Products of absolute Galois groups*

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Abstract

If the absolute Galois group G_K of a field K is a direct product $G_K = G_1 \times G_2$ then one of the factors is prosolvable and either G_1 and G_2 have coprime order or K is henselian and the direct product decomposition reflects the ramification structure of G_K. So, typically, the direct product of two absolute Galois groups is not an absolute Galois group.

In contrast, *free* (profinite) products of absolute Galois groups are known to be absolute Galois groups. The same is true about free pro-p products of absolute Galois groups which are pro-p groups. We show that, conversely, if $\mathcal{C}$ is a class of finite groups closed under forming subgroups, quotients and extensions, and if the class of pro-$\mathcal{C}$ absolute Galois groups is closed under free pro-$\mathcal{C}$ products then $\mathcal{C}$ is either the class of all finite groups or the class of all finite p-groups.

As a tool, we prove a generalization of an old theorem of Neukirch which is of interest in its own right: If K is a non-henselian field then every finite group is a subquotient of G_K, unless all decomposition subgroups of G_K are pro-p groups for a fixed prime p.

Introduction

Problem 12.19 in [FJ] asks whether it is possible that the compositum of two non-trivial Galois extensions L_1, L_2 of a hilbertian field K with $L_1 \cap L_2 = K$ is

*Mathematics Subject Classification: 12E30, 12F10, 12J10

†Heisenberg-Stipendiat der Deutschen Forschungsgemeinschaft (KO 1962/1-2)

separably closed. In [HJ] it was shown that the answer is no, since such a compositum is again hilbertian (for even stronger generalizations of Weissauer's Theorem cf. [Ha]). In this paper we show that this is not a specifically hilbertian phenomenon, but rather general. It rarely happens that the separable closure K^{sep} of a field K is the compositum of non-trivial linearly disjoint Galois extensions, i.e. that the absolute Galois group $G_K := Gal(K^{sep}/K)$ of K is a proper direct product:

Theorem A *Let K be a field with $G_K = G_1 \times G_2$ for two non-trivial normal subgroups G_1, G_2 of G_K. Then G_K is torsion-free, one of the factors is prosolvable and either G_1 and G_2 are of coprime order or K admits a non-trivial henselian valuation. For each prime p dividing the order of both factors at least one of the factors has abelian p-Sylow subgroups.*

As a consequence of Theorem A we obtain a new proof for the negative solution to the problem mentioned at the beginning (Corollary 2.4). It is another consequence of Theorem A that the class of absolute Galois groups is not closed under direct products: $G_{\mathbf{Q}} \times G_{\mathbf{Q}}$, for example, is not an absolute Galois group. In fact, the class of absolute Galois groups is not even closed under semidirect products, not even if the factors have coprime order: If G_1 and G_2 are Sylow-subgroups of $G_{\mathbf{Q}}$ w.r.t. distinct primes, then no group of the form $G_1 \rtimes G_2$ is an absolute Galois group (Proposition 2.6).

Theorem A is based on our general account ([K3]) of how valuations on a field K are reflected in G_K. We recall the necessary definitions and facts in section 1. Section 2 proves a more detailed version (Theorem 2.3) as well as a pro-p version (Proposition 2.2) of Theorem A and some consequences. Section 3 generalizes an old Theorem of Neukirch:

Theorem B *Let K be a non-henselian field. Then any finite group occurs as subquotient of G_K, unless all decomposition subgroups of G_K (w.r.t. non-trivial valuations) are pro-p groups for a fixed prime p.*

Theorem 3.1 gives a refined variant.

The last section uses this result together with the machinery from [K3] to characterize those classes of pro-$\mathcal{C}$ absolute Galois groups which are closed under free pro-$\mathcal{C}$ products:

Theorem C *Let $\mathcal{C}$ be a class of finite groups closed under forming subgroups, quotients and extensions, let $\mathcal{G}_{\mathcal{C}}$ be the class of absolute Galois groups which* *are pro-$\mathcal{C}$ groups. Then $\mathcal{G}_{\mathcal{C}}$ is closed under free pro-$\mathcal{C}$ products iff*

- *either $\mathcal{C}$ is the class of all finite groups*
- *or, for some prime p, $\mathcal{C}$ is the class of all finite p-groups.*

Acknowldgement: I would like to thank the referee for numerous very helpful comments.

1 Background from 'valois theory'

In this section we collect several facts from 'valois theory' (or 'galuation theory', if you prefer), the Galois theory of valued fields, in particular how henselianity determines and is determined by the absolute Galois group. We always consider general (Krull) valuations, not just discrete or rank-1 valuations. For background in general valuation theory see e.g. [E], or the Appendix of [DP], for a systematic developement of 'valois theory' see [K3].

For a valued field (K, v) we denote the valuation ring, its maximal ideal, the residue field and the (additive) value group by $\mathcal{O}_v$, $\mathcal{M}_v$, Kv and Γ_v respectively, always bearing in mind the three canonical exact sequences associated to any valuation:

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{M}_v & \to & \mathcal{O}_v & \stackrel{\phi_v}{\to} & Kv & \to & 0 \\ 1 & \to & \mathcal{O}_v^\times & \to & K^\times & \stackrel{v}{\to} & \Gamma_v & \to & 0 \\ 1 & \to & 1+\mathcal{M}_v & \to & \mathcal{O}_v^\times & \stackrel{\phi_v}{\to} & Kv^\times & \to & 1 \end{array}$$

v is **henselian**, if it has a unique prolongation w to the separable closure K^{sep} of K, or, equivalently, if Hensel's Lemma holds which says that simple zeros lift, i.e. that any monic $f \in \mathcal{O}_v[X]$, for which $\overline{f} \in Kv[X]$ (obtained by applying the residue map ϕ_v to the coefficients of f) has a simple zero $a \in Kv$, has a zero $x \in \mathcal{O}_v$ with $\phi_v(x) = a$. In this case the action of G_K on K^{sep} is compatible with w (i.e. $w(\sigma(x)) = w(x)$ for all $\sigma \in G_K$, $x \in K^{sep}$) and induces a canonical epimorphism $G_K \twoheadrightarrow G_{Kv}$ with kernel

$$T = \{\sigma \in G_K \mid \forall x \in \mathcal{O}_w : \sigma(x) - x \in \mathcal{M}_w\},$$

the **inertia subgroup** of G_K (w.r.t. w). And there is a canonical epimorphism

$$T \twoheadrightarrow Hom(\Gamma_w/\Gamma_v, (K^{sep}w)^\times)$$

with kernel

$$V = \{\sigma \in G_K \mid \forall x \in \mathcal{O}_w : \sigma(x) - x \in x\mathcal{M}_w\},$$

the **ramification subgroup** of G_K, which is trivial for $char\, Kv = 0$, and which is the unique Sylow-q subgroup of T if $q = char\, Kv > 0$. In particular, V is a characteristic subgroup of T and thus normal in G_K. The image T/V is torsion-free abelian, the p-Sylow sugroups of T/V being free $\mathbf{Z}_p$-modules of rank $\dim_{\mathbf{F}_p} \Gamma_v/p\Gamma_v$. Moreover, the three exact sequences split:

$$\begin{array}{ccccccccc} 1 & \to & T/V & \to & G_K/V & \to & G_{Kv} & \to & 1 \\ 1 & \to & V & \to & G_K & \to & G_K/V & \to & 1 \\ 1 & \to & T & \to & G_K & \to & G_{Kv} & \to & 1 \end{array}$$

(the first by adjoining to K a compatible system of n-th roots of elements $x \in K$ with $v(x) \notin n\Gamma_v$, the second, because the q-th cohomological dimension $cd_q G_K/V = cd_q G_{Kv} \leq 1$ for $char\, Kv = q > 0$, hence any epimorphism onto G/V with pro-q kernel splits (cf. [KPR]), and the third by 'transitivity'). So $G_K/V \cong T/V \rtimes G_{Kv}$, where the kernel of the action of G_{Kv} on the non-trivial p-Sylow subgroups of T/V (which are characteristic subgroups of T/V, and thus normal in G_K/V) is $G_{Kv(\mu_{p^\infty})}$, where μ_{p^∞} denotes the group of all p-power roots of unity. This accounts for the well known example:

Example 1.1 *Let K be a field with $char\, K = 0$ and assume that K contains all roots of unity. Let Γ be an ordered abelian group and let*

$$F := K((\Gamma)) := \{\alpha = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma \mid a_\gamma \in K \ s.t.\ supp(\alpha) \text{ is well-ordered}\}$$

be the generalized power series field with coefficients in K and exponents in Γ, where $supp(\alpha) := \{\gamma \in \Gamma \mid a_\gamma \neq 0\}$ is the 'support' of α. Then $v(\alpha) := \min supp(\alpha)$ defines a henselian valuation on F with $Fv = K$ and $\Gamma_v = \Gamma$. Hence

$$G_F \cong (\prod_{p \ prime} \mathbf{Z}_p^{\dim_{\mathbf{F}_p} \Gamma/p\Gamma}) \times G_K.$$

Any torsion-free abelian profinite group A is of the shape $A \cong \prod_{p\ prime} \mathbf{Z}_p^{\alpha_p}$ for some cardinals α_p. For each prime p we consider the p-adic rationals $Z_p := \mathbf{Q} \cap \mathbf{Z}_p$ as ordered abelian subgroup of $\mathbf{Q}$ and and let Γ_p be the lexicographically ordered direct sum of α_p copies of Z_p. Then $\Gamma_p = q\Gamma_p$ for each prime $q \neq p$

and the $\mathbf{F}_p$-dimensions of $\Gamma_p/p\Gamma_p$ is α_p. Taking Γ to be the lexicographic direct sum of all Γ_p we obtain A as absolute Galois group, and, more generally,

$$G_{K((\Gamma))} \cong A \times G_K.$$

This isomorphism holds even if K does not contain all roots of unity but only all p-power roots of unity for primes p with $\alpha_p > 0$.

Whenever v is henselian with $\Gamma_v \neq p\Gamma_v$ and $char\, Kv \neq p$, the intersection of T with a p-Sylow subgroup P of G_K is a non-trivial abelian normal subgroup of P. This property, in turn, goes as Galois code for henselianity:

Fact 1.2 (Theorem 1 of [K3]) *Let K be a field, let p be a prime, let P be a p-Sylow subgroup of G_K and assume that P is not procyclic or isomorphic to $\mathbf{Z}_2 \rtimes \mathbf{Z}/2\mathbf{Z}$.*

Then K admits a henselian valuation v with $char\, Kv \neq p$ and $\Gamma_v \neq p\Gamma_v$ iff P has a non-trivial normal abelian subgroup.

We shall also need a pro-p version of Fact 1.2. We denote by $K(p)$ the **maximal pro-p extension** of K, i.e. the compositum of all finite Galois extensions of K with Galois group a p-group. Then $(K(p))(p) = K(p)$, i.e. $K(p)$ is p-**closed**, and $G_K(p) := Gal(K(p)/K)$ is the maximal pro-p quotient of G_K. We call a valued field (K, v) p-**henselian** if v extends uniquely to $K(p)$.

Fact 1.3 (for $p = 2$ section 4 of [EN], for $p > 2$ Main Theorem of [EK], new proof in Theorem 2.15 in [K3]) *Let p be a prime, let K be a field containing a primitive p-th root ζ_p of unity (so $char\, K \neq p$) and assume that $G_K(p)$ is not procyclic and not isomorphic to $\mathbf{Z}_2 \rtimes \mathbf{Z}/2\mathbf{Z}$.*

Then K admits a p-henselian valuation v with $char\, Kv \neq p$ and $\Gamma_v \neq p\Gamma_v$ iff $G_K(p)$ has a non-trivial abelian normal subgroup.

