resulting in a category equivalence that generalizes the well-known equivalence between projective, normal, algebraic curves and function fields of transcendence degree one. This and related results found applications in the context of Rumely's Local-Global Principle [GPR] and the so-called Skolem problems [GMP4]. Earlier E. Kani in his non-standard approach to Diophantine Geometry [Kan1] established an intersection product on a valued function field making use of the regularity of R -curves of the form $C \times_k R$, where C is a smooth curve over the field $k \subset R$.

Guide for the reader

Section 1 deals with regularity and the finiteness of the weak dimension; the proof of Theorem 1 is provided. Moreover a brief introduction to non-noetherian regularity as well as to the basic properties of finitely generated algebras over Prüfer domains is given.

In Section 2 the construction of regular sequences through lifting linearly independent elements from M/M^2 to M , where M is the maximal ideal of a coherent local ring of finite weak dimension, is investigated. The results are used to prove Theorem 2 and 3 as well as the first part of Theorem 4.

Section 3 contains a geometric point of view of the results obtained in the Sections 1 and 2. A curve over a two-dimensional valuation domain is discussed as an example.

In the appendix an upper bound for the global dimension $\text{gldim } A_q$ of a regular localization of a finitely generated, flat R -algebra A over a Prüfer domain R of small cardinality is given.

Conventions: Throughout the article a **local ring** is a commutative not necessarily noetherian ring O with unity possessing a unique maximal ideal M . A local ring (O, M) is said to be **essentially finitely generated** respectively **essentially finitely presented** over the ring R if $O = A_q$, $q \in \text{Spec } A$, for some finitely generated resp. finitely presented R -algebra A . Extensions $A \subseteq B$ of rings are denoted by $B|A$. If A and B are integral domains $\text{trdeg}(B|A)$ denotes the transcendence degree of the field extension $\text{Frac } B|\text{Frac } A$. For an ideal $I \triangleleft A$ of a commutative ring A the radical of I is denoted by $\text{Rad } I$.

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1 Weak dimension

Let A be a finitely generated, flat algebra over a Prüfer domain R . The main objective in this section is to determine the weak homological dimension of the localizations $O := A_q$, $q \in \text{Spec } A$. We are interested in the case of a non-noetherian domain R , so that in general the local ring O is non-noetherian too. We shall see that the weak dimension of O is finite if and only if O is regular in the sense of Bertin [Ber]. In the regular case we express the weak dimension in terms of the dimension of the fibre ring $O \otimes_R kp$, $p := q \cap R$. Bertin's generalization of the noetherian notion of regularity is central within the present article. We therefore give a brief summary of relevant facts in Subsection 1.1. Readers acquainted with this topic may directly jump to Subsection 1.2 that contains the principal result of the present section, the determination of the weak dimension $\text{wdim } O$. Auxiliary results of independent interest on upper bounds for the weak dimension are treated in Subsection 1.3, while in the last Subsection 1.4 basic properties of algebras over a Prüfer domain are reviewed.

1.1 Non-noetherian regularity

The local ring O is called **regular** if the projective dimension $\text{pdim } I$ of every *finitely generated* ideal $I \triangleleft O$ is finite. This property in general does *not* imply the finiteness of the **global dimension**

$$\text{gldim } O := \sup(\text{pdim } V \mid V \text{ an } O\text{-module})$$

as in the noetherian case. Indeed every valuation domain is regular, however valuation domains can have infinite global dimension [Os1].

If O is a **coherent ring**, that is if every finitely generated ideal $I \triangleleft O$ is finitely presented, then flat resolutions can be used to check for regularity: the **flat dimension** $\text{fdim } V$ of an O -module V is defined to be the length $n \in \mathbb{N}$ of the shortest resolution

$$0 \rightarrow F_n \rightarrow \dots \rightarrow F_0 \rightarrow V \rightarrow 0 \tag{2}$$

of V by flat O -modules F_i . If no such n exists, then $\text{fdim } V$ is set equal to ∞ . The **weak dimension** $\text{wdim } O$ of O is defined to be

$$\text{wdim } O := \sup(\text{fdim } V \mid V \text{ an } O\text{-module}).$$

Let M be the maximal ideal of O and consider the residue field $k := O/M$ as an O -module. If O is coherent, then we have ([Gla1], Thm. 3.1.3)

$$\text{wdim } O = \text{fdim } k. \quad (3)$$

Moreover for a finitely generated ideal $I \triangleleft O$ the module F_0 in the resolution (2) of I can be chosen to be finitely generated. The coherence of O then implies that all of the modules F_i are finitely presented and thus projective. This shows: *a coherent local ring O of finite weak dimension is regular.*

Unfortunately the class of coherent local rings of finite weak dimension does not contain all coherent, regular, local rings:

Example 1.1 The localization

$$O := K[X_i \mid i \in \mathbb{N}]_P, \quad P := \sum_{i \in \mathbb{N}} K[X_i \mid i \in \mathbb{N}]X_i,$$

of the polynomial ring in countably many variables over a field K is a coherent, regular, local ring with the property $\text{wdim } O = \infty$ – see [Gla1], page 202.

Note that in general the inequality $\text{wdim } O \leq \text{gldim } O$ holds. Strict inequality is widespread for non-noetherian rings O ; in the noetherian case of course equality is present.

The formula (3) frequently is not suitable to determine $\text{wdim } O$. Instead as in the noetherian case regular sequences can be used as a device, although the theory of regular sequences in non-noetherian rings is far more complicated than in the noetherian case: let $I \triangleleft A$ be an ideal in the commutative ring A . Following Northcott [Nor] the **(classical) grade of I** is defined to be the supremum of the lengths $\ell \in \mathbb{N}$ of the finite A -regular sequences (a_1, \dots, a_ℓ) contained in I ; it is denoted by $\text{grade } I$ and can attain the value ∞ .

The function $g(I, n) := \text{grade } IA[X_1, \dots, X_n]$ assigning to each $n \in \mathbb{N}$ the grade of the lifting of the ideal $I \triangleleft A$ into the polynomial ring $A[X_1, \dots, X_n]$ is monotonically increasing. The limit

$$\text{Grade } I := \lim_{n \rightarrow \infty} \text{grade } IA[X_1, \dots, X_n] \in \mathbb{N} \cup \{\infty\}$$

is called **polynomial or non-noetherian grade of I** . In the present context it can be used to determine weak dimensions (see [Gla2], Lemma 3): *the weak dimension of a coherent, regular, local ring O with maximal ideal M satisfies*

$$\text{wdim } O = \text{Grade } M. \quad (4)$$

In particular $\text{grade } M \leq \text{wdim } O$ holds in every coherent, regular, local ring, since $\text{grade } I \leq \text{Grade } I$ holds for every ideal $I \triangleleft O$.

