Every place admits local uniformization in a finite extension of the function field

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01. 11. 2006

Abstract

We prove that every place P of an algebraic function field $F|K$ of arbitrary characteristic admits local uniformization in a finite extension $\mathcal{F}$ of F. We show that $\mathcal{F}|F$ can be chosen to be Galois, after a finite purely inseparable extension of the ground field K. Instead of being Galois, the extension can also be chosen such that the induced extension $\mathcal{F}P|FP$ of the residue fields is purely inseparable and the value group of F only gets divided by the residue characteristic. If F lies in the completion of an Abhyankar place, then no extension of F is needed. Our proofs are based solely on valuation theoretical theorems, which are of particular importance in positive characteristic. They are also applicable when working over a subring $R \subset K$ and yield similar results if R is regular and of dimension smaller than 3.

1 Introduction and main results

A place P of an algebraic function field $F|K$ is said to admit local uniformization if there exists a K-variety X having F as its field of rational functions and such that the center $x \in X$ of P on X is a regular point. In [Z1], Zariski proved the Local Uniformization Theorem for places of algebraic function fields over base fields of characteristic 0. In [Z3], he uses this theorem to prove resolution of singularities for algebraic surfaces in characteristic 0, later on generalized to positive characteristic by Abhyankar [A1]. As the resolution of singularities for algebraic varieties of arbitrary dimension in positive characteristic is still an open problem, one is interested in generalizations of the Local Uniformization Theorem to positive characteristic. In this article we prove that every place of an algebraic function field of arbitrary characteristic admits local uniformization after a finite extension of the function field. This fact already follows from the results of de Jong [dJ] who proves resolution of singularities after a finite normal extension of the function field using results on moduli spaces of stable curves. However, we will give an entirely valuation theoretical proof which will provide important additional information about the finite extension used to achieve local uniformization. Our approach also applies

*The second author thanks Peter Roquette, Mark Spivakovsky, Bernard Teissier and Frans Oort for support and inspiring conversations. Very special thanks to Hans Schoutens for many pleasant and encouraging discussions.

to the case where the restriction of the place P to K is not the identity but is centered on a regular local ring $R \subset K$, $K = \operatorname{Frac} R$, of dimension $\dim R \leq 2$–thus including the arithmetic case of a discrete valuation ring R. In the latter case and for a function field $F|K$ of transcendence degree 1, Abhyankar [A2] has proved local uniformization (under some additional assumptions). If R is a discrete valuation ring of a global field K and for arbitrary transcendence degree of $F|K$, local uniformization after a finite extension of F again follows from the results in [dJ].

Let $F|K$ be an algebraic function field equipped with a place P whose restriction $P|_K$ to K needs not be the identity. Local uniformization of P is a statement about the valuation ring $\mathcal{O}_P$ associated with P. Accordingly throughout this article places P and P' on the field F inducing the same valuation ring are identified. By abuse of language the pair $(F|K, P)$ is called a **valued function field** keeping in mind the valuation v of F associated with P. The maximal ideal of the local ring $\mathcal{O}_P$ is denoted by $\mathcal{M}_P$ and the residue field of P (or v) by $FP := \mathcal{O}_P/\mathcal{M}_P$.

Let $R \subseteq \mathcal{O}_P \cap K$ be a subring having field of fractions $\operatorname{Frac} R = K$. Given a separated, integral, finitely presented R-scheme Y with $F = K(Y)$–an **R-model of $F|K$** for short– such that P has center y on Y in the context of the resolution of singularities one searches for a birational morphism $X \to Y$ of R-models such that P is centered in a regular point x of X. Usually it is assumed that the schemes X and Y are noetherian, in the present article however we deal with the case of a non-noetherian valuation domain R too. In that case one has to replace the requirement of being regular at the center $x \in X$ by smoothness of X at the point x. For the valuation-theoretic approach presented in the sequel it is convenient to formulate the existence of the birational morphism $X \to Y$ in terms of the finite set of generators of the R-algebra $\mathcal{O}_Y(U)$ for a suitable open, affine neighborhood $U \subseteq Y$ of y. Doing so one arrives at the following notions: let $Z \subset \mathcal{O}_P$ be finite. The pair (P, Z) is called **smoothly R-uniformizable** if there exists an R-model X of $F|K$ such that $X \to \operatorname{Spec} R$ is smooth at the center $x \in X$ of P on X and Z is contained in the local ring $\mathcal{O}_{X,x}$ at x. If R is noetherian, the pair (P, Z) is said to be **R-uniformizable** if P is centered in a regular point $x \in X$ of an R-model X of $F|K$ and $Z \subset \mathcal{O}_{X,x}$ holds. The place P is called (smoothly) R-uniformizable if the pair $(P, \emptyset)$ is (smoothly) R-uniformizable. The place P is called **strongly (smoothly) R-uniformizable** if all pairs (P, Z), $Z \subset \mathcal{O}_P$ finite, are (smoothly) R-uniformizable.

A natural approach to local uniformization is to consider stratifications of a valued function field $(F|K, P)$ essentially given through the choice of appropriate transcendence bases with respect to the place P: in general the inequality

$$\operatorname{trdeg}(FP|KP) + \dim(vF/vK \otimes_{\mathbb{Z}} \mathbb{Q}) \leq \operatorname{trdeg}(F|K) \tag{1}$$

relates the transcendence degree $\operatorname{trdeg}(F|K)$ of $F|K$ with that of the residue field extension and with the rational rank of the abelian group vF/vK. The place P is called an **Abhyankar place** if in (1) equality holds. It is well-known that in every valued function field $(F|K, P)$ there exists an intermediate field $K \subset F_0 \subseteq F$ such that:

(S1) the restriction $P|_{F_0}$ is an Abhyankar place of $F_0|K$ and vF_0/vK is torsion-free,

(S2) the extension $FP|F_0P$ is algebraic and vF/vF_0 is a torsion group.