A field may have more than one henselian or p-henselian valuation. If v and w are two valuations of a field K we say that v is **finer** than w (w is **coarser** than v) if $\mathcal{O}_v \subseteq \mathcal{O}_w$ or, equivalently (!), $\mathcal{M}_w \subseteq \mathcal{M}_v$. In this case, v induces a valuation v/w on Kw, and v is henselian (resp. p-henselian) iff both w and v/w are. In particular, coarsenings of (p-)henselian valuations are (p-)henselian, the coarsest (p-)henselian valuation always being the trivial valuation.

Fact 1.4 ([EE] resp. Prop. 2.8 in [K3]) *If v and w are henselian (resp. p-henselian) valuations on a field K such that Kv is not separably (resp. p-)closed then v is comparable to, i.e. finer or coarser than w. So if K is not separably (resp. p-)closed, there is a* **canonical (p-)henselian valuation** *on K, namely the coarsest (p-)henselian valuation with separably (resp. p-)closed residue field, if there is any such, and the finest (p-)henselian valuation otherwise. In particular, K admits a non-trivial (p-)henselian valuation iff the canonical (p-)henselian valuation on K is non-trivial.*

It is immediate from the definition that if v is a (p-)henselian valuation on K and if w is any coarsening of v then $T_w \subseteq T_v$, where T_v and T_w are the corresponding inertia subgroups of G_K resp. $G_K(p)$. As a consequence of Fact 1.3 and Fact 1.4 one now obtains

Fact 1.5 (for $p = 2$, section 4 of [EN], for $p > 2$, Cor. 2.3 and Cor. 3.3 of [EK], also Cor. 2.17 of [K3]) *Let p be a prime, let K be a field containing ζ_p, and if $p = 2$ assume K to be nonreal. Let v be the finest coarsening of the canonical p-henselian valuation on K with char $Kv \neq p$ and let T denote the inertia subgroup of $G_K(p)$ w.r.t. v.*

Then

1. *There is a (unique) maximal normal abelian subgroup $N \lhd G_K(p)$ containing all normal abelian subgroups of $G_K(p)$.*
2. *$G_K(p)$ is abelian iff $G_{Kv}(p) = 1$ or $\mu_{p^\infty} \subseteq Kv$ and $G_{Kv}(p) \cong \mathbf{Z}_p$.*
3. *If $1 \neq N \neq G_K(p)$ then $N = T$.*

Fact 1.4 together with the fact that any two distinct prolongations of a valuation to a Galois extension are incomparable immediately gives

Fact 1.6 *Let L/K be a Galois (resp. pro-p Galois) extension with L not separably (resp. p-)closed, let v be coarser than the canonical henselian (resp. p-henselian) valuation on L. Then $v\mid_K$ is also (p-)henselian, and coarser than the canonical (p-)henselian valuation on K.*

Somewhat more surprising is the following fact that henselianity is also inherited from 'Sylow extensions':

Fact 1.7 ([K3], Proposition 3.1) *Let K be a field, let F be the fixed field of a non-trivial p-Sylow subgroup $P = G_F$ of G_K in K^{sep}. Let v be a coarsening* *of the canonical henselian valuation on F, and if $p = 2$ and if Fv is real closed assume v to be the coarsest henselian valuation on F with real closed residue field.*

Then $v\mid_K$ is also henselian, and coarser than the canonical henselian valuation on K.

Finally, we recall a generalization of Satz I of [N] which may be considered one of the starting points of 'valois theory'. Neukirch's Satz says that perfect fields with prosolvable absolute Galois group are henselian unless, for some fixed prime p, the absolute Galois group of all completions w.r.t. (archimedean or non-archimedean) absolute values on K are pro-p groups or pro-$\{2,3\}$ groups. The generalizsation combines [G], Satz 7.2, and [P], Proposition on p. 153 (where a different, but euqivalent hypothesis is made; cf. our Lemma 3.5):

Fact 1.8 *Let K be a field with G_K prosolvable, let $p < q$ be primes, and assume that K has separable extensions L and M such that G_L is a non-trivial pro-p group (infinite if $p = 2$ and $q = 3$), G_M is a non-trivial pro-q group and v resp. w is a non-trivial (not necessarily proper) coarsening of the canonical henselian valuation on L resp. M.*

Then $v\mid_K$ and $w\mid_K$ are comparable and the coarser valuation is henselian.

We shall generalize this fact even further in Theorem 3.1.

Valois theory was modelled as valuation theoretic analogue to Artin-Schreier theory for real fields. Since we shall also need Becker's relative variant of Artin-Schreier theory we recall it as last fact in this section:

Fact 1.9 ([B2], Chapter II, §2, Theorem 3) *Let L/K be a Galois extension such that L is p-closed for each prime p with $p \mid [L : K]$. Then for any subextension F/K of L/K the following conditions are equivalent:*

1. $1 < [L : F] < \infty$
2. *F is a relative real closure of K in L*
3. $[L : F] = 2$

In particular, the non-trivial elements of finite order in $Gal(L/K)$ are precisely the involutions.

2 Direct products as absolute Galois groups

We first observe that absolute Galois groups which are direct products are torsion-free. More generally, one has:

Lemma 2.1 *Let L/K be a Galois extension with group $G = Gal(L/K)$ such that $L = L(p)$ for each prime p with $p \mid \sharp G$. If $G = G_1 \times G_2$ for two nontrivial normal subgroups $G_1, G_2 \lhd G$ then G is torsion-free.*

Proof: By Fact 1.9, the only non-trivial torsion elements in G are involutions and the only non-trivial finite subgroups of G are of order 2.. In particular, all involutions are contained in one of the direct factors, say in G_2: if $\epsilon = (\epsilon_1, \epsilon_2) \in G$ is an involution with $\epsilon_1, \epsilon_2 \neq 1$ then ϵ_1 and ϵ_2 would generate a 4-Klein subgroup of G. If $\epsilon \in G_2$ is of order 2 then the fixed field $F = Fix\langle\epsilon\rangle$ is euclidean and, by [B1], Satz 2, this implies that all K-automorphisms of F are trivial. But $1 \neq G_1 \cong Gal(Fix\, G_2/K) \subseteq Aut_K(F)$: contradiction. So G must be torsion-free. **q.e.d.**

Let us now prove a pro-p version of Theorem A:

Proposition 2.2 *Let p be a prime and let F be a field of characteristic $\neq p$ containing a primitive p-th root of unity. Assume that $G_F(p) = P_1 \times P_2$ for two non-trivial normal subgroups P_1, P_2 of $G_F(p)$.*

Then $G_F(p)$ is torsion-free, $\mu_{p^\infty} \subseteq F$, and one of the factors P_i is abelian.

If v denotes the finest coarsening of the canonical p-henselian valuation on F with $char\, Fv \neq p$ then $\Gamma_v \neq p\Gamma_v$ and either $G_F(p)$ is abelian and $G_{Fv}(p)$ is procyclic or one of the P_i is contained in the inertia subgroup of G_F w.r.t. v.

Proof: By the previous lemma, $G_F(p)$ is torsion-free. In particular, by Fact 1.9, F is non-real if $p = 2$.

Now we show that $\mu_{p^\infty} \subseteq F$. Assume the contrary, say $\zeta_{p^n} \notin F$ for some $n > 1$. Let E/F be a maximal subextension of $F(p)/F$ with $\zeta_{p^n} \notin E$. Then $G_E(p) = Gal(F(p)/E) = \langle\sigma\rangle \cong \mathbf{Z}_p$ for some $\sigma = (\sigma_1, \sigma_2) \in P_1 \times P_2$: note that $E(\zeta_{p^n})$ is the unique Galois-extension of degree p over E, so as pro-p group with cyclic Frattini quotient, $G_E(p)$ is procyclic pro-p and torsion-free, i.e. $\cong \mathbf{Z}_p$. We hence find some $\tau \in P_1 \times P_2$ with

$$\langle\tau, \sigma\rangle = \langle\tau\rangle \times \langle\sigma\rangle \cong \mathbf{Z}_p \times \mathbf{Z}_p :$$

if $\sigma_1 = 1$ pick $\tau \in P_1 \setminus \{1\}$, if $\sigma_2 = 1$ pick $\tau \in P_2 \setminus \{1\}$ and if $\sigma_1 \neq 1 \neq \sigma_2$ take $\tau = \sigma_1$. Let L be the fixed field of $\langle \tau, \sigma \rangle$. Then, by Fact 1.5, $\mu_{p^\infty} \subseteq L$, and $L \subseteq E \not\ni \zeta_{p^n}$ gives the contradiction we are after.

If $G_F(p)$ is abelian, then each factor P_i is abelian, and, again by Fact 1.5, $G_{Fv}(p)$ is procyclic.

If $G_F(p)$ is non-abelian, then one of the factors, say P_2, is non-abelian. For each $\sigma \in P_1 \setminus \{1\}$, let K_σ be the fixed field of $\langle \sigma \rangle \times P_2$ and observe that $\langle \sigma \rangle \times 1$ is a nontrivial abelian normal subgroup of $G_{K_\sigma}(p) = \langle \sigma \rangle \times P_2$. Let v_σ be the finest coarsening of the canonical p-henselian valuation on K_σ with $char\, K_\sigma v_\sigma \neq p$. Then, by Fact 1.5.1. and 3., σ is contained in the inertia subgroup of $G_{K_\sigma}(p)$ w.r.t. v_σ. As P_2 is non-abelian, so is $G_{K_\sigma v_\sigma}$, and, therefore, the residue field of the unique prolongation of v_σ to the fixed field F_2 of P_2 (in $F(p) = K_\sigma(p)$) is not p-closed, i.e. this prolongation still is a coarsening of the canonical p-henselian valuation on F_2. Thus, by Fact 1.6, $w_\sigma := v_\sigma \mid F$ is a coarsening of the canonical p-henselian valuation on F, and so of v. Hence the inertia subgroup T of $G_F(p)$ w.r.t. v contains the inertia subgroup T_{w_σ} of $G_F(p)$ w.r.t. w_σ (cf. the remarks following Fact 1.4). But T_{w_σ} contains the inertia subgroup T_{v_σ} of G_{K_σ} w.r.t. v_σ: by definition of inertia groups, $T_{v_\sigma} = T_{w_\sigma} \cap G_{K_\sigma}$. This shows that $\sigma \in T$, and, as $\sigma \in P_1 \setminus \{1\}$ was arbitrary, that $P_1 \subseteq T$. **q.e.d.**

Now we can prove a refined version of Theorem A:

Theorem 2.3 *Let K be a field with $G_K = G_1 \times G_2$ for two non-trivial normal subgroups G_1, G_2 of G_K, let v be the canonical henselian valuation on K and let $\pi : G_K \twoheadrightarrow G_{Kv}$ be the canonical epimorphism.*

Then G_K is torsion-free, $G_{Kv} = \pi(G_1) \times \pi(G_2)$ and $(\sharp\pi(G_1), \sharp\pi(G_2)) = 1$. In particular, one of the factors G_i is prosolvable and v is non-trivial if $(\sharp G_1, \sharp G_2) \neq 1$.