1.2 A formula for the weak dimension

Throughout the whole subsection let R be a Prüfer domain with field of fractions $K \neq R$. We consider localizations $O = A_q$ of a flat R -algebra A at a prime $q \triangleleft A$. The local ring $S := R_p$, $p := q \cap R$, then is a valuation domain and the ring extension $O|S$ is faithfully flat. We are interested in calculating the weak dimension $\text{wdim } O$ in the case where A can be chosen to be finitely generated, that is the extension $O|S$ is essentially of finite type. The tools applied in this case yield an upper bound for $\text{wdim } O$ of a regular ring O belonging to a more general class of local S -algebras, namely:

(L) *For a valuation domain S with maximal ideal N let $\mathbf{L}(S)$ be the class of local, faithfully flat, coherent S -algebras O such that the maximal ideal \overline{M} of the local ring $\overline{O} := O/NO$ is finitely generated.*

In the sequel we use the brief notation (O, M) to denote a local ring O having maximal ideal M . Consequently we sometimes write $\mathbf{L}((S, N))$ for the class $\mathbf{L}(S)$ defined in (L). We are always assuming that $S \neq \text{Frac } S$ and call such a valuation domain **non-trivial**.

In Subsection 1.4 we shall prove that a finitely generated, flat R -algebra A is finitely presented and coherent – see Theorem 1.7 and its corollary. Consequently every localization A_q , $q \in \text{Spec } A$, is a member of the class $\mathbf{L}(R_{q \cap R})$. The main result of the present section allows the calculation of the weak dimension of A_q :

Theorem 1.2 *Let S be a non-trivial valuation domain. A essentially finitely presented S -algebra $O \in \mathbf{L}(S)$ is regular if and only if the weak dimension $\text{wdim } O$ is finite. If O is regular, then $\text{wdim } O = \dim \overline{O} + 1$ holds.*

The proof of Theorem 1.2 consists of three major steps:

First an upper bound for the weak dimension in the regular case is provided. This result is valid for a larger class of local S -algebras than actually considered in Theorem 1.2. It involves the Zariski cotangent space of the local ring $(\overline{O}, \overline{M})$ defined in (L): the \overline{O} -module $\overline{T} := \overline{M}/\overline{M}^2$ is a vector space over the residue field $k = O/M = \overline{O}/\overline{M}$. Its dimension is finite by assumption and yields an upper bound for $\text{wdim } O$:

Theorem 1.3 *Let S be a non-trivial valuation domain. The weak dimension of a regular local ring $O \in \mathbf{L}(S)$ satisfies $\text{wdim } O \leq \dim_k \overline{T} + 1$, $k = \overline{O}/\overline{M}$. If the local ring \overline{O} is noetherian, then the stronger inequality $\text{wdim } O \leq \dim \overline{O} + 1$ holds. In particular: a local ring $O \in \mathbf{L}(S)$ is regular if and only if its weak dimension is finite.*

Theorem 1.3 lays the basis for an inductive proof of Theorem 1.2 once we are able to reduce the weak dimension of O by a natural method. Such a method is provided by the following result well-known in the noetherian case:

Theorem 1.4 *Let (O, M) be a coherent, regular, local ring of finite weak dimension. Then for every element $t \in M \setminus M^2$ the local ring O/tO is regular (and coherent) and the formula $\text{wdim}(O/tO) = \text{wdim} O - 1$ holds. In particular $tO \in \text{Spec} O$.*

Finally we have to verify the assertion of Theorem 1.2 for the starting point of the induction, that lies at $\text{wdim} O = 1$: for $O \in \mathbf{L}(S)$ the extension $O|S$ is faithfully flat, thus by [Gla1], Thm. 3.1.1 $\text{wdim} O \geq \text{wdim} S$. On the other hand a local ring S is a non-trivial valuation domain if and only if $\text{wdim} S = 1$ – see [Gla1], Cor. 4.2.6.

Lemma 1.5 *Let (S, N) be a non-trivial valuation domain. An essentially finitely presented S -algebra $(O, M) \in \mathbf{L}(S)$ of weak dimension $\text{wdim} O = 1$ is a valuation domain with the property $\text{Rad}(NO) = M$.*

Proof. We already mentioned that a local ring of weak dimension 1 is a valuation domain. Choose some finitely presented, flat S -algebra A such that $O = A_q$ for some $q \in \text{Spec} A$ lying over N . In Subsection 1.4, Lemma 1.9 we shall see that since O is a domain A can be chosen to be a domain too. An application of [Kn1], Lemma 2.8 yields that the prime $M \cap A$ is minimal among the primes of A lying over N . Since the primes contained in $M \cap A$ are totally ordered with respect to inclusion, the minimality of M implies $\text{Rad}(NO) = M$. \square

We now directly turn to the proof of Theorem 1.2 postponing the proofs of Theorem 1.3 and 1.4 to the Subsections 1.3 and 2.1 respectively. The required properties of algebras over Prüfer domains are verified in Subsection 1.4.

Proof of Theorem 1.2. Let $(O, M) \in \mathbf{L}(S)$ be essentially finitely presented over S . Theorem 1.3 shows that regularity of O is equivalent to the finiteness of the weak dimension $\text{wdim} O$.

To verify the formula for the weak dimension we perform an induction starting with the case $\text{wdim} O = 1$: Lemma 1.5 yields $\text{Rad}(NO) = M$, consequently $\dim \overline{O} = 0$ as asserted.

Assume next that $\text{wdim} O > 1$ holds: the ring \overline{O} is noetherian, Theorem 1.3 thus yields $\text{wdim} O \leq \dim \overline{O} + 1$, hence $\dim \overline{O} \geq 1$. We choose a foreimage $t \in M$ of some $\bar{t} \in \overline{M} \setminus \overline{M}^2$ under the natural homomorphism $O \rightarrow \overline{O}$. We get $t \notin M^2$, hence an application of Theorem 1.4 yields that the local ring O/tO is coherent, regular, and possesses weak dimension

$$\text{wdim}(O/tO) = \text{wdim} O - 1. \quad (5)$$

Claim: O/tO is essentially finitely presented over S/p , $p := tO \cap S$, and $O/tO \in \mathbf{L}(S/p)$ – note that S/p can be a field.

The domain O/tO is a local extension of the valuation domain S/p hence faithfully flat. We get $O/tO \in \mathbf{L}(S/p)$. By assumption $O = A_q$, $q \in \text{Spec} A$, for some finitely presented S -algebra A . Consequently $O/tO = B_{q_B}$ for $B := A/(tO \cap A)$ and $q_B := q/(tO \cap A)$. The domain B is a finitely generated S/p -algebra and thus finitely presented – [Nag], Thm. 3, [RG], Cor. 3.4.7.

We are now in the position to perform the induction step in which we have to distinguish between two cases:

S/p is no field: by induction hypothesis we then get

$$\text{wdim}(O/tO) = \dim(O/tO \otimes_{S/p} S/N) + 1, \quad (6)$$

where $O/tO \otimes_{S/p} S/N = \overline{O}/\overline{t} \overline{O}$, $\overline{t} := t + NO$. The Principal Ideal Theorem yields the inequality $\dim(\overline{O}/\overline{t} \overline{O}) \geq \dim \overline{O} - 1$. The combination of the equations (5) and (6) thus leads to $\text{wdim} O \geq \dim \overline{O} + 1$. The reversed inequality $\text{wdim} O \leq \dim \overline{O} + 1$ is given by Theorem 1.3.