The field F_0 can be choosen to be a rational function field–see Theorem 2.1 of [K–K] and Proposition 2.3 of the present article. Note also that if P is not itself an Abhyankar place, then $F|F_0$ has positive transcendence degree.

In [K–K] valued function fields of the type appearing in (S1) are investigated: it is proved that an Abhyankar place P_0 of a function field $F_0|K$ is strongly R-uniformizable, where $R \subseteq K$ is a regular, local Nagata ring of Krull dimension $\dim R \le 2$ dominated by $\mathcal{O}_{P_0}$, provided that the extension $FP_0|KP_0$ is separable and the valuation ring $\mathcal{O}_{P_0} \cap K$ is defectless.

In the present article we study valued function fields $(E|K, P)$ as arising in (S2) with respect to smooth uniformizability of P over the possibly non-noetherian valuation ring $\mathcal{O}_P \cap K$. We show that given a finite set $Z \subset \mathcal{O}_P$ there exists a finite extension $\mathcal{E}|E$, a finite extension $\mathcal{K}|K$ within $\mathcal{E}$ and a prolongation $\mathcal{P}$ of P to $\mathcal{E}$ such that the pair $(\mathcal{P}, Z)$ is smoothly $(\mathcal{O}_\mathcal{P} \cap \mathcal{K})$-uniformizable.

The extension $\mathcal{E}|E$ can be choosen to be Galois. However for certain applications of local uniformization, e.g. to the model theory of fields in the spirit of [J–R] (cf. also [K5]), it is important to have a valuation theoretical control on the extension $\mathcal{E}|E$ and the residue field extension $\mathcal{E}\mathcal{P}|EP$ that we cannot obtain in the Galois case: we want to have $\mathcal{E}\mathcal{P}$ to be as close to EP as possible, but in positive characteristic we may expect that we have to take a purely inseparable extension into the bargain. Therefore instead of choosing a suitable extension $\mathcal{E}|E$ within the separable closure $E^{\rm sep}$ of E we do the same within a separably tame hull of E: a valued field (L, P) is called **separably tame** if it is henselian and its separable algebraic closure $L^{\rm sep}$ equals the absolute ramification field of (L, P). A **separably tame hull of the valued field** (E, P) is a field extension $E^{\rm st}|E$ equipped with an extension $P^{\rm st}$ of P such that $(E^{\rm st}, P^{\rm st})$ is separably tame, $E^{\rm st}|E$ is separable-algebraic, $(v^{\rm st}E^{\rm st}/vE)$ is a p-group, and $E^{\rm st}P^{\rm st}|EP$ is a purely inseparable extension. Here p denotes the characteristic of EP respectively $p = 1$ in the case of characteristic 0. $v^{\rm st}$ is the valuation associated to the place $P^{\rm st}$. For basic properties of separably tame fields and the existence of separably tame hulls, see Subsection 2.3.

Theorem 1.1 *Let $(E|K, P)$ be a separable, valued function field such that vE/vK is a torsion group and $EP|KP$ is algebraic. Let $Z \subset \mathcal{O}_P$ be a finite set. Let $\mathcal{P}$ be an extension of P to the separable closure $E^{\rm sep}$ of E. Then there exists a finite extension $\mathcal{E}|E$ within $E^{\rm sep}$ and a finite extension $\mathcal{K}|K$ within $\mathcal{E}$ such that the function field $\mathcal{E}|\mathcal{K}$ possesses an $\mathcal{O}_\mathcal{K}$-model X, $\mathcal{O}_\mathcal{K} := \mathcal{O}_\mathcal{P} \cap \mathcal{K}$, with the properties:*

- *$X \to \operatorname{Spec} \mathcal{O}_\mathcal{K}$ is smooth at the center $x \in X$ of $\mathcal{P}$ on X,*
- *every $z \in Z$ can be expressed as $z = uz'$ with some $u \in \mathcal{O}_{X,x}^\times$ and $z' \in \mathcal{O}_\mathcal{K}$.*

The extension $\mathcal{E}|E$ can be choosen to be either Galois or to be a subextension of a separably tame extension $E^{\rm st}|E$ within $E^{\rm sep}$–for example a separably tame hull of (E, P). If $\mathcal{E}|E$ is choosen to be Galois, then $\mathcal{K}|K$ can be choosen to be Galois too.

If $E_0|K$ is a subextension of $E|K$ such that $\operatorname{trdeg} E_0|K = \operatorname{trdeg} E|K - 1$ and $E|E_0$ is separable, then $\mathcal{E}$ can be chosen to be a compositum $E.\mathcal{E}_0$, where $\mathcal{E}_0|E_0$ is a finite extension that is Galois respectively is contained in $E^{\rm st}$.