If p is a prime dividing $(\sharp G_1, \sharp G_2)$ then $char\, K \neq p$, $\mu_{p^\infty} \subseteq K(\zeta_p)$, the p-Sylow subgroups of G_1 or of G_2 are abelian, and $\Gamma_{v_p} \neq p\Gamma_{v_p}$ for the finest coarsening v_p of v with residual characteristic $\neq p$ (and hence also $\Gamma_v \neq p\Gamma_v$).

Proof: Again, by Lemma 2.1, G_K is torsion-free.

For $i = 1, 2$, we define K_i to be the fixed field of G_i, and T_i to be the inertia subgroup of G_i w.r.t. the unique prolongation of v to K^{sep}. Then $T_1 \times T_2$ is the inertia subgroup T of G_K: clearly, $T_1 \times T_2 \subseteq T$; the canonical restriction isomorphisms $G_1 \to Gal(K_2/K)$ and $G_2 \to Gal(K_1/K)$ carry inertia onto inertia and the compositum of the corresponding inertia subfields of K_1/K and K_2/K (which is the fixed field of $T_1 \times T_2$) is again inert, i.e. $T \subseteq T_1 \times T_2$.

Thus $ker\, \pi = ker\, (\pi \mid_{G_1}) \times ker\, \pi \mid_{G_2})$ and so $G_{Kv} = \pi(G_K) = \pi(G_1) \times \pi(G_2)$.

Now let p be a prime with $p \mid (\sharp G_1, \sharp G_2)$. Then, for any p-Sylow subgroups P_1 of G_1, P_2 of G_2, $P := P_1 \times P_2$ is a p-Sylow subgroup of G_K. Since G_K (and hence P) is torsion-free, P contains subgroups of the shape $\mathbf{Z}_p \times \mathbf{Z}_p$. Let F be the fixed field of P. Then $cd_p G_F = cd_p P \geq cd_p(\mathbf{Z}_p \times \mathbf{Z}_p) = 2 > 1$, so $char\, F = char\, K \neq p$. As G_F is a pro-p group, $\zeta_p \in F$: adjoining ζ_p to any field of characteristic $\neq p$ is an extension of degree $< p$, because ζ_p is a zero of the polynomial $X^{p-1} + X^{p-2} + \ldots + X + 1$. Proposition 2.2 now implies that $\mu_{p^\infty} \subseteq F$ (so also $\mu_{p^\infty} \subseteq K(\zeta_p)$), that P_1 or P_2 is abelian, and that the finest coarsening w_p of the canonical henselian valuation on F with residual characteristic $\neq p$ has non-p-divisible value group. By Fact 1.7, $w_p \mid_K$ is a coarsening of the canonical henselian valuation on K, and so a coarsening of v_p. In particular, $\Gamma_{v_p} \neq p\Gamma_{v_p}$ and $\Gamma_v \neq p\Gamma_v$, so v is certainly non-trivial if the orders of G_1 and G_2 have a common factor.

Now $\pi(G_1)$ and $\pi(G_2)$ must be of coprime order: otherwise, the residue field Kv of v is not separably closed, so v is the finest henselian valuation on K, but, by what we have just seen, the canonical henselian valuation on Kv would also be non-trivial, giving rise to a proper henselian refinement of v: contradiction.

Finally, for $i = 1$ or 2, $\sharp\pi(G_i)$ is odd, and so, by Feit-Thompson, $\pi(G_i)$ is pro-solvable. But then so is $G_i \cong T_i \rtimes \pi(G_i)$. **q.e.d.**

Corollary 2.4 *Let K be a field such that $K^{sep} = L_1 L_2$ for non-trivial incomparable Galois extensions L_1, L_2 of K. Then K has a non-trivial henselian valuation or $(\sharp G_{L_1}, \sharp G_{L_2}) = 1$. In both cases, K is not hilbertian.*

Proof: If G_{L_1} and G_{L_2} are not of coprime order, then by the Theorem, $L_1 \cap L_2$ admits a non-trivial henselian valuation, since $G_{L_1 \cap L_2} = G_{L_1} \times G_{L_2}$. Hence the canonical henselian valuation on $L_1 \cap L_2$ is nontrivial, and, by Fact 1.6, its restriction to K remains henselian ($L_1 \cap L_2/K$ is Galois).

If K is henselian it cannot be hilbertian ([FJ], Ch. 14, Exercise 8). If K is not henselian, then G_{L_1} and G_{L_2} are of coprime order, say $2 \not| \sharp G_{L_1}$. If K were hilbertian, then, by Weissauer's Satz 9 ([We]), a proper finite separable extension of L_1 would also be hilbertian, yet allowing no separable quadratic extension: contradiction. **q.e.d.**

Products of coprime order

Let us recall that a profinite group G is **projective** if any projection $\pi : H \twoheadrightarrow G$ splits, or, equivalently, if all p-Sylow subgroups of G are free pro-p groups. Direct products of projective absolute Galois groups of coprime order are again absolute Galois groups. More generally:

Observation 2.5 *If G_1 and G_2 are any projective profinite groups of coprime order and if G is any semidirect product $G = G_1 \rtimes G_2$ then G is an absolute Galois group of some field.*

Proof: G is again projective since any Sylow-subgroup of G is either in G_1 or conjugate to a Sylow-subgroup of G_2. And any projective group is an absolute Galois group ([LvD]). **q.e.d.**

However, not any direct or semidirect product of absolute Galois groups of coprime order is an absolute Galois group:

Proposition 2.6 *Let p, q be distinct primes, let G_1 resp. G_2 be a pro-p resp. pro-q group without nontrivial abelian normal subgroup, but containing a torsion-free nonabelian metabelian subgroup $H_i \leq G_i$ for $i = 1, 2$ (e.g. if G_1 and G_2 are Sylow subgroups of $G_{\mathbf{Q}}$).*

Then $G := G_1 \rtimes G_2$ (no matter what the action is) cannot be an absolute Galois group.

Proof: Let us first explain the 'e.g.'-bracket: Given a prime p we may choose a prime $l \neq p$ and recall that the p-Sylow subgroups of $G_{\mathbf{Q}_l}$ are non-abelian subgroups of the shape $\mathbf{Z}_p \rtimes \mathbf{Z}_p$. As $G_{\mathbf{Q}_l} \hookrightarrow G_{\mathbf{Q}}$ any p-Sylow subgroup G_1 of $G_{\mathbf{Q}}$ contains non-abelian metabelian subgroups. On the other hand, G_1 contains no non-trivial abelian normal subgroup, since, by Fact 1.2, this would give a non-trivial henselian valuation on $\mathbf{Q}$. But there aren't any.

Now suppose $G = G_K$ for some field K. Then H_2 is a non-abelian metabelian Sylow-q subgroup of $G_1 \rtimes H_2$. By Fact 1.2, the fixed field L of $G_1 \rtimes H_2$ admits a henselian valuation v with $char\, Lv \neq q$, $\Gamma_v \neq q\Gamma_v$ and, as H_2 is non-abelian, $q \mid \sharp G_{Lv}$: H_2 is a q-Sylow subgroup of G_L, so if $q \not| \sharp G_{Lv}$, then $L^{sep}/Fix\, H_2$ is totally and tamely ramified, so $H_2 = G_{Fix\, H_2}$ is abelian. Let L_1 be the fixed field of G_1 and let v_1 be the unique prolongation of v to L_1. Let M be the fixed field of H_1. Then, again by Fact 1.2, M admits a henselian valuation w with $char\, Mw \neq p$, $\Gamma_w \neq p\Gamma_w$ and, as H_1 is non-abelian, $p \mid \sharp G_{Mw}$. Therefore, w is a coarsening of the canonical henselian valuation on M and thus comparable to the unique prolongation of v to M. Hence the restriction w_1 of w to L_1 is comparable to v_1.

If w_1 is coarser than v_1 then w_1 is henselian with residual characteristic not p and with value group not p-divisible. But then the inertia subgroup of G_1 is a non-trivial abelian normal subgroup, contradicting our hypothesis.

If w_1 is finer than v_1 then $p \mid \sharp G_{L_1 v_1}$, and so v_1 is a coarsening of the canonical henselian valuation on L_1, hence, by Fact 1.7, v_1 restricts to a henselian valuation v_K on K. Note that $v_K = v\mid_K$ inherits from v the properties regarding residual characteristic and value group: $char\, Kv \neq q$ and $\Gamma_{v_K} \neq q\Gamma_{v_K}$. Since $G_2 = G_{L_2}$ is a q-Sylow subgroup of G_K, these properties pass to the unique prolongation v_2 of v_K to L_2. But this would imply that G_2 contains a nontrivial abelian normal subgroup (the inertia subgroup w.r.t. v_2), again contradicting our hypothesis. **q.e.d.**

While the Sylow subgroups of $G_{\mathbf{Q}}$ encode the existence of valuations with non-divisible value group, the p-Sylow subgroups G_p of $G_{\mathbf{Q}_p}$ can be realised as absolute Galois groups of fields having no valuations with non-divisible value group (cf. [MW]). We have no answer to the following

Question 2.7 *Is $G_p \times G_q$ an absolute Galois group if $p \neq q$?*

Note that, by Proposition 2.2, $G_p \times G_p$ cannot be an absolute Galois group, because G_p is not abelian.

3 Generalizing a theorem of Neukirch

In this section we stay the course set by Neukirch, Geyer and Pop to prove the perhaps ultimate generalization of Fact 1.8, the refined variant of Theorem B promised in the Introduction. The crucial new ingredients are Lemma 3.2 and Lemma 3.3 which were hard to find, though easy to prove.

Theorem 3.1 *Let K be a field, let L and M be algebraic extensions of K with non-trivial henselian valuations, and assume that G_L is a non-trivial pro-p group and G_M is a non-trivial pro-q group, where $p < q$ are primes. Let v resp. w be non-trivial (not necessarily proper) coarsenings of the canonical henselian valuation on L resp. M, and, if $p = 2$ and Lv is real closed, assume v to be the coarsest henselian valuation on L with real closed residue field.*

Then either any finite group occurs as subquotient of G_K or $v_K := v\mid_K$ and $w_K := w\mid_K$ are comparable and the coarser valuation is henselian on K.

It is clear that Theorem 3.1 *does* generalize Fact 1.8, since a prosolvable group cannot have any finite group as subquotient. And, obviously, Theorem B is an immediate consequence.

For the proof of Theorem 3.1 we need four simple lemmas.

Lemma 3.2 *Let $p < q$ be primes. Then there are infinitely many $k \in \mathbf{N}$ such that*

$$\begin{array}{ll} p \mid q^k + p - 1 & \textit{and} \\ l \not| \, q^k + p - 1 & \textit{for all primes } l < p. \end{array}$$

Proof: If $(p-1) \mid k$ then $q^k \equiv 1 \bmod p$ and so the first condition is satisfied. If l is a prime $< p$ and $(l-1) \mid k$ then $q^k \equiv 1 \bmod l$ and the second condition is satisfied: $p-1 \not\equiv -1 \bmod l$ as $l \neq p$. Hence any multiple k of $\prod_{l \leq p \text{ prime}}(l-1)$ will do. **q.e.d.**

Recall that a transitive subgroup G of the symmetric group S_n is called **imprimitive**, if there is an **imprimitivity domain** for G, i.e. a subset $\Delta \subseteq \{1, \ldots, n\}$ with $1 < \sharp\Delta < n$ such that

$$\forall \sigma \in G: \ \sigma(\Delta) = \Delta \text{ or } \sigma(\Delta) \cap \Delta = \emptyset.$$

One easily checks that in this case $\sharp\Delta \mid n$ (cf. e.g. [Hu], II, Satz 1.2b)). G is called **primitive** if it is not imprimitive.