S/p is a field: the ring O/tO then is noetherian, therefore the homological dimensions satisfy $\text{wdim} O/tO = \text{gldim} O/tO$. Equation (5) thus yields

$$\text{wdim} O = \text{wdim}(O/tO) + 1 = \text{gldim}(O/tO) + 1 = \dim(O/tO) + 1. \quad (7)$$

The local ring O_{tO} is regular, coherent and by formula (3) has weak dimension $\text{wdim} O = 1$. Since $O_{tO} \in \mathbf{L}(S)$ is essentially finitely presented over S , an application of Lemma 1.5 yields the minimality of the prime tO among the primes of O containing NO . In Subsection 1.4 we shall see that the ring $\overline{O} = O/NO$ is equidimensional – see Corollary 1.11. We get $\dim \overline{O} = \dim(O/tO)$ and thus the assertion by plugging into equation (7). \square

1.3 Upper bounds for the weak dimension

In the present subsection we provide a proof of Theorem 1.3 essentially by applying the following variation of a result of W. Vasconcelos [Vas], Cor. 5.12:

Proposition 1.6 *Let (O, M) be a coherent, regular, local ring with the property $M = \text{Rad} I$ for some finitely generated ideal $I \triangleleft O$. If I is generated by $\ell \in \mathbb{N}$ elements, then $\text{wdim} O \leq \ell$ holds. If I is generated by a regular sequence of length ℓ , then the weak dimension satisfies $\text{wdim} O = \ell$.*

Proof. Let $M = \text{Rad } I$ for some ideal $I \triangleleft O$ generated by $\ell \in \mathbb{N}$ elements. The equation $\text{wdim } O = \text{Grade } M$ ([Gla2], Lemma 3) reduces the proof of the first statement in the theorem to verifying the inequality $\text{Grade } M \leq \ell$. According to [Nor], Ch. 5, Thm. 12 for every ideal $J \triangleleft O$ the equation

$$\text{Grade } J = \text{Grade}(\text{Rad } J) \quad (8)$$

holds. Moreover $\text{Grade } J \leq \ell$ whenever J is generated by ℓ elements ([Nor], Ch. 5, Thm. 13). Hence we get $\text{Grade } M = \text{Grade } I \leq \ell$ as asserted. If I is generated by a regular sequence of length ℓ , then we clearly have $\text{Grade } I \geq \ell$, which yields the second assertion. \square

Within the subsequent proof and at other points of this article we need to consider those primes $p \triangleleft S$ of a valuation domain S that have the property

$$p = \bigcup_{p' \in \text{Spec } S: p' \subset p} p'. \quad (9)$$

We call a prime p with the property (9) a **limit prime of S** . Note that instead of taking the union of all primes $p' \subset p$ we can use only those primes p' of the form $\text{Rad}(tS)$ for some $t \in S$ – in the sequel we denote this set by $\mathcal{R}(S)$.

Proof of Theorem 1.3. Let $(O, M) \in \mathbf{L}((S, N))$ be regular; we first prove the inequality $\text{wdim } O \leq \dim \overline{T} + 1$. Choose foreimages $\bar{t}_1, \dots, \bar{t}_\ell \in \overline{M}$ of a basis of \overline{T} . For each \bar{t}_i choose a foreimage $t_i \in M$ under the natural homomorphism $O \rightarrow \overline{O}$. The ideal $I := \sum_{i=1}^{\ell} Ot_i$ then satisfies the equation $M = I + NO$ since by Nakayama's Lemma \overline{M} is generated by $\bar{t}_1, \dots, \bar{t}_\ell$.

If N is no limit prime, then $N = \text{Rad}(tS)$ for some $t \in N$. The ideal $J := I + tO$ is generated by $\ell + 1$ elements and satisfies $\text{Rad } J = M$. Proposition 1.6 thus yields $\text{wdim } O \leq \ell + 1$ as asserted.

In the remaining case the comment subsequent to equation (9) for the maximal ideal N gives us

$$M = \bigcup_{p \in \mathcal{R}(S)} (I + pO).$$

Using [Gla2], Lemma 3 and a basic property of Grade yields $\text{wdim } O = \text{Grade } M = \sup(\text{Grade } J \mid J \triangleleft O \text{ finitely generated})$. Hence in order to prove the assertion in the present case it suffices to verify the inequality $\text{Grade } J \leq \ell + 1$ for every finitely generated ideal $J \triangleleft O$. Such an ideal is contained in some $I + pO$, $p \in \mathcal{R}(S)$. Utilizing the relation (8) we get

$$\text{Grade}(I + pO) \leq \text{Grade}(\text{Rad}(I + tO)) = \text{Grade}(I + tO) \leq \text{Grade}(I + pO)$$

and therefore $\text{Grade } J \leq \text{Grade } (I + pO) = \text{Grade } (I + tO) \leq \ell + 1$.

In the case of a noetherian ring \overline{O} we start by choosing a system of parameters $\overline{t}_1, \dots, \overline{t}_\ell \in \overline{M}$, $\ell = \dim \overline{O}$. The ideal $I := \sum_{i=1}^{\ell} Ot_i$ generated by foreimages $t_i \in M$ of the elements \overline{t}_i under the map $O \rightarrow \overline{O}$ then satisfies $\text{Rad } (I + NO) = M$ since by construction $\text{Rad } ((I + NO)/NO) = \overline{M}$. From here on the proof proceeds almost identically to the proof of the preceding case. \square

1.4 Properties of algebras over Prüfer domains

Let R be a Prüfer domain with field of fractions $\text{Frac } R =: K \neq R$. In the sequel we discuss various properties of R -algebras, that are relevant for the present article. Partially they have already been used in Subsection 1.2.

Every finitely generated ideal $I \triangleleft R$ is invertible and thus finitely presented – a Prüfer domain is coherent. While in general a finitely presented algebra over a coherent ring needs not be coherent, Prüfer domains show a smoother behavior in that respect:

Theorem 1.7 *Every finitely generated, flat algebra A over a Prüfer domain R is finitely presented and coherent.*

Proof. A finitely generated, flat algebra over a domain is finitely presented – [RG], Cor. 3.4.7. The coherence of A now follows from the coherence of the polynomial ring $R[X_1, \dots, X_n]$ over a Prüfer ring ([Sab], Prop. 3, [Gla1], Cor. 7.3.4.) and the coherence of factor rings of a coherent ring with respect to a finitely generated ideal. \square

Corollary 1.8 *Let (O, M) be a local, flat, essentially finitely presented R -algebra and $p := M \cap R$, then $O \in \mathbf{L}(R_p)$.*

Proof. Choose a finitely presented R -algebra A such that $O = A_q$, $q \in \text{Spec } A$, holds. The set of primes $q' \in \text{Spec } A$ such that $A_{q'}|_{R_{q' \cap R}}$ is flat, is open and by assumption non-empty. Thus we can replace A by a finitely presented, flat R -algebra. The stability of coherence under localization and Theorem 1.7 yield the coherence of O , which is the property that was missing to ensure $O \in \mathbf{L}(R_p)$. \square

A coherent, regular, local ring O is a domain – [Gla1], Lemma 4.2.3. The subsequent auxiliary result is motivated by this fact and has been applied in the proof of Theorem 1.2.