Let us return to a stratification $K \subset F_0 \subset F$ satisfying the conditions (S1) and (S2) and assume in addition that $F|F_0$ is separable: Theorem 1.1 yields finite extensions $\mathcal{F}|F$,

$\mathcal{F}_0|F_0$ of a certain type such that $(\mathcal{P}|_{\mathcal{F}}, Z)$ is smoothly $(\mathcal{O}_{\mathcal{P}} \cap \mathcal{F}_0)$-uniformizable for every finite set $Z \subset \mathcal{O}_P$. The place $\mathcal{P}|_{\mathcal{F}_0}$ is an Abhyankar place of the function field $\mathcal{F}_0|K$, thus the results on local uniformization of Abhyankar places obtained in [K–K] apply. Using a descend property of smooth algebras we can combine these facts to get general results about the uniformizability of pairs (P, Z). Utilizing the statement in Theorem 1.1 concerning factorizations of the elements $z \in Z$ we can even extent these results to include monomialization of all $z \in Z$: let $\mathcal{O}$ be a commutative ring and $H \subseteq \mathcal{O}$. An element $a \in \mathcal{O}$ is called an **$\mathcal{O}$-monomial in** H if

$$a = u \prod_{i=1}^{d} h_i^{\mu_i},\ u \in \mathcal{O}^{\times},\ h_i \in H,\ \mu_i \in \mathbb{N}_0,\ i = 1, \dots, d,$$

holds, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

In the case of a valued function field $(F|K, P)$ with $P|_K = \mathrm{id}_K$, in which case we also say that P **is a place of** $F|K$, the combination of Theorem 1.1 with the results of [K–K] yields:

Theorem 1.2 *Let P be a place of the function field $F|K$ and let $Z \subset \mathcal{O}_P$ be a finite set. Let $\mathcal{P}$ be an extension of P to the algebraic closure $\tilde{F}$ of F. Then there exist a finite purely inseparable extension $\mathcal{K}|K$ and a finite separable extension $\mathcal{F}|F.\mathcal{K}$ such that the pair $(\mathcal{P}|_{\mathcal{F}}, Z)$ is smoothly $\mathcal{K}$-uniformizable.*

More precisely let F_0 be an intermediate field of $F|K$ with the properties: $P|_{F_0}$ is an Abhyankar place of $F_0|K$, $F|F_0$ is separabel, vF/vF_0 is a torsion group and $FP|F_0P$ is algebraic. Then there exist a finite purely inseparable extension $\mathcal{K}|K$, a finite separable extension $\mathcal{F}|F.\mathcal{K}$, a finite extension $\mathcal{F}_0|F_0.\mathcal{K}$ within $\mathcal{F}$ and a morphism $f : X \to X_0$ of $\mathcal{K}$-models of $\mathcal{F}|\mathcal{K}$ and $\mathcal{F}_0|\mathcal{K}$ with the properties:

- *f is smooth at the center x of $\mathcal{P}$ on X,*
- *$X_0|\mathcal{K}$ is smooth at $f(x)$,*
- $\dim \mathcal{O}_{X,x} \geq \dim \mathcal{O}_{X_0,f(x)} = \dim(vF \otimes \mathbb{Q})$,
- *all $z \in Z$ are $\mathcal{O}_{X,x}$-monomials in a regular parameter system of $\mathcal{O}_{X,x}$.*

The extension $\mathcal{F}|F.\mathcal{K}$ can be choosen to be either Galois or to be a subextension of a given separably tame field F^{st} such that $F.\mathcal{K} \subseteq F^{\mathrm{st}} \subseteq (F.\mathcal{K})^{\mathrm{sep}}$. In the first case the extension $\mathcal{F}_0|F_0.\mathcal{K}$ can be choosen to be Galois too.

The fact that in Theorem 1.2 one can choose F^{st} to be a separably tame hull of $F.\mathcal{K}$ has interesting consequences concerning the valuation theoretical control of $\mathcal{F}|F$ mentioned earlier:

Corollary 1.3 *If $\mathrm{char}\, K = p > 0$, then the valued extension $(\mathcal{F}|F, \mathcal{P}|_{\mathcal{F}})$ can be choosen such that $\mathcal{F}\mathcal{P}|FP$ is a finite purely inseparable extension and $v\mathcal{F}/vF$ is a finite p-group. In particular one gets $\mathcal{F}\mathcal{P} = FP$ if FP is perfect and $v\mathcal{F} = vF$ if vF is p-divisible.*

If $\mathrm{char}\, K = 0$, then one can take $\mathcal{F}$ to lie in the henselization of (F, P). In particular $v\mathcal{F} = vF$ and $\mathcal{F}\mathcal{P} = FP$ holds.

The assertion of the corollary in the case $p > 0$ is a direct consequence of Theorem 1.2 and the definition of the separably tame hull. The assertion in the case $p = 0$ can be considered as a weak version of Zariski's result on local uniformization [Z1]. It is a consequence of the fact that the henselization of (F, P) is a separably tame hull–see Lemma 2.8.

We turn to the case $P|_K \neq \mathrm{id}_K$, where we have to assume that the valued field $(K, P|_K)$ is defectless, that is that the fundamental equality of valuation theory holds in every finite extension $L|K$–see Section 2. If the valuation associated to $P|_K$ is discrete, then defectlessness of $(K, P|_K)$ is equivalent to $\mathcal{O}_P \cap K$ being a Nagata ring.

Theorem 1.4 *Let $(F|K, P)$ be a valued function field such that $P|_K \neq \mathrm{id}_K$, $(K, P|_K)$ is defectless and KP is perfect. Let $\mathcal{P}$ be an extension of P to the separable closure F^{sep} of F. Let $R \subseteq \mathcal{O}_P$ be a noetherian, regular local ring with maximal ideal $M = \mathcal{M}_P \cap R$. Assume that $\mathrm{Frac}\, R = K$, $\dim R \leq 2$ and that R is a Nagata ring if $\dim R = 2$.*

Then for every finite set $Z \subset \mathcal{O}_P$ there exists a finite separable extension $\mathcal{F}|F$ such that the pair $(\mathcal{P}|_{\mathcal{F}}, Z)$ is R-uniformizable.