Lemma 3.3 *Let q be a prime, let k and r be integers ≥ 1 with $r < q$, and assume that $l \not| \, q^k + r$ for all primes $l < r$. Set $n = q^k + r$ and assume that G is a transitive subgroup of S_n containing a q^k-cycle. Then G is primitive.*

Proof: Assume to the contrary that $\Delta \subseteq \{1, \ldots, n\}$ is an imprimitivity domain for G. Let $\sigma \in G$ be a q^k-cycle, say, acting on $\{1, 2, \ldots, q^k\}$ as '+1' (except that $\sigma(q^k) = 1$).

We first claim that $\Delta \subseteq \{1, 2, \ldots, q^k\}$. Otherwise pick $j \in \Delta$ with $j > q^k$. Then $\sigma(j) = j \in \Delta$, and so $\sigma(\Delta) = \Delta$. On the other hand $\sharp\Delta > r$ since $1 < \sharp\Delta \mid n$ and $l \not| \, n$ for any prime $l \leq r$ (if $l = r$ then $l \not| \, n$ as $r < q$). So there is some $i \in \Delta$ with $i \leq q^k$. But then $\sigma(i) = i + 1 \in \Delta$ etc., so

$$\{1, 2, \ldots, q^k\} = \{i, \sigma(i), \ldots, \sigma^{q^k - 1}(i)\} \subseteq \Delta,$$

and hence $\frac{n}{2} < q^k \leq \sharp\Delta \mid n$. This is only possible when $\sharp\Delta = n$, which is not allowed for an imprimitivity domain. This contradiction proves the claim.

Now $\langle\sigma\rangle$ acts transitively on $\{1, 2, \ldots, q^k\}$ and so either Δ is also an imprimitivity domain for the subgroup $\langle\sigma\rangle \leq S_{q^k}$ or $\sharp\Delta = q^k$. In any case, $\sharp\Delta$ is a non-trivial q-power, and so $\sharp\Delta \not| n$: contradiction. **q.e.d.**

Lemma 3.4 *Let $1 \leq m \leq n$ be integers and let G be an m-transitive subgroup of S_n. Then G has a copy of S_m as subquotient.*

Proof: Let $H := \{\sigma \in G \mid \sigma(\{1, \ldots, m\}) = \{1, \ldots, m\}\}$. Then H is a subgroup of G and, by m-transitivity of G, the canonical homomorphism

$$\begin{array}{ccc} H & \to & S_m \\ \sigma & \mapsto & \sigma\mid_{\{1,\ldots,m\}} \end{array}$$

is onto. **q.e.d.**

Lemma 3.5 *Let (K, v) be a valued field with henselization (K^h, v^h), let p be a prime with $p \mid \sharp G_{K^h}$ and let L be the fixed field of a p-Sylow subgroup of G_{K^h}.*

(a) Then the following are equivalent:

1. *The (unique) prolongation v_L of v^h to L is a coarsening of the canonical henselian valuation on L*
2. *For some non-separably closed algebraic extension L'/L, the (unique) prolongation of v^h to L' is a coarsening of the canonical henselian valuation on L'*
3. *$p \mid \sharp G_{K^h u}$ for any proper coarsening u of v^h.*

(b) Moreover, these conditions pass to any coarsening of v, and, if $p > 2$ to any finite extension of K.

In the terminology of [P], condition 3. says that v equals its 'p, K-core'. Note that condition 3. is, in general, stronger than the condition that $p \mid \sharp G_{Kw}$ for any proper coarsening of v.

Proof of Lemma 3.5:

(a) **1.** $\Rightarrow$ **2.** is trivial

2. $\Rightarrow$ **3.** Let L'/L be as in 2., let u be a proper coarsening of v^h, so (K^h, u) is again henselian, and let u' be the unique prolongation of u to L'. Then

u' is a proper coarsening of the canoncial henselian valuation on L'. By the definition of the canonical henselian valuation, $L'u'$ is not separably closed. Hence $p \mid \sharp G_{L'u'}$ and, since $G_{L'u'}$ is a subgroup of $G_{K^h u}$, also $p \mid \sharp G_{K^h u}$.

3. $\Rightarrow$ 1. Condition 3. is inherited from (K^h, v^h) to (L, v_L), because G_L is a p-Sylow subgroup of G_{K^h}. And, by Fact 1.4, 3. for (L, v_L) implies 1.

(b) is immediate from condition 3. and the fact that, for $p > 2$ and any field F, the condition $p \mid \sharp G_F$ remains valid for any finite extension of F. **q.e.d.**

Proof of Theorem 3.1: If one of the valuations v_K, w_K is henselian, say v_K is, then v_K is comparable to w_K: the unique prolongation v_M of v_K to M is comparable to w, since w is a coarsening of the canonical henselian valuation on M (Fact 1.4). Hence v_K is comparable to w_K.

Now assume that v_K and w_K are both non-henselian, and that G is a given finite group. We have to show that G is a subquotient of G_K. It suffices to show this for some algebraic extension of K.

Let us begin with the essential

Case 1: *v_K and w_K are independent*

We will even show that in this case G is a subquotient of $G_{L \cap M}$. So we may assume that $K = L \cap M$. This does not affect the independence of v_K and w_K. By [He], Corollary 1.2 and Proposition 1.4, (L, v) is a henselization of (K, v_K), (M, w) is a henselization of (K, w_K) and K is dense in L resp. M w.r.t. the v- resp. w-topology.

There is a finite subextension K'/K of L/K such that w_K has two distinct prolongations to K': by Fact 1.6, G_L is not a p-Sylow subgroup of G_K, because v_K is non-henselian. So we can choose $K' \subseteq L$ with $p \mid [K' : K] < \infty$. Let N/K be the Galois hull of K'/K and let $N^w = N \cap M$. Then N^w is the decomposition subfield of N/K w.r.t. w and w_K has $[N^w : K]$ many prolongations to N. Assume that w_K has only one prolongation $w_{K'}$ to K'. $K'N^w$ is the decomposition subfield of N/K' w.r.t. w. In particular, $w_{K'}$ (and so, by assumption, also w_K) has $[K'N^w : K']$ many prolongations to N. Hence $[N^w : K] = [K'N^w : K']$, and so $[K'N^w : N^w] = [K' : K]$. Since G_M is a pro-q group, $[N : N_w]$ is a q-power and, thus, so is $[K'N^w : N^w]$. But this contradicts $p \mid [K' : K]$, and the assumption that w_K has only one prolongation to K' was false.

Let (M', w') and $(M^\sharp, w^\sharp)$ be henselisations of K' w.r.t. two distinct prolongations $w'_{K'}$ and $w^\sharp_{K'}$ of w_K to K' (such distinct prolongations are always incomparable, since K'/K is algebraic). Then M' and $M^\sharp$ are finite extensions of some henselisations of (K, w_K) so they are conjugate (over K) to finite extensions of M. Therefore, $G_{M'}$ and $G_{M^\sharp}$ are non-trivial pro-q groups, since $q > 2$. If w (and hence w_K, w' etc.) is a rank-1 valuation then $w'_{K'}$ and $w^\sharp_{K'}$ are independent. If not, we may pass from w to a proper non-trivial coarsening for which then, by Fact 1.4, the residue field is not separably closed.

After these adjustments, and after replacing K once again by an algebraic extension we may now assume that

1. $K = L \cap M' \cap M^\sharp$
2. (K, v_K) is dense in its henselisation (L, v)
3. v_K is independent of w'_K and of $w^\sharp_K$
4. (M', w') is a henselisation of (K, w'_K) with $G_{M'}$ a non-trivial pro-q group
5. $(M^\sharp, w^\sharp)$ is a henselisation of $(K, w^\sharp_K)$ with $G_{M^\sharp}$ a non-trivial pro-q group
6. - either w'_K and $w^\sharp_K$ are independent, (K, w'_K) is dense in (M', w') and $(K, w^\sharp_K)$ is dense in $(M^\sharp, w^\sharp)$ (independent case)
   - or w'_K and $w^\sharp_K$ are (dependent, but) incomparable, and the residue fields $Kw'_K = M'w'$ and $Kw^\sharp_K = M^\sharp w^\sharp$ admit Galois extensions of degree q (dependent case)

We will apply [Wh], Theorem 2, by which any field admitting Galois extensions of prime degree $q > 2$ admits a cyclic Galois extension of degree q^k for any integer $k \geq 1$.

By Lemma 3.2, we may choose $k \in \mathbf{N}$ such that $p \mid q^k + p - 1$, $l \not| q^k + p - 1$ for all primes $l < p$ and $q^k + p - q \geq \sharp G$. Write $n := q^k + p - 1 = m \cdot p$.

Let $g_1 \in L[X]$, $h'_1 \in M'[X]$ and $h^\sharp_1 \in M^\sharp[X]$ be the irreducible polynomials of a primitive element of a cyclic Galois extension of L of degree p resp. of M' of degree q^k resp. of $M^\sharp$ of degree q, where, in the dependent case, the last two cyclic extensions are chosen purely inert, and the primitive element is chosen as a unit inducing a primitive element of the residual extension. Choose elements $a_1, \ldots, a_{p-1} \in \mathcal{O}_{w'}$ with distinct residues in $M'w'$ and elements $b_1, \ldots, b_{n-q} \in \mathcal{O}_{w^\sharp}$ with distinct residues in $M^\sharp w^\sharp$: this is possible because $G_{M'w'}$ and $G_{M^\sharp w^\sharp}$ are pro-q groups, and hence the fields $M'w'$ and $M^\sharp w^\sharp$ are infinite. Define $g := g_1^m \in L[X]$, $h' := h'_1 \cdot (X - a_1) \cdots (X - a_{p-1}) \in$

$M'[X]$ and $h^\sharp := h_1^\sharp \cdot (X - b_1) \cdots (X - b_{n-q}) \in M^\sharp[X]$, and observe that $\deg g = \deg h' = \deg h^\sharp = n$.

Approximate g w.r.t. v, h' w.r.t. w' and $h^\sharp$ w.r.t. $w^\sharp$ by a single monic polynomial $f \in K[X]$ well enough to guarantee that, by Krasner's or by Hensel's Lemma, f decomposes into irreducible factors over L resp. M' resp. $M^\sharp$ like g resp. h' resp. $h^\sharp$, and such that the splitting field of f and g over L resp. that of f and h' over M' resp. that of f and $h^\sharp$ over $M^\sharp$ coincide. In the independent case such an approximation is possible because then all three valuations on K are independent. In the dependent case, one can still simultaneously approximate g w.r.t. v arbitrarily well (by assumption 3. above) and approximate h' resp. $h^\sharp$ well enough by a monic polynomial $f \in (\mathcal{O}_{w'_K} \cap \mathcal{O}_{w^\sharp_K})[X]$ such that the corresponding polynomials in the residue field remain the same (by weak approximation for incomparable valuations).

Then f is an irreducible separable polynomial over K: The degree d of any irreducible factor of f must be a multiple of p because of the decomposition in L; so, in particular, $d > 1$. And the decomposition of f in M' gives $d \geq q^k$. Then $q^k \leq d \leq q^k + p - 1$, so $d = q^k + p - 1$ because $q^k + p - 1$ is the only p-divisible integer between q^k and $q^k + p - 1$. Separability is guaranteed e.g. by f approximating the separable polynomial h' sufficiently well.