Lemma 1.9 *Let R be a non-trivial valuation domain. Every essentially finitely presented domain $O \in \mathbf{L}(R)$ can be written as $O = A_q$ for some finitely generated domain A .*

Proof. Choose a finitely presented, flat R -algebra A such that $O = A_q$ for some $q \in \text{Spec } A$. The flatness of $A|R$ implies $q_0 \cap R = 0$ for every minimal prime ideal $q_0 \subseteq q$. Since O is a domain we have $A_q = (A/q_0)_{q/q_0}$. \square

We finish this subsection with a dimension-theoretic property of algebras over Prüfer domains:

Theorem 1.10 *Let A be a domain that is finitely generated over the Prüfer domain R . Then the rings $A \otimes_R kp$, $p \in \text{Spec } R$, $kp = \text{Frac}(R/p)$, are equidimensional of dimension $\dim(A \otimes_R K)$, $K = \text{Frac } R$.*

Proof. Let $q \in \text{Spec } A$ be minimal among the primes lying over $p \in \text{Spec } R \setminus 0$. [Nag], Lemma 2.1 yields $\text{trdeg}(A|R) = \text{trdeg}(A/q|R/p)$. The assertion now follows from the fact that $\text{trdeg}(A|R) = \dim(A \otimes_R K)$ and $\text{trdeg}(A/q|R/p) = \dim(A/q \otimes_{R/p} kp)$ hold. \square

Corollary 1.11 *Let (S, N) be a non-trivial valuation domain. The factor ring $\overline{O} = O/NO$ of an essentially finitely presented, local domain $O \in \mathbf{L}((S, N))$ is equidimensional.*

Proof. Lemma 1.9 shows that $O = A_q$, $q \in \text{Spec } A$, for some domain A finitely generated over S . \square

2 Regular sequences

In the present section we are concerned with the construction of maximal regular sequences within the maximal ideal M of a coherent local ring O of finite weak dimension. These sequences are obtained through lifting linearly independent elements from the O/M -vector space M/M^2 to M . We study this approach in Subsection 2.1, where we also provide the proof of Theorem 1.4 that was postponed in Subsection 1.2. In Subsection 2.2 we apply the results to a regular, local, essentially finitely presented, flat S -algebra O over the valuation domain S . We shall see that in this case regular sequences of the maximal possible length $\text{wdim } O$ exist within M . As a consequence the fibre ring $O/(M \cap S)O$ is shown to be Cohen-Macaulay, and even regular in the case that $M \cap S$ is *not* finitely generated.

2.1 Coherent local rings of finite weak dimension

For a local ring (O, M) the O -module $T := M/M^2$ forms a vector space over the residue field $k := O/M$. If O is a noetherian regular ring, then a regular sequence (t_1, \dots, t_ℓ) in M can be constructed by choosing the elements $t_i \in M, i = 1, \dots, \ell$, to be k -linearly independent modulo M^2 . The ideal $q \triangleleft O$ generated by t_1, \dots, t_ℓ is prime and the factor ring O/q is regular too. In the sequel we show that this method carries over to coherent local rings of finite weak dimension:

Theorem 2.1 *In a coherent local ring (O, M) of finite weak dimension every set $\{t_1, \dots, t_\ell\} \subset M$ such that $\{t_1 + M^2, \dots, t_\ell + M^2\} \subset T$ is k -linearly independent has the properties:*

- (1) (t_1, \dots, t_ℓ) is a regular sequence,
- (2) $q := \sum_{i=1}^{\ell} Ot_i$ is a prime ideal,
- (3) $\text{wdim } O_q = \ell$ and $\text{wdim } O/q = \text{wdim } O - \ell$.

Note however that in the situation given in Theorem 2.1 the vector space T may have strange properties:

Example 2.2 If the maximal ideal of a valuation domain (S, N) is not finitely generated it satisfies $N = N^2$, thus $T = 0$ holds although $\text{grade } N = 1$. The localization $O := S[X]_q$ of the polynomial ring $S[X]$ at $q := XS[X] + N[X]$ is coherent ([Gla1], Thm. 7.3.3) and satisfies $\text{wdim } O = 2$ according to Hilbert's Szyzygy Theorem [Vas], Thm. 0.14. On the other hand $T = k(X + M^2)$.

This example demonstrates that we cannot expect to obtain maximal regular sequences within M through lifting a k -basis of T as in the noetherian case. Nevertheless we pursue the approach of lifting as far as possible.

As one expects the proof of Theorem 2.1 is an induction on the number ℓ of k -linearly independent elements. The case of one element $t \in M \setminus M^2$ is treated in Theorem 1.4 whose proof was postponed in Subsection 1.2; we provide this proof now:

Proof of Theorem 1.4. Every coherent, regular, local ring is a domain [Gla1], Cor. 6.2.4, a fact that we need twice throughout the current proof.

The weak dimension of the coherent ring O/tO can be computed through formula (3) in Subsection 1.1: $\text{wdim}(O/tO) = \text{fdim}_{O/tO}(k), k = O/M$. In order to prove regularity of O/tO it therefore suffices to show $\text{fdim}_{O/tO}(M/tO) \neq \infty$.

Claim: $\text{fdim}_{O/tO}(M/tM) \neq \infty$ implies $\text{fdim}_{O/tO}(M/tO) \neq \infty$.

Choose a set $\mathcal{B} \subset T$ such that $\mathcal{B} \cup \{\bar{t}\}, \bar{t} := t + M^2$ is a k -basis of T . For each $\bar{b} \in \mathcal{B}$ choose a foreimage in M ; let $B \subset M$ be the set of these foreimages. The ideal

$$I := \sum_{b \in B} bO + M^2$$

has the properties $M = I + tO$ and $I \cap tO = tM$. Only the inclusion \subseteq of the second property requires a verification: an element $a \in I \cap tO$ can be expressed as

$$tr = a = \sum_{b \in B} r_b b + s, \quad r, r_b \in O, \quad s \in M^2.$$

Taking this equation modulo M^2 and using the k -linear independence of $\mathcal{B} \cup \{\bar{t}\}$ yields $r, r_b \in M$ as desired.

Utilizing the properties of I we see that the exact sequence

$$0 \rightarrow tO/tM \rightarrow M/tM \rightarrow M/tO \rightarrow 0$$

splits and we obtain $M/tM \cong M/tO \oplus tO/tM$ as O/tO -modules. Since for arbitrary modules N, P the inequality $\text{fdim } N \leq \text{fdim } (N \oplus P)$ holds, the claim is proved.

We now have to verify $\text{fdim}_{O/tO}(M/tM) \neq \infty$: the maximal ideal M by assumption possesses a flat resolution

$$0 \rightarrow F_m \rightarrow \dots \rightarrow F_0 \rightarrow M \rightarrow 0$$

of finite length. Tensoring with O/tO yields a resolution

$$0 \rightarrow F_m \otimes_O (O/tO) \rightarrow \dots \rightarrow F_0 \otimes_O (O/tO) \rightarrow M/tM \rightarrow 0$$

of M/tM by flat O/tO -modules; here we use the fact that t is no zero-divisor.

The formula for the weak dimension of O/tO follows from [Gla1], Thm. 3.1.4 (2), where we again need the fact that t is no zero-divisor. \square

Proof of Theorem 2.1. We perform an induction on the number ℓ of elements in the sequence.