More precisely let F_0 be an intermediate field of $F|K$ with the properties: $P|_{F_0}$ is an Abhyankar place of $F_0|K$, $F|F_0$ is separabel, vF_0/vK is torsion-free, vF/vF_0 is a torsion group and $FP|F_0P$ is algebraic. Then there exist a finite extension $\mathcal{F}|F$, a finite extension $\mathcal{F}_0|F_0$ within $\mathcal{F}$ and a morphism $f : X \to X_0$ of R-models of $\mathcal{F}|K$ and $\mathcal{F}_0|K$ with the properties:

- *f is smooth at the center x of $\mathcal{P}$ on X,*
- *$\mathcal{O}_{X_0,f(x)}$ is regular,*
- *$\dim \mathcal{O}_{X,x} \geq \dim \mathcal{O}_{X_0,f(x)}$, where*

$$\dim \mathcal{O}_{X_0,f(x)} = \begin{cases} \dim(vF/vK \otimes \mathbb{Q}) + 1 & \text{if } \dim R = 1 \text{ or } \mathrm{trdeg}\,(KP|R/M) > 0 \\ \dim(vF/vK \otimes \mathbb{Q}) + 2 & \text{in the remaining cases} \end{cases}$$

- *all $z \in Z$ are $\mathcal{O}_{X,x}$-monomials in a regular parameter system of $\mathcal{O}_{X,x}$.*

The extension $\mathcal{F}|F$ can be choosen to be either Galois or to be a subextension of a given separably tame field F^{st} such that $F \subseteq F^{\mathrm{st}} \subseteq F^{\mathrm{sep}}$. In the first case the extension $\mathcal{F}_0|F_0$ can be choosen to be Galois too.

In [K–K] we have shown that Abhyankar places admit local uniformization without any extension of the function field. In [K5] a construction of places P on a function field $F|K$ is given that yields non-Abhyankar places which are still "very close to" Abhyankar places in the following sense: the valued field (F, P) lies in the completion of a subfield $(F_0, P|_{F_0})$ such that $P|_{F_0}$ is an Abhyankar place. Therefore, it is important to know that also the latter places admit local uniformization without any extension of the function field. Here by "completion" we mean the completion with respect to the uniformity induced by the valuation: (F, P) lies in the completion of $(F_0, P|_{F_0})$ if for every $a \in F$ and $\alpha \in vF$ there is some $b \in F_0$ such that $v(a - b) \geq \alpha$.

Theorem 1.5 *Let $(F|K, P)$ be a valued function field with the property that (F, P) lies in the completion of a subfunction field $(F_0, P|_{F_0})$ such that $P|_{F_0}$ is an Abhyankar place of $F_0|K$, vF_0/vK is torsion-free and $F_0P|KP$ is separable.*

1. *If $P|_K = \mathrm{id}_K$, then P is strongly smoothly K-uniformizable and the conclusions of Theorem 1.2 concerning the existence and properties of the morphism $f : X \to X_0$ hold with $\mathcal{F}_0 = F_0$ and $\mathcal{F} = F$.*

2. *Let $R \subset K$ be a subring of K satisfying the requirements stated in Theorem 1.4. If $P|_K \neq \mathrm{id}_K$ and $(K, P|_K)$ is defectless, then P is strongly R-uniformizable and the conclusions of Theorem 1.4 concerning the existence and properties of the morphism $f : X \to X_0$ hold with $\mathcal{F}_0 = F_0$ and $\mathcal{F} = F$.*

The results stated so far–in particular Theorem 1.5–raise the question for necessary conditions for local uniformization without extending the function field. At least in the case of smooth uniformizability a condition in the same spirit as the major premise in Theorem 1.5 can be given: a valued function field $(F|K, P)$ is called **inertially generated** if it admits a transcendence basis T such that (F, P) lies in the absolute inertia field of $(K(T), P|_{K(T)})$. If it admits a transcendence basis T such that (F, P) lies in the henselization of $(K(T), P|_{K(T)})$, then we call it **henselian generated**.

Theorem 1.6 *Let $(F|K, P)$ be a valued function field such that P is smoothly $\mathcal{O}_K$-uniformizable. Then $(F|K, P)$ is inertially generated. In particular $F|K$ and $FP|KP$ are separable. If in addition $FP = KP$, then $(F|K, P)$ is even henselian generated.*

2 Valuation theoretical preliminaries

In this section we review relevant facts from valuation theory in order to make the present article sufficiently self-contained. For basic facts from valuation theory we refer the reader to [EN], [R], [W] and [Z–S].

2.1 Some fundamentals

In the present article we formulate most of the results using the notion of a place of a field rather than that of a valuation to stress their geometric nature. It is well-known that the two notions essentially are synonymous to each other. Consequently by abuse of language we call a pair (F, P) consisting of a field F and a place P of F a **valued field**, keeping in mind the valuation associated to P, which we denote by v or sometimes v_P if explicit reference to the place P is required. A **valued field extension** is a pair $(F|K, P)$, where (F, P) is a valued field and F is an extension field of K. The field K is always understood to be equipped with the place $P|_K$, where we frequently suppress mentioning the restriction explicitely, that is we write P instead of $P|_K$. If $F|K$ is finite respectively finitely generated, then we speak of a finite respectively finitely generated, valued field extension $(F|K, P)$. The valuation ring of the valuation v associated to P is denoted by $\mathcal{O}_P$ and its maximal ideal by $\mathcal{M}_P$. Additionally when considering intermediate fields $K \subseteq M \subseteq F$ of a valued field extension $(F|K, P)$ we use $\mathcal{O}_M := \mathcal{O}_P \cap M$ for the valuation ring of $v|_M$.