Now let G_f be the Galois group of the splitting field of f over K. G_f acts transitively on the n distinct roots of f and may thus be considered a transitive subgroup of S_n. The decomposition subgroup of G_f w.r.t. w' acts on the roots of f like the Galois group of the splitting field of h' (i.e. of h'_1) over M'. In particular, G_f contains a q^k-cycle. By the choice of k, no prime $l \leq p - 1$ divides $n = q^k + p - 1$. Hence, Lemma 3.3 implies that G is a primitive subgroup of S_n.

G_f also contains a cycle of length q, because the decomposition subgroup of G_f w.r.t. $w^\sharp$ does. By [Hu], II, Satz 4.5a), a primitive subgroup of S_n containing a cycle of prime length q is $(n - q + 1)$-transitive. In particular, by Lemma 3.4, it has a copy of S_{n-q+1} as subquotient, and so also one of G ($\sharp G \leq n - q + 1 = q^k + p - q$). This completes the proof of Case 1.

The rest of the proof proceeds along the general valuation theoretic lines of the proof of [P], Proposition p. 153.

Case 2: *v_K and w_K are incomparable*

Let u be the finest common coarsening of v_K and w_K and let $\overline{v_K}$ resp. $\overline{w_K}$ be the valuations induced by v_K resp. w_K on the residue field Ku of u. Then $\overline{v_K}$ and $\overline{w_K}$ are independent.

By Lemma 3.5, the condition that v is a non-trivial coarsening of the canonical henselian valution on L is equivalent to condition 3. of the Lemma that for the henselization (K^h, v_K^h) of (K, v_K) one has $p \mid \sharp G_{Kv'}$ whenever v' is a proper coarsening of v_K^h on K^h. But this condition 3. is inherited by $(Ku, \overline{v_K})$ which, again by the equivalence in Lemma 3.5, means that the fixed field $(\overline{L}, \overline{v})$ of a p-Sylow subgroup of a henselization of $(Ku, \overline{v_K})$ satisfies the same assumption as the one made in our Theorem on the valued field (L, v) (note that, by construction, $\overline{v_K} = \overline{v} \mid_{Ku}$). And, of course, the assumption on (M, w) has a corresponding counter part for suitable $(\overline{M}, \overline{w})$.

Since $\overline{v_K}$ and $\overline{w_K}$ are independent and have both non-separably closed henselizations, they are both non-henselian. Applying now Case 1 to Ku, $(\overline{L}, \overline{v})$ and $(\overline{M}, \overline{w})$ in place of K, (L, v) and (M, w) gives us G as subquotient of G_{Ku}, and hence as subquotient of G_K.

Case 3: *v_K and w_K are comparable*
By Lemma 3.5, the assumptions of the Theorem regarding (L, v) and (M, w) are still valid when passing to non-trivial coarsenings of v and w. So we may assume that $v_K = w_K$ and that $p \cdot q \mid \sharp G_{K^h u}$ for any proper coarsening u of v_K^h, where (K^h, v_K^h) is a henselization of (K, v_K).

Because (K, v_K) is non-henselian, there is a finite subextension K'/K of K^h/K such that besides $v' := v_K^h \mid_{K'}$ there is also another prolongation w' of v_K to K'. v' and w' are then as distinct prolongations of v_K to the *algebraic* extension K'/K incomparable.

The (L, v, p)-assumption on K passes to K' with v_K replaced by v', and, as $q > 2$, the (M, w, q)-assumption passes to K' with w_K replaced by w'. Hence Case 2 applies to K', i.e. $G_{K'}$ has G as subquotient, and then so does G_K. **q.e.d.**

4 Free pro-$\mathcal{C}$ products of absolute Galois groups

In this section we prove a generalization of Theorem C from the Introduction (Theorem 4.2).

Throughout the section, $\mathcal{C}$ denotes a class of finite groups which is closed under subgroups, quotients and direct products. Under these assumptions it is well known that the free pro-$\mathcal{C}$ product of pro-$\mathcal{C}$ groups exists ([RZ], Section 9.1), and the question arises whether the class $\mathcal{G}_{\mathcal{C}}$ of pro-$\mathcal{C}$ absolute Galois groups is closed under free pro-$\mathcal{C}$-products.

The answer is known to be yes, if $\mathcal{C}$ is the class of all p-groups ([He], Theorem 3.2), or if $\mathcal{C}$ is the class of all finite groups ([Mv], [Er], [K2]).

If $\mathcal{C}$ is a class of abelian groups then $\mathcal{G}_{\mathcal{C}}$ contains only torsion-free abelian groups or the group $\mathbf{Z}/2\mathbf{Z}$. Since any torsion-free abelian group is an absolute Galois group and since the free pro-$\mathcal{C}$ product is just the direct product, $\mathcal{G}_{\mathcal{C}} \setminus \{\mathbf{Z}/2\mathbf{Z}\}$ is closed under free pro-$\mathcal{C}$ products. And, by Lemma 2.1, $\mathcal{G}_{\mathcal{C}}$ is closed under free pro-$\mathcal{C}$ products iff $\mathcal{C}$ contains only odd order groups.

If, however, the class $\mathcal{C}$ is the class of all nilpotent groups or the class of all metabelian groups or the class of all odd-order groups, or the class of all solvable groups, the following Theorem 4.2 shows that the answer is no (for metabelian groups, this also follows from the classification of all metabelian absolute Galois groups in [K1]).

For the formulation of Theorem 4.2 we need the following definition: Given an odd prime p we call a set Π of primes **Galois-p admissible** if for each pro-p absolute Galois group G there is a field F with $G_F \cong G$ such that for all $q \in \Pi$, F contains a primitive q-th root ζ_q of unity (so, in particular, $char\, F \neq q$). Since any absolute Galois group can be realized in characteristic 0, and since for any prime q and any field F of characteristic $\neq q$, $[F(\zeta_q) : F] \mid q-1$, a set Π of primes is Galois-p-admissible whenever $\Pi \subseteq \{q \text{ prime } \mid q \not\equiv 1 \text{mod } p\}$. We don't know whether any set of primes is p-admissible. We have not even an answer to the following

Question 4.1 *Given a pro-p absolute Galois group G and a prime $q \equiv 1 \text{mod } p$, is there a field F with $G_F \cong G$ and $\zeta_q \in F$?*

For a set Π of primes we use the standard terminology of calling a (pro)finite group G a **(pro-)Π group** if any prime dividing $\sharp G$ is in Π. G is a **(pro-)Π' group** if no prime dividing $\sharp G$ is in Π.

Let us now give the group-theoretic characerization of classes $\mathcal{C}$ of finite groups for which $\mathcal{G}_{\mathcal{C}}$ is closed under free pro-$\mathcal{C}$ products:

Theorem 4.2 *Let $\mathcal{C}$ be a class of finite groups closed under forming subgroups, quotients and direct products, and let $\mathcal{G}_{\mathcal{C}}$ be the class of absolute Galois groups which are pro-$\mathcal{C}$ groups.*

Then $\mathcal{G}_{\mathcal{C}}$ is non-trivial and closed under free pro-$\mathcal{C}$ products iff one of the following four cases holds:

1. *$\mathcal{C}$ is the class of all finite groups.*
   In this case the free pro-$\mathcal{C}$ product is the free profinite product, and $\mathcal{G}_{\mathcal{C}}$ is the class of all absolute Galois groups.

2. *There is a prime p such that*

    - *$\mathcal{C}$ contains all finite p-groups*
    - *for each prime $q \neq p$, the exponent of q-groups in $\mathcal{C}$ is bounded*
    - *if $p \neq 2$, all groups in $\mathcal{C}$ have odd order*
    - *each group in $\mathcal{C}$ has a (unique) normal p-Sylow subgroup*

    In this case the free pro-$\mathcal{C}$ product of two pro-p groups is the free pro-p product, and $\mathcal{G}_{\mathcal{C}}$ is the class of all pro-p absolute Galois groups.

3. *There is a non-empty set Π of odd primes such that*

    - *$\mathcal{C}$ contains all finite abelian Π-groups*
    - *for each prime $q \notin \Pi$, the exponent of q-groups in $\mathcal{C}$ is bounded*
    - *each $G \in \mathcal{C}$ is of the form $G = A \rtimes H$, where A is an abelian Π-group and H a Π'-group of odd order*

    In this case, the free pro-$\mathcal{C}$ product of two abelian pro-Π groups is the direct product, and $\mathcal{G}_{\mathcal{C}}$ is the class of all torsion-free abelian pro-Π groups.

4. *There is an odd prime p and a non-empty Galois-p admissible set Π of odd primes $\neq p$ such that*

    - *$\mathcal{C}$ contains all finite p-groups and all abelian Π-groups*
    - *for each prime $q \notin \Pi \cup \{p\}$, the exponent of q-groups in $\mathcal{C}$ is bounded*
    - *each $G \in \mathcal{C}$ is of the form $G = (A \times P) \rtimes H$, where A is an abelian Π-group, P is a p-group and H is a $(\Pi \cup \{p\})'$-group.*

    In this case, for any pair P_1, P_2 of pro-p groups and any pair A_1, A_2 of abelian pro-Π groups,

    $$(A_1 \times P_1) \star_{\mathcal{C}} (A_2 \times P_2) = (A_1 \times A_2) \times (P_1 \star_p P_2),$$

    and $\mathcal{G}_{\mathcal{C}}$ is the class of groups of the form $A \times P$, where A is any torsion-free abelian pro-Π group and P is any pro-p absolute Galois group.

Theorem C from the Introduction is an immediate consequence of the above Theorem 4.2: The class $\mathcal{C}$ of finite groups in Theorem C is also assumed to be closed under extensions. So if for some prime q, $\mathcal{C}$ contains non-trivial q-groups, it also contains non-abelian q-groups and q-groups of arbitrarily high exponent. Hence case 3 and 4 cannot occur, and in case 2, $\mathcal{C}$ contains only p-groups, i.e. $\mathcal{C}$ is exactly the class of all p-groups.