In the case $\ell = 1$, point 1 is clear and point 2 as well as the second part of point 3 are the content of Theorem 1.4. An application of formula (3) in Subsection 1.1 yields $\text{wdim } O_{t_1O} = 1$, since the maximal ideal of O_{t_1O} is principal.

Assume now that $\ell > 1$ holds and consider the coherent, regular, local ring $O' := O/t_1O$ of weak dimension $\text{wdim } O' = \text{wdim } O - 1$.

The elements $t'_i := t_i + t_1 O$, $i = 2, \dots, \ell$, are k -linearly independent modulo $(M')^2$, where $M' := M/t_1 O$. Therefore by induction hypothesis the sequence (t'_2, \dots, t'_ℓ) is regular, so that the sequence (t_1, \dots, t_ℓ) is regular too. Moreover the ideal $q := \sum_{i=1}^{\ell} O t_i$ is prime since this holds for $q' := q/t_1 O \triangleleft O'$; we get

$$\text{wdim}(O/q) = \text{wdim}(O'/q') = \text{wdim } O' - (\ell - 1) = \text{wdim } O - \ell.$$

As a localization of a regular ring $O'_{q'} = O_q/t_1 O_q$ is regular too. We apply [Gla1], Thm. 3.1.4 (2) and obtain

$$\ell - 1 = \text{wdim } O'_{q'} = \text{wdim } O_q - 1$$

as asserted. □

Corollary 2.3 *For every coherent local ring (O, M) of finite weak dimension the inequalities*

$$\dim_k T \leq \text{grade } M \leq \text{wdim } O$$

hold.

Remark 2.4 We have already seen in Example 2.2 that the first inequality in Corollary 2.3 can be strict. An example for the strict inequality $\text{grade } M < \text{wdim } O$ seems to be missing in the literature.

Corollary 2.5 *For a coherent local ring (O, M) of finite weak dimension the equation $\dim_k T = \text{wdim } O$ holds if and only if M is finitely generated. M is then generated by a regular sequence of length $\text{wdim } O$.*

Proof. Let $q \triangleleft O$ be an ideal generated by a set t_1, \dots, t_ℓ of foreimages of a k -basis of T . Assume that $\dim_k T = \text{wdim } O$ holds. Theorem 2.1 then yields $\text{wdim } O/q = 0$, which implies that O/q is a field. Assume now that M is finitely generated. The regular local ring $O' := O/q$ has the property $(M/q)/(M/q)^2 = 0$. Since M/q is finitely generated Nakayama's Lemma yields $q = M$. □

Remark 2.6 Based on the work of Vasconcelos [Vas] the regularity of a coherent local ring (O, M) with finitely generated maximal ideal M has been investigated in [TZT]. One of the main results of this article states that the regularity of O is equivalent to the existence of a regular sequence generating M . In [THT] this equivalence is generalized to indecomposable, semilocal, coherent rings.

2.2 Regular local algebras over a valuation domain

In a valuation domain (S, N) radical ideals are prime and the primes of the form $p = \text{Rad}(tS)$, $t \in S$, are precisely the non-limit primes – see (9) in Subsection 1.3. Utilizing this property of prime ideals Theorem 2.1 applied to a regular, essentially finitely presented algebra $(O, M) \in \mathbf{L}(S)$ yields distinguished regular sequences within M :

Theorem 2.7 *Let (S, N) be a non-trivial valuation domain. The maximal ideal of a regular, essentially finitely presented S -algebra $(O, M) \in \mathbf{L}(S)$ contains a maximal regular sequence (t_1, \dots, t_d) of length $d = \text{wdim } O$.*

More precisely if $t_1, \dots, t_\ell \in M$ are elements such that $(t_1 + M^2, \dots, t_\ell + M^2)$ forms a k -basis of $T = M/M^2$, then:

- (1) *If M is finitely generated, then (t_1, \dots, t_ℓ) is a maximal regular sequence of length $\ell = \text{wdim } O$ and $M = \sum_{i=1}^{\ell} Ot_i$.*
- (2) *If M is not finitely generated, then for every $t \in N \setminus 0$ the sequence (t_1, \dots, t_ℓ, t) is maximal regular of length $\ell + 1 = \text{wdim } O$. The ideal $\text{Rad}(\sum_{i=1}^{\ell} Ot_i + Ot)$ is a prime ideal. If $t \in N$ satisfies $\text{Rad}(tS) = N$, then the equation $M = \text{Rad}(\sum_{i=1}^{\ell} Ot_i + Ot)$ holds.*

Remark 2.8 Note that Theorem 2.7 implies $\text{grade } M = \text{wdim } O$ for regular, essentially finitely presented S -algebras $(O, M) \in \mathbf{L}(S)$. In [Vas] Vasconcelos showed that a maximal regular sequence within a coherent, regular, local ring (O, M) satisfying $\text{grade } M = \text{wdim } O$ may have a length smaller than $\text{wdim } O$. The author does not know whether this can happen for the local rings considered in the theorem.

As an important consequence of Theorem 2.7 we obtain information about the structure of the fibre ring $\overline{O} = O/NO$ and get a result that is analogous to the noetherian case if N is finitely generated but possesses a surprising non-noetherian component otherwise:

Theorem 2.9 *The fibre ring \overline{O} of a regular, local, essentially finitely presented S -algebra $(O, M) \in \mathbf{L}(S)$ over a non-trivial valuation domain (S, N) has the properties:*

- (1) \overline{O} is Cohen-Macaulay.
- (2) If N is not finitely generated, then \overline{O} is regular.

Proof. For a non-finitely generated ideal $N \triangleleft S$ the second assertion implies the first. We thus prove the first assertion only in the case $N = tS$.

We have to show that $\dim \overline{O} = \text{grade } \overline{M}$ holds; to this end we apply the formula [Alf], Cor. 2.10 that describes the behavior of polynomial grade in essentially finitely presented ring extensions like $O|S$:

$$\text{Grade } {}_O M = \text{Grade } {}_S N + \text{Grade } {}_{\overline{O}} \overline{M}; \quad (10)$$

the indices denote the base ring with respect to which polynomial grade has to be calculated. Theorem 1.2 yields $\text{Grade } {}_O M = \text{wdim } O = \dim \overline{O} + 1$. Furthermore $\text{Grade } {}_S N = 1$ since S is a valuation domain. Finally $\text{Grade } {}_{\overline{O}} \overline{M} = \text{grade } \overline{O} \overline{M}$ because \overline{O} is noetherian. Equation (10) consequently leads to the claim $\dim \overline{O} = \text{grade } \overline{O} \overline{M}$.

Assume next that N is not finitely generated and consider the natural homomorphism

$$\tau : T \rightarrow \overline{T} := (M/NO)/(M/NO)^2. \quad (11)$$

Since by assumption $N = N^2$ holds we get $\text{Ker } \tau = (M^2 + NO)/M^2 = 0$. The surjectivity of τ thus yields $\dim_k T = \dim_k \overline{T}$, $k = O/M$. On the other hand by Theorem 2.7, (2) and Theorem 1.2

$$\dim_k T = \text{wdim } O - 1 = \dim(\overline{O})$$

and thus $\dim(\overline{O}) = \dim_k \overline{T}$, which implies the regularity of \overline{O} . \square

Remark 2.10 Alfonsi gives a homological definition of non-noetherian grade for arbitrary modules. The proof for the equivalence of his definition with the one given by Northcott in the case of ideals can be found in [Gla1], Thm. 7.1.8.