Throughout the article we identify places P and P' of F if they are inducing the same valuation ring of F. If that valuation ring is the field F itself we call P a **trivial place**. A trivial place is equivalent to the identity map of F. In particular if $(F|K, P)$ is a valued field extension such that $P|_K$ is an isomorphism of K, then we will assume that $P|_K = \mathrm{id}_K$ and call P a **place of** $F|K$.

Places operate on the right: the image of $f \in F$ under P is denoted fP; consequently FP is the residue field $\mathcal{O}_P/\mathcal{M}_P$. The value group of the valuation v associated to P is denoted by vF thus using the common convention $v(0) = \infty$.

For a valued extension $(L|K, P)$ the degree $f := [LP : KP]$ is called **inertia degree** and $e := (vL : vK)$ is the **ramification index**. If $L|K$ is finite, then f and e are finite too. More precisely if $P_1, \ldots, P_g$ are the distinct extensions of $P|_K$ to L, then the **fundamental inequality**

$$[L : K] \ \geq \ \sum_{i=1}^{g} e_i f_i \ , \tag{2}$$

with $f_i = [LP_i : KP]$ and $e_i = (v_{P_i}L : vK)$, holds.

A valued field (K, P) is called **defectless** (or **stable**) if equality holds in (2) for every finite extension $L|K$. As a consequence of the "Lemma of Ostrowski" ([EN], [R]) a valued field with $\mathrm{char}\, KP = 0$ is defectless.

The effect of extending a place P of a field K to its separable closure K^{sep} is described through the following fact:

Lemma 2.1 *Let K be an arbitrary field and P a non-trivial place on K^{sep}. Then $v(K^{\mathrm{sep}})$ is the divisible hull $vK \otimes_{\mathbb{Z}} \mathbb{Q}$ of vK, and $K^{\mathrm{sep}}P$ is the algebraic closure of KP.*

For a proof see [K4], Lemma 2.16.

The valued extension $(L|K, P)$ is called **immediate** if $vL = vK$ and $LP = KP$.

A valued field (K, P) is called **henselian** if it satisfies Hensel's Lemma; see [R] or [W]. The place P then possesses a unique extension P' to every algebraic extension field L of K and (L, P') is henselian too.

In general for every valued field (K, P) there exists a henselian field (K^h, P^h) and an embedding $i : K \to K^h$ such that $P = P^h \circ i$ with the following universal property: for every henselian extension (L, P') of (K, P) there exists a unique embedding $j : K^h \to L$ such that $P^h = P' \circ j$. The valued field (K^h, P^h) is uniquely determined up to a valuation-preserving K-isomorphism and is called the **henselization of** (K, P). It can be contructed using ramification theory: define the **decomposition group** of an extension P^{sep} of P to K^{sep} as $G_d := \{\sigma \in \mathrm{Gal}\,(K^{\mathrm{sep}}|K) : P^{\mathrm{sep}} \circ \sigma = P^{\mathrm{sep}}\}$. The fixed field of G_d then is a henselization of (K, P). The decomposition group contains the normal subgroup $G_i := \{\sigma \in G_d : (\sigma(a) - a)P^{\mathrm{sep}} = 0\}$ called the **inertia group** of P^{sep}. The fixed field K^i of G_i equipped with the place $P^i := P^{\mathrm{sep}}|_{K^i}$ is henselian and is called the **absolute inertia field of** (K, P); in the context of the present article the following property is relevant:

Lemma 2.2 *Let P be a place of $F|K$ and let (F^i, P^i) denote the absolute inertia field of (F, P). Then $K^{\mathrm{sep}} \subset F^i$ holds. Further, if $FP|K$ is algebraic, then $(F.K^{\mathrm{sep}})P^i$ is the separable closure of FP.*

Proof: By assumption $P|_K = \mathrm{id_K}$, hence $K \subseteq FP$. By general ramification theory we know that F^iP^i is separable-algebraically closed, thus $K^{\mathrm{sep}} \subseteq F^iP^i$. Using Hensel's Lemma one can then construct a K-embedding $K^{\mathrm{sep}} \hookrightarrow F^i$. Further $K^{\mathrm{sep}}P^i \subseteq (F.K^{\mathrm{sep}})P^i$

and $K^{\mathrm{sep}} \subseteq K^{\mathrm{sep}}P^i$ by Lemma 2.1. As $F.K^{\mathrm{sep}}|F$ is algebraic, so is $(F.K^{\mathrm{sep}})P^i|FP$. Therefore, if $FP|K$ is algebraic, then $(F.K^{\mathrm{sep}})P^i$ is algebraic over K^{sep} and hence separable-algebraically closed. Since $(F.K^{\mathrm{sep}})P^i \subset F^iP^i = (FP)^{\mathrm{sep}}$, it follows that $(F.K^{\mathrm{sep}})P^i = (FP)^{\mathrm{sep}}$. □

2.2 Transcendence bases of separable valued function fields

The goal of the present section is to prove the existence of a transcendence basis of a valued function field $(F|K,P)$ that reflects basic properties of P itself:

Proposition 2.3 *Let $(F|K,P)$ be a valued function field and assume that $F|K$ is separable. Then there exists a separating transcendence basis of $F|K$ containing elements $x_1,\dots,x_\rho,y_1,\dots,y_\tau$ such that:*

- *The images of $vx_1,\dots,vx_\rho$ under the natural map $vF \to vF/vK \otimes \mathbb{Q}$ form a basis of the $\mathbb{Q}$-vector space on the right side,*
- *$y_1P,\dots,y_\tau P$ form a transcendence basis of $FP|KP$.*

Here v is the valuation of F associated to P.