Before we prove Theorem 4.2 let us single out a few auxiliary results. The first is about the subgroups generated in a free pro-$\mathcal{C}$ product by subgroups of the factors. If the class $\mathcal{C}$ of finite groups is in addition extension closed then for any pair of pro-$\mathcal{C}$ groups G_1, G_2 with subgroups $H_1 \leq G_1$, $H_2 \leq G_2$, the subgroup generated by H_1 and H_2 in $G_1 \star_{\mathcal{C}} G_2$ is the free pro-$\mathcal{C}$ product: $\langle H_1, H_2\rangle \cong H_1 \star_{\mathcal{C}} H_2$ ([RZ], Corollary 9.1.7). Without this additional assumption one only has the following

Lemma 4.3 *For $i = 1$ and 2, let G_i be a pro-$\mathcal{C}$ group, let $N_i \lhd G_i$ be a normal subgroup with a complement H_i in G_i, and consider H_1 and H_2 as subgroups of $G_1 \star_{\mathcal{C}} G_2$ in the obvious way. Then $\langle H_1, H_2\rangle \cong H_1 \star_{\mathcal{C}} H_2$.*

Proof: Let $\phi_i : G_i \twoheadrightarrow H_i$ be epimorphisms with kernel N_i ($i = 1, 2$), let ϕ_i' be the isomorphism $\phi_i \mid_{H_i}$, let $\phi : G_1 \star_{\mathcal{C}} G_2 \twoheadrightarrow H_1 \star_{\mathcal{C}} H_2$ be the (unique) extension of ϕ_1, ϕ_2 (considered as maps into $H_1 \star_{\mathcal{C}} H_2$) and let $\phi' = \phi \mid_{\langle H_1, H_2\rangle}$. Then the image of ϕ' is still $H_1 \star_{\mathcal{C}} H_2$, and the partial inverses $\phi_i'^{-1}$ of ϕ' extend (uniquely) to an epimorphismn $\psi : H_1 \star_{\mathcal{C}} H_2 \twoheadrightarrow \langle H_1, H_2\rangle$ with $\phi' \circ \psi \mid_{H_i} = id_{H_i}$. So, by uniqueness, $\phi' \circ \psi = id_{H_1 \star_{\mathcal{C}} H_2}$. Hence $\psi = \phi'^{-1}$ and ϕ' is the isomorphism looked for. **q.e.d.**

In the next Lemma we prove the p-part of Theorem 4.2:

Lemma 4.4 *Let p be a prime, assume that $\mathcal{G}_{\mathcal{C}}$ contains a nontrivial pro-p group and that $\mathcal{G}_{\mathcal{C}}$ is closed under free pro-$\mathcal{C}$ products.*

Then $\mathbf{Z}_p \in \mathcal{G}_{\mathcal{C}}$ and the class $\mathcal{C}_p$ of p-groups in $\mathcal{C}$ is either the class of all finite p-groups or $p > 2$ and $\mathcal{C}_p$ is the class of all abelian p-groups.

Proof: If A and B are pro-$\mathcal{C}$ groups which are also pro-p groups we denote the maximal pro-p quotient of $A \star_{\mathcal{C}} B$ by $A \star_{\mathcal{C},p} B$.

Let us first show that $\mathbf{Z}_p \in \mathcal{G}_{\mathcal{C}}$. By assumption there is a field K such that G_K is a non-trivial pro-p group. If $p > 2$ or if $p = 2$ and K is not formally real, choose $1 \neq \sigma \in G_K$. By Artin-Schreier theory, $\langle\sigma\rangle$ is infinite, and so as procyclic pro-p group $\cong \mathbf{Z}_p$, as claimed. If $p = 2$, $\mathbf{Z}/2\mathbf{Z} \in \mathcal{G}_{\mathcal{C}}$, and so $\mathbf{Z}/2\mathbf{Z} \star_{\mathcal{C}} \mathbf{Z}/2\mathbf{Z} \in \mathcal{G}_{\mathcal{C}}$, say $G_F \cong \mathbf{Z}/2\mathbf{Z} \star_{\mathcal{C}} \mathbf{Z}/2\mathbf{Z}$. Then, by Fact 1.9, $G_F(2) = \mathbf{Z}/2\mathbf{Z} \star_{\mathcal{C},2} \mathbf{Z}/2\mathbf{Z}$ is infinite. This implies that G_F has an infinite 2-Sylow subgroup, say G_E, and so the previous case applies to the non-real field $K = E(\sqrt{-1})$. Note also that for $p = 2$, $\mathcal{C}$ contains non-abelian 2-groups: otherwise $G_F(2) = \mathbf{Z}/2\mathbf{Z} \star_{\mathcal{C},2} \mathbf{Z}/2\mathbf{Z} \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, which contradicts Fact 1.9.

As $\mathcal{C}$ is closed under direct products it follows that for any $n \in \mathbf{N}$, $\mathbf{Z}_p^n \in \mathcal{G}_{\mathcal{C}}$, and so any finite abelian p-group is in $\mathcal{C}$.

Now let us assume that $\mathcal{C}$ contains a non-abelian p-group. We have to show that $\mathcal{C}$ contains all finite p-groups. By assumption, there is a field K with

$$G_K \cong (\mathbf{Z}_p \times \mathbf{Z}_p) \star_{\mathcal{C}} (\mathbf{Z}_p \times \mathbf{Z}_p).$$

Let E and F be the fixed fields of the $\mathbf{Z}_p \times \mathbf{Z}_p$-factors, so $G_K = G_E \star_{\mathcal{C}} G_F$. As $cd_p(G_K) \geq cd_p(\mathbf{Z}_p \times \mathbf{Z}_p) = 2$, *char* $K \neq p$ and, by fact 1.5, $\mu_{p^\infty} \subseteq E$ and $\mu_{p^\infty} \subseteq F$, so $\mu_{p^\infty} \subseteq K = E \cap F$. Let E' and F' be the fixed fields of, say, the second $\mathbf{Z}_p$-factor of G_E resp. G_F. Then the first $\mathbf{Z}_p$-factor of G_E resp. G_F is a normal complement of $G_{E'}$ resp. $G_{F'}$ in G_E resp. G_F, and so, by the previous lemma,

$$G_{E' \cap F'} = \langle G_{E'}, G_{F'} \rangle \cong G_{E'} \star_{\mathcal{C}} G_{F'} \cong \mathbf{Z}_p \star_{\mathcal{C}} \mathbf{Z}_p,$$

and, therefore, $G_{E' \cap F'}(p) \cong \mathbf{Z}_p \star_{\mathcal{C},p} \mathbf{Z}_p$.

By [Wa1], Theorem 4.1, 4.5 or Corollary 4.6 for $p = 2$, and by [Wa2], Lemma 7 and Corollary 1 for $p > 2$, $G_{E' \cap F'}(p)$ is either metabelian, so isomorphic to $\mathbf{Z}_p \rtimes \mathbf{Z}_p$, or $G_{E' \cap F'}(p)$ is the free pro-p group of rank 2, and hence contains free pro-p groups of arbitrary finite rank. In the last case we are done: any finite p-group is the quotient of some free pro-p group of finite rank and so in $\mathcal{C}$. The metabelian case, however, cannot occur: Since $\mathcal{C}$ contains a non-abelian p-group which we may take to be of rank 2, $\mathbf{Z}_p \star_{\mathcal{C},p} \mathbf{Z}_p$ cannot be abelian. Now assume $G_{E' \cap F'} \cong \mathbf{Z}_p \rtimes \mathbf{Z}_p$. Then, by Fact 1.3, $E' \cap F'$ has a p-henselian valuation w with residual characteristic $\neq p$ and with $\Gamma_w \neq p\Gamma_w$. Hence, the finest coarsening v of the canonical p-henselian valuation on $E' \cap F'$ with residual characteristic $\neq p$ is a refinement of w and so also $\Gamma_v \neq p\Gamma_v$. As $G_{E' \cap F'} \cong \mathbf{Z}_p \rtimes \mathbf{Z}_p$, this implies that $G_{(E' \cap F')v}(p)$ is $\cong 1$ or $\cong \mathbf{Z}_p$. Moreover, $\mu_{p^\infty} \subseteq K = E \cap F \subseteq E' \cap F'$. By Fact 1.5.2, however, this implies that $G_{E' \cap F'}(p) \cong \mathbf{Z}_p \star_{\mathcal{C},p} \mathbf{Z}_p$ is abelian. But this contradicts that $\mathcal{C}$ contains non-abelian p-groups which, again, may be chosen of rank 2, i.e. as quotients of $\mathbf{Z}_p \star_{\mathcal{C},p} \mathbf{Z}_p$. **q.e.d.**

The next lemma gives a partial turnabout of Theorem A: a vague version of Theorem A says that Sylow subgroups of absolute Galois groups which decompose in a direct product tend to be abelian. The next lemma says that abelian Sylow subgroups of pro-$\mathcal{C}$ absolute Galois groups tend to be direct factors.

Lemma 4.5 *Assume that $\mathcal{G}_\mathcal{C}$ is closed under free pro-$\mathcal{C}$ products and that p is a prime for which all p-groups in $\mathcal{C}$ are abelian and for which $\mathcal{G}_\mathcal{C}$ contains non-trivial pro-p groups.*

Then any $G \in \mathcal{G}_\mathcal{C}$ is of the form $G = G_p \times H$ with G_p a p-Sylow subgroup of G and all groups in $\mathcal{C}$ have odd order.

Proof: It suffices to prove the following
Claim: *For all primes $q \neq p$ with $\mathbf{Z}_q \in \mathcal{G}_\mathcal{C}$,*

$$\mathbf{Z}_p \star_\mathcal{C} \mathbf{Z}_q = \mathbf{Z}_p \times \mathbf{Z}_q.$$

If the claim is proved we choose for any $G \in \mathcal{G}_\mathcal{C}$ q-Sylow subgroups G_q and infer from the claim for any $\sigma \in G_p$, $\tau \in G_q$, $[\sigma, \tau] = 1$, because there is an epimorphism $\mathbf{Z}_p \star_\mathcal{C} \mathbf{Z}_q \twoheadrightarrow \langle \sigma, \tau \rangle$ and $\mathbf{Z}_p \star_\mathcal{C} \mathbf{Z}_q$ is abelian. Hence for $p \neq q$, $\langle G_p, G_q \rangle = G_p \times G_q$ and so $G = G_p \times H$, where $H = \langle G_q \mid q \neq p \rangle$. Moreover, if $\mathcal{C}$ contains groups of even order then $\mathbf{Z}/2\mathbf{Z} \in \mathcal{G}_\mathcal{C}$, so by the previous lemma $\mathbf{Z}_2 \in \mathcal{G}_\mathcal{C}$ and $p \neq 2$. Hence, by assumption, $\mathbf{Z}_p \star_\mathcal{C} (\mathbf{Z}/2\mathbf{Z}) \in \mathcal{G}_\mathcal{C}$, but, using the claim, $\mathbf{Z}_p \star (\mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}_p \times (\mathbf{Z}/2\mathbf{Z})$ which, by Theorem A, is not an absolute Galois group: absolute Galois groups which are direct products are torsion-free.

To prove the claim let $q \neq p$ be a prime with $\mathbf{Z}_q \in \mathcal{G}_\mathcal{C}$. By the previous lemma, also $\mathbf{Z}_p \in \mathcal{G}_\mathcal{C}$. Since $\mathcal{C}$ is closed under direct products, $\mathbf{Z}_p \times \mathbf{Z}_p$ and $\mathbf{Z}_q \times \mathbf{Z}_q$ are pro-$\mathcal{C}$ groups which, by Example 1.1, occur as absolute Galois groups. Hence, by assumption, we find a field K with

$$G_K \cong (\mathbf{Z}_p \times \mathbf{Z}_p) \star_\mathcal{C} (\mathbf{Z}_q \times \mathbf{Z}_q).$$

Let E and F be the fixed fields of the factors $G_E = \mathbf{Z}_p \times \mathbf{Z}_p$ and $G_F = \mathbf{Z}_q \times \mathbf{Z}_q$. Let v be the finest coarsening of the canonical henselian valuation on E with *char* $Ev \neq p$ and let $v_K = v\mid_K$.

Then, by Fact 1.2, v is non-trivial and $\Gamma_v \neq p\Gamma_v$. Moreover, v_K is henselian: As $\mathbf{Z}_p \times \mathbf{Z}_p \leq G_K$, the p-Sylow subgroups of G_K are not pro-cyclic and not isomorphic to $\mathbf{Z}_2 \rtimes \mathbf{Z}/2\mathbf{Z}$, but they are abelian, because all

p-groups in $\mathcal{C}$ are. By Fact 1.2 and Fact 1.4, the canonical henselian valuation u on K is then non-trivial. The unique prolongation of u to E is therefore finer than v, so v_K is a coarsening of u, and, therefore, must be henselian.