The remainder of the subsection is devoted to the proof of Theorem 2.7.

Consider a set $P := \{t_1, \dots, t_\ell\} \subset M$ such that $t_1 + M^2, \dots, t_\ell + M^2 \in T$ are k -linearly independent. According to Theorem 2.1 the sequence (t_1, \dots, t_ℓ) is regular and the ideal

$$q_P := \sum_{i=1}^{\ell} O t_i \quad (12)$$

is a prime ideal of O . The proof of Theorem 2.7 strongly relies on the particular distribution of these primes within $\text{Spec } O$.

Proposition 2.11 *Let (S, N) be a non-trivial valuation domain and assume that $(O, M) \in \mathbf{L}(S)$ is essentially finitely presented and regular. Then the prime ideals defined by equation (12) share the property $q_P \cap S \in \{0, N\}$. The equation $q_P \cap S = N$ implies that N is principal.*

Proof. We perform an induction on the cardinality ℓ of P .

Assume first that $q_p = tO$ and let $p := tO \cap S$; moreover assume that $p \neq 0$ holds. The regular local ring $O_{tO} \in \mathbf{L}(S_p)$ is essentially finitely presented over S_p . Equation (3) in Subsection 1.1 yields $\text{wdim } O_{tO} = 1$, that is O_{tO} is a valuation domain.

Claim: there exists a prime $Q \in \text{Spec } O$ containing tO with the properties $Q \cap S = N$ and O_Q is a valuation domain.

We choose a domain A finitely generated over S such that $O = A_q$ for some prime $q \triangleleft A$ (Lemma 1.9). Lemma 1.5 shows that the prime $q_t := tO \cap A$ is minimal among the primes of A lying over p . An application of [Nag], Lemma 2.1 yields:

$$\text{trdeg}(A/q_t|S/p) = \text{trdeg}(A|S). \quad (13)$$

Let $\bar{q}_0 \in \text{Spec}(A/q_t)$ be minimal among the primes containing N/p ; \bar{q}_0 can be chosen such that $\bar{q}_0 \subseteq q/q_t$ holds. Again utilizing [Nag], Lemma 2.1 the prime $q_0 \in \text{Spec } A$ with $q_0/q_t = \bar{q}_0$ satisfies:

$$\text{trdeg}(A/q_0|S/N) = \text{trdeg}(A/q_t|S/p). \quad (14)$$

Combining (13) and (14) we can choose a transcendence basis (x_1, \dots, x_m) of $\text{Frac } A|\text{Frac } S$ within A such that $(x_1 + q_0, \dots, x_m + q_0)$ forms a transcendence basis of $\text{Frac } A/q_0|\text{Frac } S/N$. This choice yields

$$q_0 \cap S[x_1, \dots, x_m] = N[x_1, \dots, x_m].$$

The localization $B := S[x_1, \dots, x_m]_{N[x_1, \dots, x_m]}$ is a valuation domain contained in the local ring $A_{q_0} = O_Q$, $Q := q_0O$. Coherent regular rings are normal ([Gla1], page 205), thus the coherent regular ring O_Q contains the integral closure B' of B in $\text{Frac } A$. Since B' is a Prüfer domain O_Q is indeed a valuation domain.

In a valuation domain a finitely generated prime either equals 0 or the maximal ideal. Consequently we get $tO_Q = QO_Q$ and thus $tO \cap S = N$. As we have seen in the proof of the preceding claim the valuation domain O_Q dominates the valuation domain B and the extension $\text{Frac } O_Q|\text{Frac } B$ is finite. General valuation theory yields that finite generation of the maximal ideal QO_Q descends to the maximal ideal $N[x_1, \dots, x_m]B \triangleleft B$. Let $f \in N[x_1, \dots, x_m]B$ be a generator of $N[x_1, \dots, x_m]B$ and let $I \triangleleft S$ be generated by the coefficients of f , then $I = cS$ for some $c \in I$ and the factorization $f = sf^*$, where $f^* \notin N[x_1, \dots, x_m]$ eventually yields $N = I$ as asserted.

We now turn to the case $P = \{t_1, \dots, t_\ell\}$ with $\ell > 1$.

We already know that $t_1O \cap S \in \{0, N\}$. Since the equation $t_1O \cap S = N$ forces $q_P \cap S = N$ and since in this case we also know that N must be principal it suffices to treat the case $t_1O \cap S = 0$.

Consider the set $P_1 := \{t_2 + t_1O, \dots, t_\ell + t_1O\} \subset O/t_1O$, and recall that $O/t_1O \in \mathbf{L}(S)$ is regular. The elements of P_1 are k -linearly independent modulo $(M/t_1O)^2$. By induction hypothesis we can thus assume that the prime ideal

$$q_{P_1} := \sum_{i=2}^{\ell} (O/t_1O)(t_i + t_1O) \triangleleft O/t_1O$$

satisfies $q_{P_1} \cap S \in \{0, N\}$ and that in the second case N is principal. The assertions about q_P now follow. \square

Remark 2.12 The conclusions of Proposition 2.11 hold for every principal prime ideal tO . The requirement $t \notin M^2$ is not needed in the proof of that particular case.

Proof of Theorem 2.7. Let $P := \{t_1, \dots, t_\ell\}$ be chosen such that its elements modulo M^2 form a k -basis of T .

If M is finitely generated, then $M = q_P$ holds by Nakayama's lemma. Moreover by Theorem 2.1, (3), the sequence (t_1, \dots, t_ℓ) possesses the maximal possible length $\ell = \text{wdim } O$.

Assume now that M and thus N are not finitely generated. We have seen in the proof of Theorem 2.9 that in this case the natural map $\tau : T \rightarrow \overline{T}$ is an isomorphism – see (11). As a consequence the elements $t_1 + NO, \dots, t_\ell + NO$ generate \overline{M} , hence $M = q_P + NO$ holds. We next first apply Theorem 2.1 followed by Theorem 1.3 to obtain

$$\text{wdim } (O/q_P) \leq \dim \overline{T} + 1 - \dim T.$$

Hence using $\dim T = \dim \overline{T}$ we arrive at the inequality $\text{wdim } (O/q_P) \leq 1$. The equation $\text{wdim } (O/q_P) = 0$ implies that O/q_P is a field and thus that $M = q_P$ is finitely generated, which has been excluded. Consequently O/q_P is a valuation domain S' and $\ell = \text{wdim } O - 1$ by Theorem 2.1 (3).

Proposition 2.11 tells us that $q_P \cap S = 0$, because N is not finitely generated. We conclude that S' is a local extension of S and that therefore (t_1, \dots, t_ℓ, t) is a regular sequence for every non-zero $t \in N$. Moreover this sequence possesses the maximal possible length $\text{wdim } O$.