Remark 2.4 We do not know whether in addition to the assertion of the proposition, the y_i can be chosen such that $y_1P,\dots,y_\tau P$ form a separating transcendence basis of $FP|KP$ if the latter extension is separable.

To prove Proposition 2.3 we start with the case of a valued rational function field $(K(z)|K,P)$. The inequality (1) then implies that the following three cases appear:

1. $K(z)P|KP$ is an algebraic extension and $vK(z)/vK$ is a torsion group,
2. $K(z)P|KP$ is a transcendental extension and $vK(z)/vK$ is a torsion group,
3. $K(z)P|KP$ is an algebraic extension and $vK(z)/vK$ is no torsion group.

The first case can be characterized in terms of the behavior of the valued rational function field $(\tilde{K}(z)|\tilde{K},\tilde{P})$, where $\tilde{K}$ denotes the algebraic closure of K and $\tilde{K}(z)$ is equipped with an arbitrary extension of the place P: using Lemma 2.1 we then see that the case 1 is equivalent to $(\tilde{K}(z)|\tilde{K},\tilde{P})$ being immediate. We use this fact in combination with the following easy to prove

Lemma 2.5 *The valued extension $(L|K,P)$ is immediate if and only if for every $z \in L$, the set $\{v(z-a) \mid a \in K\}$ has no maximal element.*

We then get–see [K4]:

Lemma 2.6 *For a valued rational function field $(K(z)|K,P)$ such that $(\tilde{K}(z)|\tilde{K},\tilde{P})$ is not immediate, there exists a monic irreducible polynomial $f \in K[X]$ that has a root in the set $\{a \in \tilde{K} : va = \max(v(z-b) \mid b \in \tilde{K})\}$. If f has least degree among all such polynomials, then the following statements hold:*

1. *If $vK(z)/vK$ is no torsion group, then $vf(z) + vK$ is no torsion element.*

2. *If $K(z)P|KP$ is transcendental, then there is some $e \in \mathbb{N}$ and some $d \in K$ such that $(df(z)^e)P$ is transcendental over KP.*

We deduce:

Lemma 2.7 *In the situation of Lemma 2.6 there exists a non-constant polynomial $h \in K[z]$ such that $K(z)|K(h)$ is separable and either $vh + vK$ is no torsion element in $vK(z)/vK$ or hP is transcendental over KP.*

Proof: If vz is no torsion element of $vK(z)/vK$ or zP is transcendental over KP, then $h := z$ fullfills the requirements. Otherwise we treat the cases 2 and 3 separately:

case 3: let $f \in K[X]$ be the irreducibel polynomial as defined in Lemma 2.6. We consider $h(X) := Xf(X)$: since $h'(z) = zf'(z) + f(z) \neq 0$ the element z is a simple root of $h(X) - h(z) \in K(h(z))[X]$. Moreover $v(h(z)) + vK = vz + v(f(z)) + vK$; since $vz + vK$ is torsion by assumption, $h(z)$ fullfills all requirements by Lemma 2.6.

case 2: we consider $h(X) := Xdf(X)^e$, where $f \in K[X], d, e \in \mathbb{N}$ are choosen as in Lemma 2.6. Using the same argument as in the preceeding case we see that z is a simple root of $h(X) - h(z) \in K(h(z))[X]$. Moreover $h(z)P = (zdf(z)^e)P$ is transcendental over KP by Lemma 2.6 and since zP is assumed to be algebraic over KP. □

Proof of Proposition 2.3: let $z_1, \ldots, z_n$ be a separating transcendence basis of $F|K$ and set $K_0 := K$, $K_i := K(z_1, \ldots, z_i)$. Let $\tilde{P}$ be an extension of P to the algebraic closure $\tilde{F}$ of F. For $i = 1, ..., n$ we consider the extension $(\tilde{K}_{i-1}(z_i)|\tilde{K}_{i-1}, \tilde{P})$, where $\tilde{K}_i$ denotes the algebraic closure of K_i in $\tilde{F}$.

If $(\tilde{K}_{i-1}(z_i)|\tilde{K}_{i-1}, P)$ is not immediate we choose $h_i \in K_{i-1}[z_i]$ as in Lemma 2.7. Otherwise, we set $h_i := z_i$. The elements $h_1, \ldots, h_n$ then form a separating transcendence basis of $F|K$. The invariants $\rho := \dim_{\mathbb{Q}}(vF/vK) \otimes \mathbb{Q}$ and $\tau := \operatorname{trdeg} FP|KP$, where v is the valuation associated to P, satisfy:

$$\rho \;=\; \sum_{i=0}^{n-1} \dim_{\mathbb{Q}}(vK_{i+1}/vK_i \otimes \mathbb{Q}) \quad \text{and} \quad \tau \;=\; \sum_{i=0}^{n-1} \operatorname{trdeg} K_{i+1}P|K_iP \;.$$

In view of the fact that

$$\dim_{\mathbb{Q}}(vK_{i+1}/vK_i) \otimes \mathbb{Q} \;+\; \operatorname{trdeg} K_{i+1}P|K_iP \;\leq\; \operatorname{trdeg} K_{i+1}|K_i \;=\; 1 \;,$$

we find that for precisely ρ many values of i, vh_i will be rationally independent modulo vK_{i-1}. Collecting all of these $h_i(z_i)$ and calling them $x_1, \ldots, x_\rho$ we thus obtain that $vx_1, \ldots, vx_\rho$ is a maximal set of elements in vF rationally independent modulo vK.