Let v_F be the unique prolongation of v_K to F and let w be the finest coarsening of the canonical henselian valuation on F with $char\, Fw \neq q$. Again, by Fact 1.2 and Fact 1.4, w is non-trivial and $\Gamma_w \neq q\Gamma_w$. Moreover, w is comparable to v_F.

Case 1: $char\, Ev (= char\, Fv_F) = q$

In this case w is a proper coarsening of v_F, $char\, Fw = 0$, and $w_K := w\mid_K$ is also henselian. Let T_w be the inertia subgroup of G_K w.r.t. w_K and choose $\mathbf{Z}_q$-extensions F_1, F_2 of F such that $G_{F_1} \subseteq T_w$ and $G_F = G_{F_1} \times G_{F_2} \cong \mathbf{Z}_q \times \mathbf{Z}_q$: this is possible because either $G_F = T_w$ or $T_w \cong \mathbf{Z}_q$, $G_{Fw} \cong \mathbf{Z}_q$ and $G_F \cong T_w \times G_{Fw}$ (cf. Fact 1.5). Choose $\mathbf{Z}_p$-extensions E_1, E_2 of E with $\zeta_q \in E_1$ (so $\mu_{q^\infty} \subseteq E_1$) and such that $G_E = G_{E_1} \times G_{E_2} \cong \mathbf{Z}_p \times \mathbf{Z}_p$: this is possible as $E(\zeta_q)/E$ is cyclic, so we may lift a generator of $Gal(E(\zeta_q)/E)$ to some $\sigma \in G_E = \mathbf{Z}_p \times \mathbf{Z}_p$ and we may choose $\tau \in G_E$ such that $\{\sigma, \tau\}$ becomes an $\mathbf{F}_p$-base in the Frattini quotient $\mathbf{F}_p \times \mathbf{F}_p$ of G_E and set $G_{E_1} = \langle \tau \rangle$ and $G_{E_2} = \langle \sigma \rangle$.

Now let $L = E_1 \cap F_1$, let w_L be the unique prolongation of w_K to L and let $T_L = T_w \cap G_L$ be the inertia subgroup of G_L w.r.t. w_L. Then T_L is abelian as $char\, Fw = char\, Lw_L = 0$ and $\mu_{q^\infty} \subseteq Lw_L$: by Fact 1.5, $\mu_{q^\infty} \subseteq F \subseteq F_1$, so $\mu_{q^\infty} \subseteq E_1 \cap F_1 = L$. But then the pro-$q$ subgroup G_{F_1} of T_L is in the center of G_L: $G_L \cong T_L \rtimes G_{Lw_L}$ and, as $\mu_{q^\infty} \subseteq Lw_L$, G_{Lw_L} acts trivially on the q-Sylow subgroup Q of T_L (which as characteristic subgroup of T_L is normal in G_L), so with T_L being abelian, Q is in the center of G_L. In particular, $\langle G_{E_1}, G_{F_1} \rangle = G_{E_1} \times G_{F_1}$. By Lemma 4.3, this proves the claim in case 1:

$$\mathbf{Z}_p \star_{\mathcal{C}} \mathbf{Z}_q \cong \langle G_{E_1}, G_{F_1} \rangle = G_{E_1} \times G_{F_1} \cong \mathbf{Z}_p \times \mathbf{Z}_q.$$

Case 2: $char\, Ev \neq q$

In this case we work with v instead of w. Let T_v be the inertia subgroup of K w.r.t. v_K, choose $\mathbf{Z}_p$-extensions E_1, E_2 of E with $G_E \cong G_{E_1} \times G_{E_2} \cong \mathbf{Z}_p \times \mathbf{Z}_p$ and $G_{E_1} \subseteq T_v$. Choose $\mathbf{Z}_q$-extensions F_1, F_2 of F with $G_F = G_{F_1} \times G_{F_2} \cong \mathbf{Z}_q \times \mathbf{Z}_q$ and $\zeta_p \in F_1$ (so $\mu_{p^\infty} \subseteq F_1$), and let, again, $L = E_1 \cap F_1$ with prolongation v_L of v_K to L. Let V be the ramification subgroup of G_L w.r.t. v_L. If $V = 1$, we argue — mutatis mutandis — as in case 1 to prove the claim.

If $V \neq 1$, then V is a pro-l group with $l = char\, Kv \neq p, q$, and we can only conclude that

$$\mathbf{Z}_p \star_{\mathcal{C}} \mathbf{Z}_q \cong G_L \cong V \rtimes G/V \cong V \rtimes (\mathbf{Z}_p \times \mathbf{Z}_q).$$

In this case $\mathbf{Z}_l \in \mathcal{G}_{\mathcal{C}}$ and $l \mid \sharp(\mathbf{Z}_p \star_{\mathcal{C}} \mathbf{Z}_q)$. We shall make a detour in order to show that this cannot be.

Let K' be a field with

$$G_{K'} \cong (\mathbf{Z}_p \times \mathbf{Z}_p) \star_{\mathcal{C}} (\mathbf{Z}_q \times \mathbf{Z}_q) \star_{\mathcal{C}} (\mathbf{Z}_l \times \mathbf{Z}_l)$$

and let E', F' and M be the fixed fields of the three abelian factors and let v' be defind as v above. Then, again, $v'_{K'} := v' \mid_{K'}$ is henselian. As $l \mid \sharp(\mathbf{Z}_p \star_{\mathcal{C}} \mathbf{Z}_q)$, the above argument shows that $char\, K'v' = l$. Now let u be the finest coarsening of the canonical henselian valuation on M with $char\, Mu \neq l$. Then, as in case 1, $char\, Mu = 0$, $u_{K'}$ is henselian and $\mathbf{Z}_p \star_{\mathcal{C}} \mathbf{Z}_l = \mathbf{Z}_p \times \mathbf{Z}_l$. Similarly, working with the prolongation of $u_{K'}$ to $F' \cap M$, we get $\mathbf{Z}_q \star_{\mathcal{C}} \mathbf{Z}_l = \mathbf{Z}_q \times \mathbf{Z}_l$ again as in case 1.

Returning to K and L above this implies that

$$\mathbf{Z}_p \star_{\mathcal{C}} \mathbf{Z}_q \cong G_L = V \rtimes (\mathbf{Z}_p \times \mathbf{Z}_q) \cong V \times \mathbf{Z}_p \times \mathbf{Z}_q.$$

The two inclusion maps ϕ_p resp. ϕ_q embedding the $\mathbf{Z}_p$- resp. $\mathbf{Z}_q$-factor of G_L into G_L will then, however, have two distinct extensions to a homomorphism $\phi : G_L \to G_L$, one the identity map on G_L, the other projection on the last two coordinates with kernel V. This contradicts the uniqueness condition in the universal property for free pro-$\mathcal{C}$ products. So $V = 1$ after all, and the claim is proved in all cases. **q.e.d.**

The next two lemmas are rather easy:

Lemma 4.6 *Let p be a prime. Assume that $\mathcal{C}$ contains all finite p-groups and that each $G \in \mathcal{C}$ has a unique normal p-Sylow subgroup. Let P_1 and P_2 be pro-p groups. Then $P_1 \star_{\mathcal{C}} P_2 = P_1 \star_p P_2$.*

Proof: Let G be any pro-$\mathcal{C}$ group. Then G has a unique p-Sylow subgroup G_p. Let $\phi_i : P_i \to G$ be given homomorphisms ($i = 1, 2$). Then the images of ϕ_i lie in the pro-p group G_p and there is a unique homomorphism $\phi : P_1 \star_p P_2 \to G_p$ with $\phi \mid_{P_i} = \phi_i$ ($i = 1, 2$). Since any homomorphism from $P_1 \star_p P_2$ to G extending the ϕ_i has image in $\langle im\, \phi_1, im\, \phi_2 \rangle \subseteq G_p$, ϕ is also the unique homomorphism from $P_1 \star_p P_2$ to G extending ϕ_1 and ϕ_2. So $P_1 \star_p P_2$ satisfies the universal property for the free pro-$\mathcal{C}$ product $P_1 \star_{\mathcal{C}} P_2$, and, by uniqueness of the free pro-$\mathcal{C}$ product (up to isomorphism), equality holds. **q.e.d.**

Lemma 4.7 *Let Π be a set of primes and assume that each $G \in \mathcal{C}$ is of the form $G = A \rtimes H$ with A an abelian Π-group and H a Π'-group. Let A_1 and A_2 be abelian pro-Π groups which are also pro-$\mathcal{C}$ groups. Then $A_1 \star_{\mathcal{C}} A_2 = A_1 \times A_2$.*

Proof: Let G be any pro-$\mathcal{C}$ group. Then $G = A \rtimes H$ with A an abelian pro-Π group and H a pro-Π' group. Let $\phi_i : A_i \to G$ be any given homomorphisms ($i = 1, 2$). Then the images of ϕ_i are in the abelian pro-Π group A, and we can proceed as in the previous lemma. **q.e.d.**

The last ingredient for the proof of Theorem 4.2 is Corollary 4.9, a purely group theoretic result which, however, happens to follow from a more general field theoretic fact:

Proposition 4.8 *Let p be a prime, let E and F be fields whose absolute Galois groups G_E, G_F are non-trivial pro-p groups, not both of order 2. Then $G_E \star_p G_F$ has no non-trivial abelian normal subgroup.*

Note that for $G_E \cong G_F \cong \mathbf{Z}/2\mathbf{Z}$, by [BEK] and by [EV], Proposition 4,

$$G_E \star_2 G_F \cong \mathbf{Z}/2\mathbf{Z} \star_2 \mathbf{Z}/2\mathbf{Z} \cong \mathbf{Z}_2 \rtimes \mathbf{Z}/2\mathbf{Z}.$$

Proof: By [He], Theorem 3.2, there is a field K with two independent valuations v, w such that $G_K \cong G_E \star_p G_F$, where G_E resp. G_F becomes a decomposition subgroup of G_K w.r.t. (some prolongation to K^{sep} of) v resp. w. G_K is obviously not procyclic, and also not $\cong \mathbf{Z}_2 \rtimes \mathbf{Z}/2\mathbf{Z}$: all infinite subgroups of this last group are of finite index, but G_E or G_F *is* infinite, and so, by Fact 1.6, either v or w would be henselian and so the deomposition subgroup would be G_K rather than the smaller group G_E resp. G_F.

We therefore may apply Fact 1.2. Assuming that G_K has a non-trivial abelian normal subgroup this implies that K has a non-trivial henselian valuation u. Since the henselisation of (K, v) is not separably closed, this implies that u and v must be dependent, and the same holds for u and w. Hence, v and w must be dependent as well, but, by construction, they are not. **q.e.d.**

By [LvD], any projective profinite group is an absolute Galois group, and by Example 1.1, so is any torsion-free abelian profinite group. As a consequence we obtain

Corollary 4.9 *Let A and B be non-trivial pro-p groups each of which is either projective or torsion-free abelian. Then $A \star_p B$ has no non-trivial abelian normal subgroup.*

Proof of Theorem 4.2: In each of the four cases an assertion ('In this case ...') is made about free pro-$\mathcal{C}$ products of certain pro-$\mathcal{C}$ groups and about the structure of the groups in $\mathcal{G}_{\mathcal{C}}$ ('... and $\mathcal{G}_{\mathcal{C}}$ is the class of ...'). Let us first prove these assertaions.