If $N = \text{Rad}(tS)$ for some $t \in N$, then $M = q_P + NO = \text{Rad}(q_P + tO)$ as asserted. For an arbitrary $t \in N \setminus 0$ the ideal $\text{Rad}(tS')$ is prime, hence the same holds for $\text{Rad}(q_P + tO)$. \square

Corollary 2.13 *Under the assumptions made in Theorem 2.7 for the local extension $(O, M)|(S, N)$ the maximal ideal M is finitely generated if and only if N is finitely generated (that is principal).*

3 Families over Prüfer domains

The results presented so far in this article grew out of an attempt to understand the geometric structure of certain non-noetherian schemes:

Definition 3.1 Let R be a Prüfer domain. An **R -family** is a separated, faithfully flat R -scheme \mathcal{X} of finite type such that the fibres $\mathcal{X}_p := \mathcal{X} \times_R kp$, $p \in \text{Spec } R$, are equidimensional kp -schemes of a common dimension $n \in \mathbb{N}$. An *integral* R -family \mathcal{X} is called **R -variety**. The common dimension n of the fibres is called the **relative dimension of \mathcal{X} over R** .

Throughout this section assume again R to be a Prüfer domain with field of fractions $K \neq R$. As an immediate consequence of the Theorems 1.7 and 1.10 we get:

Theorem 3.2 *Let R be a Prüfer domain.*

- (1) *An R -family \mathcal{X} is of finite presentation and its structure sheaf $\mathcal{O}_{\mathcal{X}}$ is coherent.*
- (2) *A separated faithfully flat R -scheme \mathcal{X} of finite type possessing an equidimensional generic fibre is an R -family.*

For an arbitrary scheme \mathcal{X} we define the **regular locus** $\text{Reg } \mathcal{X}$ to be the set of points $P \in \mathcal{X}$ such that the local ring $\mathcal{O}_{\mathcal{X}, P}$ at P is regular (in the sense of Bertin). Theorem 1.2 allows to give a description of the regular locus of an R -family:

Theorem 3.3 *Let \mathcal{X} be an R -family of relative dimension $n \in \mathbb{N}$ over the Prüfer domain R . The regular locus of \mathcal{X} is characterized through*

$$\text{Reg } \mathcal{X} = \{P \in \mathcal{X} \mid \text{wdim } \mathcal{O}_{\mathcal{X}, P} \leq n + 1\}.$$

The upper bound $n + 1$ is attained at the closed points of the fibres \mathcal{X}_p , where $p \in \text{Spec } R \setminus 0$.

A rough estimate of the extend of the singular locus $\mathcal{X} \setminus \text{Reg } \mathcal{X}$ of an R -family \mathcal{X} is provided by the next result.

Theorem 3.4 *The regular locus of an R -family \mathcal{X} satisfies:*

$$\bigcup_{p \in \operatorname{Spec} R} \operatorname{Reg} \mathcal{X}_p \subseteq \operatorname{Reg} \mathcal{X}.$$

Proof. For a point $P \in \operatorname{Reg} \mathcal{X}_p$ the exact sequence

$$0 \rightarrow p\mathcal{O}_{\mathcal{X},P} \rightarrow \mathcal{O}_{\mathcal{X},P} \rightarrow \mathcal{O}_{\mathcal{X}_p,P} \rightarrow 0$$

is a flat resolution of the $\mathcal{O}_{\mathcal{X},P}$ -module $\mathcal{O}_{\mathcal{X}_p,P}$: since $\mathcal{O}_{\mathcal{X},P}$ is a flat R -module the ideal $p\mathcal{O}_{\mathcal{X},P} \cong p \otimes_R \mathcal{O}_{\mathcal{X},P}$ is a flat $\mathcal{O}_{\mathcal{X},P}$ -module. We can thus apply [Gla1], Thm. 3.1.1 and obtain:

$$\operatorname{wdim} \mathcal{O}_{\mathcal{X},P} \leq \operatorname{wdim} \mathcal{O}_{\mathcal{X}_p,P} + \operatorname{fdim} \mathcal{O}_{\mathcal{X}_p,P} \leq \dim \mathcal{O}_{\mathcal{X}_p,P} + 1,$$

where in the last inequality we use the equation $\operatorname{wdim} \mathcal{O}_{\mathcal{X}_p,P} = \dim \mathcal{O}_{\mathcal{X}_p,P}$ for the noetherian, regular, local ring $\mathcal{O}_{\mathcal{X}_p,P}$. \square

It is well-known that the regular locus of a reduced scheme Y of finite type over a field k is open and non-empty. Therefore the significance of Theorem 3.4 is increased through the following fact proved in [Kn2]: denote the set of maximal ideals of the Prüfer domain R by $\operatorname{MaxSpec} R$ and define

$$\operatorname{fgSpec} R := \{p \in \operatorname{Spec} R \mid pR_p \text{ is finitely generated.}\}.$$

According to [Kn2], Cor. 2.8 a normal R -variety \mathcal{X} of relative dimension 1 has the property: the fibres $\mathcal{X} \times_R kp$, $p \notin \operatorname{fgSpec} R$, are reduced. If $\mathcal{X}|\operatorname{Spec} R$ is proper, then all fibres over primes $p \notin \operatorname{MaxSpec} R$ are reduced. The valuation-theoretic proof given in [Kn2] for this fact easily carries over to the case of arbitrary relative dimension.

We have seen in Subsection 2.2 that the set $\operatorname{fgSpec} R$ plays a particular role in the context of regularity too:

Theorem 3.5 *The fibres of an R -family \mathcal{X} over the Prüfer domain R possess the following properties:*

- (1) *If \mathcal{X} is regular, then all fibres \mathcal{X}_p are Cohen-Macaulay, while the fibres \mathcal{X}_p , $p \notin \operatorname{fgSpec} R$, are regular kp -schemes.*
- (2) *If the structure morphism $\mathcal{X} \rightarrow \operatorname{Spec} R$ is closed and all fibres \mathcal{X}_p , $p \in \operatorname{MaxSpec} R$, are regular, then \mathcal{X} is regular.*

Proof. Assertion 1 follows from Theorem 2.9.

Assertion 2: by assumption about the structure morphism the Zariski closure of every point $P \in \mathcal{X}$ intersects some fibre \mathcal{X}_p , $p \in \operatorname{MaxSpec} R$. Therefore

$$\mathcal{O}_{\mathcal{X},P} = (\mathcal{O}_{\mathcal{X},Q})_q, \quad q \in \operatorname{Spec} \mathcal{O}_{\mathcal{X},Q},$$

for some $Q \in \mathcal{X}_p$. By assumption $\mathcal{O}_{\mathcal{X},Q}$ is regular, therefore $\mathcal{O}_{\mathcal{X},P}$ is regular too. \square

We conclude this section with the discussion of an example:

Example 3.6 Let R be a 2-dimensional valuation domain with field of fractions K and assume that the spectrum of R satisfies $\text{Spec } R = \text{fgSpec } R = \{0, p, N\}$. Choose elements $s, t \in N$ such that $pR_p = sR_p$ and $N = tR$ hold.

Consider the projective R -family

$$\mathcal{C} := \text{Proj}(R[X, Y, Z]/FR[X, Y, Z]), \quad F := sX^3 + tZ^3 - ZY^2. \quad (15)$$

The generic fibre $C := \mathcal{C} \times_R K$ is an elliptic curve – of course we have to assume $\text{char } K \neq 2, 3$ at this point. Consequently the scheme \mathcal{C} is an R -variety of relative dimension 1 (Theorem 3.2) and $C \subset \text{Reg } \mathcal{C}$.