Similarly, we find that for precisely τ many values of i, the residues h_iP will be transcendental over $K_{i-1}P$. Collecting all of these h_i and calling them $y_1, \ldots, y_\tau$ we thus obtain that $y_1P, \ldots, y_\tau P$ form a transcendence basis of $FP|KP$. □

2.3 Separably tame fields

The **absolute ramification field** K^r of a valued field (K,P) is defined to be the fixed field of the group $G_r := \{\sigma \in G_i : (\frac{\sigma(a)}{a} - 1)P^{\mathrm{sep}} = 0\}$, where P^{sep} is an extension of P to the separable closure K^{sep} of K. Let $p := \max(1, \mathrm{char}\, KP)$. By general ramification theory, $K^{\mathrm{sep}}|K^r$ is a p-extension (cf. [EN]). Moreover K^r contains the henselization K^h of (K,P) and is therefore henselian. The valued field (K,P) is called **separably tame** if it is henselian and satisfies $K^{\mathrm{sep}} = K^r$–see also [K8].

Lemma 2.8 *Every henselian field of residue characteristic 0 is a separably tame field.*

Let $L|K$ be an algebraic extension. General ramification theory yields $L^r = L.K^r$ (cf. [EN]). Hence if (K,P) is a separably tame field, then $L^r = L.K^r = L.K^{\mathrm{sep}} = L^{\mathrm{sep}}$. This proves:

Lemma 2.9 *Every algebraic extension of a separably tame field is separably tame.*

A finite, separable extension L of a separably tame field (K,P) is a subextension of $K^r|K$ and thus, it satisfies the fundamental equality (cf. [EN]). This shows that every finite separable extension of a separably tame field is defectless. A valued field with this property is called a **separably defectless field**. So we note:

Lemma 2.10 *Every separably tame field is separably defectless.*

The valued field (K,P) is called **separable-algebraically maximal** if it admits no proper immediate separable-algebraic extension. Since the henselization is an immediate separable-algebraic extension (cf. [R]), we have:

Lemma 2.11 *Every separable-algebraically maximal field is henselian.*

A finite, separable, immediate extension $L|K$ of a henselian, separably defectless field (K,P) is trivial: $[L:K] = \mathrm{e} \cdot \mathrm{f} = 1 \cdot 1$. Consequently:

Lemma 2.12 *Every henselian, separably defectless field is separable-algebraically maximal.*

Combined with Lemma 2.10 this yields:

Corollary 2.13 *Every separably tame field is separable-algebraically maximal.*

The subclass of separably tame fields within the class of separable-algebraically maximal fields can be characterized through conditions the value group and the residue field must satisfy–see also [K6]:

Proposition 2.14 *Suppose that P is a non-trivial place on K. Then (K,P) is separably tame if and only if it is separable-algebraically maximal, vK is p-divisible and KP is perfect.*

Proof: Assume first that (K,P) is separably tame. By Corollary 2.13, (K,P) then is separable-algebraically maximal and by Lemma 2.1, $vK^r = vK^{\rm sep}$ is divisible and $K^rP = K^{\rm sep}P$ is algebraically closed, where v denotes the valuation associated with the unique extension of P to $K^{\rm sep}$. This extension is denoted by P again. General ramification theory tells us that every element of vK^r/vK has order prime to p, and that $K^rP|KP$ is separable. Thus, vK is p-divisible and KP is perfect.

For the proof of the converse we start with the fact that for every henselian field (K,P) there exists a subfield W of $K^{\rm sep}$ such that $W.K^r = K^{\rm sep}$ and $W|K$ is linearly disjoint from $K^r|K$. This fact follows from Theorem 2.2 of [K–P–R] by Galois correspondence. Moreover Proposition 4.5 (ii) of [K–P–R] yields that vW is the p-divisible hull of vK and that WP is the perfect hull of KP. In the present setting, as vK is p-divisible and KP is perfect, we conclude that $(W|K,P)$ is immediate. But as (K,P) is separable-algebraically maximal and $W|K$ is separable-algebraic, it follows that $W = K$. Consequently $K^r = W.K^r = K^{\rm sep}$ showing that (K,P) is separably tame. □

Corollary 2.15 *If the valued field (K,P) has p-divisible value group and perfect residue field, then every maximal immediate separable-algebraic extension of (K,P) is a separably tame field. If* $\operatorname{char} KP = 0$ *then already the henselization (K^h,P^h) is a separably tame field.*

Proof: Let $(L,P')|(K,P)$ be a maximal immediate separable-algebraic extension. Then (L,P') is separably-algebraically maximal, thus Proposition 2.14 yields the first assertion.

The henselization (K^h,P^h) of (K,P) is an immediate separable-algebraic extension. Lemmas 2.8 shows that (K^h,P^h) is separably tame. □

We next turn to the question under which conditions a subfield of a separably tame field inherits this property.