They are obvious in Case 1. In Cases 2, 3 and 4, the assertions about free pro-$\mathcal{C}$ products of pro-p groups (Case 2), pro-abelian groups (Case 3) and direct products of pro-p and pro-abelian groups (Case 4) follow immediately from Lemma 4.6, Lemma 4.7 and Lemma 4.3. And the assertions about the structure of the groups in $\mathcal{G}_{\mathcal{C}}$ are simple consequences of the three following well known facts (the first from Artin-Schreier theory, the second and third from Example 1.1):

- the only torsion elements in absolute Galois groups are involutions
- any torsion-free abelian profinite group A is an absolute Galois group
- if Π is a set of primes and if F is a field of characteristic 0 containing all primitive p-th roots of unity ($p \in \Pi$), if G_F is a pro-Π' group and if A is any torsion-free abelian pro-Π group then $G_{F((\Gamma))} \cong A \times G_F$, where Γ is chosen as in Examples 1.1: note that F contains all p-power roots of unity for $p \in \Pi$.

It now easily follows that in all four cases, $\mathcal{G}_{\mathcal{C}}$ is non-trivial, i.e. contains non-trivial groups, and is closed under free pro-$\mathcal{C}$ products. We already know that the free profinite product of absolute Galois groups is an absolute Galois group (this gives Case 1), and that the free pro-p product of pro-p absolute Galois groups is a pro-p absolute Galois group (this gives Case 2). That any torison-free abelian group is an absolute Galois group and that direct products of torsion-free abelian groups are again torsion-free abelian, gives Case 3. And, similarly, for Case 4: by Galois-p admissibility of Π, we can realize any pro-p absolute Galois group P by a field F of characteristic 0 containing all ζ_q ($q \in \Pi$).

For the converse direction, assume that $\mathcal{G}_\mathcal{C}$ is closed under free pro-$\mathcal{C}$ products and that $\mathcal{G}_\mathcal{C}$ is non-trivial. Then there are primes p for which $\mathcal{G}_\mathcal{C}$ contains non-trivial pro-p absolute Galois groups. Let

$$\Sigma := \{p \text{ prime} \mid \mathcal{C} \text{ contains all finite } p\text{-groups}\},$$

$$\Pi := \{q \text{ prime} \mid \text{the } q\text{-groups } \in \mathcal{C} \text{ are exactly all abelian } q\text{-groups}\}.$$

Then, by Lemma 4.4, $\Pi \cup \Sigma \neq \emptyset$, each $G \in \mathcal{G}_\mathcal{C}$ is a pro-$(\Pi \cup \Sigma)$ group and $2 \notin \Pi$. In particular, the exponent of all l-groups in $\mathcal{C}$ with $l \notin \Pi \cup \Sigma$ are bounded: otherwise $\mathbf{Z}_l$ is a pro-$\mathcal{C}$ group in $\mathcal{G}_\mathcal{C}$.

If $\sharp\Sigma > 1$ choose two distinct primes $p, q \in \Sigma$, choose fields E and F with $G_E \cong \mathbf{Z}_p \star_p (\mathbf{Z}_p \times \mathbf{Z}_p)$ and $G_F \cong \mathbf{Z}_q \star_q (\mathbf{Z}_q \times \mathbf{Z}_q)$, and let K be a field with $G_K \cong G_E \star_\mathcal{C} G_F$. We may assume that $K = E \cap F$. Let L be the fixed field of $\mathbf{Z}_p \times \mathbf{Z}_p \leq G_E$ and let M be the fixed field of $\mathbf{Z}_q \times \mathbf{Z}_q \leq G_F$, let v and w be the finest coarsening of the canonical henselian valuation on L and M with residue characteristic $\neq p$ resp. $\neq q$, and let $v_K := v \mid_K$ and $w_K := w \mid_K$. Then, by Fact 1.2, v and w are non-trivial, $\Gamma_v \neq p\Gamma_v$ and $\Gamma_w \neq q\Gamma_w$. However, $v_E := v \mid_E$ and $w_F := w \mid_F$ are non-henselian, because, by Corollary 4.9, G_E and G_F have no non-trivial abelian normal subgroup and, for henselian v_E resp. w_F the inertia subgroups would have to be of this kind ($char\, Ev_E = char\, Lv \neq p$ and $char\, Fw_F = char\, Mw \neq q$). Hence v_K and w_K are non-henselian, and so, by Theorem 3.1, any finite group occurs as subquotient of G_K. As G_K is a pro-$\mathcal{C}$ group this means that $\mathcal{C}$ contains all finite groups, and we are in Case 1.

If $\sharp\Sigma = 1$, say $\Sigma = \{p\}$, and if $\Pi = \emptyset$, then we are in Case 2: If $p \neq 2$ then all groups in $\mathcal{C}$ have odd order (otherwise the non-$(\Pi \cup \Sigma)$ group $\mathbf{Z}/2\mathbf{Z}$ is in $\mathcal{G}_\mathcal{C}$). Moreover, each $G \in \mathcal{C}$ has normal p-Sylow subgroups: if P_1 and P_2 are distinct p-Sylow subgroups of G we can choose epimorphisms $\phi_i : G_i \twoheadrightarrow P_i$ $(i = 1, 2)$ for suitable $G_i \in \mathcal{G}_\mathcal{C}$, because $\mathcal{G}_\mathcal{C}$ contains all pro-p absolute Galois groups and so, in particular, all free pro-p groups. But then the unique homomorphism $\phi : G_1 \star_\mathcal{C} G_2 \to G$ with $\phi \mid G_i = \phi_i$ $(i = 1, 2)$ has image $\langle P_1, P_2 \rangle$ which is not a p-group. So $G_1 \star_\mathcal{C} G_2$ is not a pro-p group: contradiction.

If $\sharp\Sigma = 1$, say $\Sigma = \{p\}$ and if $\Pi \neq \emptyset$ then we are in Case 4: Any $G \in \mathcal{C}$ has normal Sylow-q subgroups G_q for each $q \in \Pi \cup \{p\}$ (as in the previous case). As any Π-group and any p-group in $\mathcal{C}$ is epimorphic image of a pro-Π group resp. pro-p group in $\mathcal{G}_\mathcal{C}$, Lemma 4.5 implies that all Π-groups in $\mathcal{C}$ are abelian and all $(\Pi \cup \{p\})$-groups in $\mathcal{C}$ are of the form $A \times P$ with A an abelian Π-group and P a p-group. Hence $\langle G_q \mid q \in \Pi \cup \{p\}\rangle$ is of this shape $A \times P$, and by a well-known theorem of Zassenhaus (e.g. [Hu], I. 18.1), it has a complement H in G, so $G = (A \times P) \rtimes H$. Moreover, again by Lemma 4.5, all groups in $\mathcal{C}$ have odd order.

If, finally, $\Sigma = \emptyset$ then $\Pi \neq \emptyset$ and, arguing exactly as in the previous case, we are in Case 3. **q.e.d.**

References

[B1] E. Becker, *Euklidische Körper und euklidische Hüllen von Körpern*, J. reine angew. Math. **268-269** (1974), 41-52.

[B2] E. Becker, *Hereditarily-pythagorean fields and orderings of higher level*, IMPA-monografias de matemática **29**, Rio de Janeiro 1978.

[BEK] S.V. Bredikhin, Y.L. Ershov, V.E. Kalnei, *Fields with two linear orderings*, Math. Notes USSR **7-8** (1970), 319-325.

[DP] C.N. Delzell, A. Prestel, *Positive Polynomials*, Springer 2001.

[E] O. Endler, *Valuation Theory*, Springer 1972.

[EE] O. Endler, A.J. Engler, *Fields with henselian valuation rings*, Math. Zeitschr. **152** (1977), 191-193.

[EK] A.J. Engler, J. Koenigsmann, *Abelian subgroups of pro-p Galois groups*, Trans. AMS **350(6)** (1998), 2473-2485.

[EN] A.J. Engler, J.B. Nogueira, *Maximal abelian normal subgroups of Galois pro-2 groups*, J. of Algebra **166(3)** (1994), 481-505.

[EV] A.J. Engler, T.M. Viswanathan, *Digging holes in algebraic closures à la Artin – II*, Contemp. Math. **8** (1982), 351-360.

[Er] Y.L. Ershov, *Free products of absolute Galois groups*, Doklady Mathematics **56(3)** (1997), 915-917.

[FJ] M. D. Fried, M. Jarden, *Field Arithmetic*, Ergebnisse der Mathematik (3), vol. **11**, Springer, Heidelberg 1986.

[G] W.-D. Geyer, *Unendliche algebraische Zahlkörper, über denen jede Gleichung auflösbar von beschränkter Stufe ist*, J. Number Theory **1** (1969), 346-374.

[Ha] D. Haran, *Hilbertian fields under separable algebraic extensions*, Inventiones math. **137** (1999), 113-126.

[HJ] D. Haran, M. Jarden, *Compositum of Galois extensions of Hilbertian fields*, Ann. scient. Èc. Norm. Sup. **ser. 4, no. 24** (1991), 739-748.

[He] B. Heinemann, *On finite intersections of 'henselian valued' fields*, manuscr. math. **52** (1985), 37-61.

[Hu] B. Huppert, *Endliche Gruppen I*, Springer Grundlehren **134**, 1967.

[K1] J. Koenigsmann, *Solvable absolute Galois groups are metabelian*, Inventiones math. **144** (2001), 1-22.

[K2] J. Koenigsmann, *Relatively projective groups as absolute Galois groups*, Israel J. Math. **127** (2002), 93-129.

[K3] J. Koenigsmann, *Encoding valuations in absolute Galois groups*, Fields Institute Communications **33** (2003), 107-132.

[KPR] F.-V. Kuhlmann, M. Pank, P. Roquette. *Immediate and purely wild extensions of valued fields*, manuscr. math. **55** (1986), 39-67.

[LvD] A. Lubotzky, L.v.d. Dries, *Subgroups of free profinite groups and large subfields of* $\tilde{\mathbf{Q}}$, Israel J. Math. **39** (1981), 25-45.

[Mv] O.V. Mel'nikov, *On free products of absolute Galois groups* (1997, Russian), English transl. in Siberian Math. J. **40(1)** (1999), 95-99.

[MW] J. Mináč, R. Ware, *Demuškin groups of rank* $\aleph_0$ *as absolute Galois groups*, manuscr. math. **73** (1991), 411-421.

[N] J. Neukirch, *Über eine algebriasche Kennzeichnung der Henselkörper*, J. reine angew. Math. **231** (1968), 75-81.

[P] F. Pop, *On Grothendieck's conjecture of birational anabelian geometry*, Annals of Math. **138** (1994), 145-182.

[RZ] L. Ribes, P. Zalesskii, *Profinite groups*, Ergebnisse der Mathematik (3), vol. **40**, Springer, Heidelberg 2000.

[Wa1] R. Ware, *Quadratic forms and profinite* 2*-groups*, J. Algebra **58** (1979), 227-237.

[Wa2] R. Ware, *Galois groups of maximal* p*-extensions*, Trans. AMS **333(2)** (1992), 721-728.

[We] R. Weissauer, *Der Hilbertsche Irreduzibilitätssatz*, J. reine angew. Math. **334** (1982), 203-220.

[Wh] G. Whaples, *Algebraic extensions of arbitrary fields,* Duke Math. J. **24** (1957), 201-204.

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