The fibre $\mathcal{C}_p = \mathcal{C} \times_R kp$ is defined through the homogenous polynomial

$$(t + p)Z^3 - ZY^2 \in kp[X, Y, Z].$$

Assuming that kp is perfect and using the Jacobian criterion we see that the point $Q := [1 : 0 : 0] \in \mathbb{P}_{kp}^2$ is the only singular point of \mathcal{C}_p . It lies in the affine open subscheme $U \subset \mathcal{C} \subset \mathbb{P}_R^2$ defined through the condition $X \neq 0$. The coordinate ring of U is given through

$$R[y, z] := R[Y, Z]/GR[Y, Z], \quad G := s + tZ^3 - ZY^2 \in R[Y, Z], \quad (16)$$

hence

$$\mathcal{O}_{\mathcal{C},Q} = R[y, z]_q, \quad q := R[y, z]y + R[y, z]z + pR[y, z].$$

The relation $s = zy^2 - tz^3$ (16) shows that $qR_p[y, z]$ and thus the maximal ideal of $\mathcal{O}_{\mathcal{C},Q}$ are generated by y, z .

We claim that (y, z) is a regular sequence in $R_p[y, z]$ and thus in $\mathcal{O}_{\mathcal{C},Q}$: indeed

$$R_p[y, z]/yR_p[y, z] \cong R_p[Z]/(tZ^3 + s)R_p[Z]$$

and the polynomial $tZ^3 + s$ is prime in $R_p[Z]$. Consequently $y \in R_p[y, z]$ is prime and z is no zero-divisor on $R_p[y, z]/yR_p[y, z]$.

We conclude that $Q \in \text{Reg } \mathcal{C}$: the Koszul complex of the maximal ideal $\mathcal{M}_{\mathcal{C},Q}$ yields a flat resolution of length 2, therefore $\text{wdim } \mathcal{O}_{\mathcal{C},Q} \leq 2$ by Formula (3) in Subsection 1.1.

Since $\text{Reg } \mathcal{C}_p = \mathcal{C}_p \setminus Q$ Theorem 3.4 implies $\mathcal{C}_p \subset \text{Reg } \mathcal{C}$.

The closed fibre $\mathcal{C}_N := \mathcal{C} \times_R kN$ is defined through the homogenous polynomial

$$ZY^2 \in kN[X, Y, Z].$$

The singular locus of $\mathcal{C}_N \subset \mathbb{P}_{kN}^2$ thus equals the doubled line given through $Y = 0$. For the sake of simplicity we assume kN to be algebraically closed. The singular points of the form $[\alpha : 0 : 1] \in \mathcal{C}_N$, $\alpha \in kN$, lie in the affine open subscheme $V \subset \mathcal{C}$ defined through $Z \neq 0$. The coordinate ring of V thus is given through

$$R[x, y] := R[X, Y]/HR[X, Y], \quad H := sX^3 - Y^2 + t \in R[X, Y]. \quad (17)$$

The point $Q' := [\alpha : 0 : 1]$ corresponds to a prime ideal

$$q' := R[x, y](x - a) + R[x, y]y + R[x, y]t, \quad a \in R^\times.$$

Using equation (17) we get

$$t = y^2 - s((x - a) + a)^3 = y^2 - s(x - a)((x - a)^2 + 3a(x - a) + 3a^2) - sa^3,$$

which shows that $t + sa^3 \in q'$. Since the element $t + sa^3$ generates N , we see that q' is generated by $x - a$ and y .

We claim that $(x - a, y)$ is a regular sequence in $R[x, y]$, thus – as performed earlier – proving the regularity of $\mathcal{O}_{\mathcal{C}, Q'}$: indeed the isomorphism

$$R[x, y]/(x - a)R[x, y] \cong R[Y]/(Y^2 - (t + sa^3))$$

shows that $x - a$ is a prime element of $R[x, y]$ and that y is no zerodivisor on $R[x, y]/(x - a)R[x, y]$.

Let $Q'' := [1 : 0 : 0] \in \mathcal{C}_N$; so far we have shown that $\mathcal{C}_N \setminus Q'' \subset \text{Reg } \mathcal{C}$. The point Q'' itself is *not* regular: consider

$$\mathcal{O}_{\mathcal{C}, Q''} = R[y, z]_{q''}, \quad q'' := R[y, z]y + R[y, z]z + R[y, z]t;$$

if $\mathcal{O}_{\mathcal{C}, Q''}$ were regular, then according to the Theorems 1.2 and 2.7 two among the three elements y, z, t would be prime. However t is not prime, because the local ring

$$\mathcal{O}_{\mathcal{C}, Q''}/t\mathcal{O}_{\mathcal{C}, Q''} = \mathcal{O}_{\mathcal{C}_N, Q''}$$

is no domain. In addition the factor ring

$$\mathcal{O}_{\mathcal{C}, Q''}/z\mathcal{O}_{\mathcal{C}, Q''} = (R[y, z]/zR[y, z])_{q''}$$

is no domain too.

4 Appendix: global dimension

In general the finiteness of the weak dimension of a (local) ring O has no influence on its global dimension. However Jensen [Jen] and Osofsky [Os2] give an upper bound for the global dimension in terms of the weak dimension provided that O satisfies some cardinality condition. We use this generalization to obtain an upper bound for the global dimension $\text{gldim } O$ of certain regular local rings $O \in \mathbf{L}(S)$.

Let κ be an infinite cardinal number. A commutative ring O is said to be κ -**noetherian** if every ideal $I \triangleleft O$ can be generated by a set of cardinality at most κ and at least one ideal of O cannot be generated by a set of cardinality less than κ . For every $m \in \mathbb{N}$ denote by \aleph_m the m -th successor cardinal of the cardinality \aleph_0 of natural numbers.

Theorem 4.1 *Let (S, N) be a non-trivial \aleph_m -noetherian valuation domain. Every essentially finitely generated regular S -algebra $O \in \mathbf{L}(S)$ then satisfies $\text{gldim } O \leq \text{wdim } O + m + 1 \leq \dim \bar{O} + m + 1$.*

Proof. The second inequality follows from the first and Theorem 1.3.

As for the first inequality we know from [Jen], Thm. 2 respectively [Os2], Cor. 1.4 that an \aleph_m -noetherian ring O satisfies

$$\text{gldim } O \leq \text{wdim } O + m + 1,$$

hence it remains to show that the local ring O is \aleph_l -noetherian for some natural number $l \leq m$.

Since a localization of a κ -noetherian ring is κ' -noetherian for some $\kappa' \leq \kappa$, it suffices to prove that an algebra A finitely generated over a \aleph_m -noetherian ring S is $\aleph_{m'}$ -noetherian for some $m' \leq m$.

As an S -module the algebra A can be generated by countably many elements, for example by the monomials in a set a_1, \dots, a_s of S -algebra generators of A . By [Os2], Lemma 1.1 this implies that every S -submodule of A can be generated by \aleph_m elements. Thus every ideal $I \triangleleft A$ as an S -module and hence as an A -module too can be generated by \aleph_m elements, which completes the proof. \square

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