Proposition 2.16 *Let (L,P) be a separably tame field. Assume that the subfield $K \subset L$ is separable-algebraically closed in L and that $LP|KP$ is an algebraic extension, then (K,P) is a separably tame field, the value group vK is pure in vL and $KP = LP$.*

Proof: The field K is separable-algebraically closed in the henselian field L, thus henselian too. Hensel's Lemma shows that KP is separable-algebraically closed in LP. If (K,P) is separably tame, then KP is perfect by Lemma 2.14. Consequently we get $KP = LP$. In this situation Hensel's Lemma yields that vL/vK is a p-group. On the other hand we know from Lemma 2.14 that vK is p-divisible. This shows that vK is pure in vL.

Altogether it remains to show that (K,P) is separably tame. Considering Lemma 2.8 from now on we can assume $p > 0$. In order to prove that $K^r = K^{\rm sep}$ holds, as in the proof of Proposition 2.14 we choose a field $W \subseteq K^{\rm sep}$ such that $W.K^r = K^{\rm sep}$ and W, K^r are linearly disjoint over K. We then have to show that $W = K$ holds.

Let $K'|K$ be a finite subextension of $W|K$. The degree of $K'|K$ is a power of p, since the Galois group $\operatorname{Gal}(K^{\rm sep}|K^r)$ is known to be a p-group and $[K':K] = [K'.K^r : K^r]$ by linear disjointness of W and K^r over K.

The fields K' and L are linearly disjoint over K, since $K'|K$ is separable and K is separable-algebraically closed in L. Consequently $L' := L.K'$ satisfies $[L':L] = [K':K]$ and K' is separable-algebraically closed in L'. The extension $L'|L$ is separable and since

L is assumed to be separably tame we have $L^{\text{sep}} = L^r$. We conclude $L' \subset L^r$. Since L is henselian and the value group $v(L)$ is divisible by Propositon 2.14, general ramification theory yields that $L'P|LP$ is separable and that $[L' : L] = [L'P : LP]$ holds.

Next utilizing Hensel's Lemma we see that $K'P$ is separable-algebraically closed in $L'P$. By hypothesis $LP|KP$ is an algebraic extension, therefore the same is true for $L'P|KP$ and thus for $L'P|K'P$. As a subextension of $WP|KP$ the extension $K'P|KP$ is purely inseparable. Now let $M|KP$ be a finite extension such that $M \subseteq LP$. Then

$$[M.K'P : KP]_{\text{sep}} = [M.K'P : K'P]_{\text{sep}}[K'P : KP]_{\text{sep}} = 1$$

thus proving that $LP = L'P$. Consequently $L = L'$ and thus $K = K'$ as desired. □

Remark: The preceeding proof is adapted from a proof that was given by F. Pop for the case of tame fields.

A extension $(K^{\text{st}}, P^{\text{st}})$ of the valued field (K, P) is called a **separably tame hull of** (K, P) if it is a separably tame field with the following properties:

- $K^{\text{st}}|K$ is separable-algebraic,
- $v^{\text{st}}K^{\text{st}}/vK$ is a p-group,
- $K^{\text{st}}P^{\text{st}}|KP$ is a purely inseparable extension.

These properties combined with Proposition 2.14 imply that $v^{\text{st}}K^{\text{st}}$ is the p-divisible hull of vK and that $K^{\text{st}}P^{\text{st}}$ is the perfect hull of KP.

A separably tame hull of a valued field (K, P) always exists: in the case $p = 1$ we can take the henselization (K^h, P^h) of (K, P). Otherwise let W be an intermediate field of $K^{\text{sep}}|K^h$ such that W and K^r are linearly disjoint over K^h and $W.K^r = K^{\text{sep}}$. Every maximal immediate separable-algebraic extension K^{st} of W then is a separably tame hull of (K, P) by Corollary 2.15. Unfortunately the separably tame hulls of (K, P) are not unique up to valuation preserving isomorphism over K. However the failure of uniqueness does not matter for our use of separably tame hulls.

2.4 Kaplansky approximation

For a polynomial $f \in K[z]$ in one variable over a field K of arbitrary characteristic the i-th formal derivative $f^{[i]} \in K[z]$ can be defined such that the following Taylor expansion holds (cf. [KA]):

$$f(z) \;=\; f(a) + \sum_{i=1}^{\deg f} f^{[i]}(a)(z-a)^i \;.$$

Let v be a valuation on $K(z)$. In this section we provide a result that allows to compute the value $vf(z)$ in terms of values derived from the Taylor polynomials after a suitable linear transformation of the variable z. Of course this is possible only if the valuation v satisfies certain conditions; they were studied by Ostrowski and Kaplansky [KA]: let $(K(z)|K, P)$ be an immediate transcendental extension. The element z induces the open sets $B(z, \alpha) := \{a \in K : \; v(z-a) \geq \alpha\}$, $\alpha \in vK$, in the uniform topological space (K, v). Note that by the triangle inequality $B(z, \alpha)$ is a ball in K. These balls are interesting because of the particular behavior of maps $f : B(z, \alpha) \to K$ induced by polynomials $f \in K[z]$.

Lemma 2.17 *Let $(K(z)|K,P)$ be an immediate transcendental extension. Assume that (K,P) is a separable-algebraically maximal field or that $(K(z),P)$ lies in the completion of (K,P). Then:*

$$\forall f \in K[z] \;\; \exists \alpha,\beta \in vK \;\; \forall a \in B(z,\beta):\; vf(a)=\alpha. \tag{3}$$

Kaplansky proved that if (3) does not hold, then there is a proper immediate algebraic extension of (K,P). If $(K(z),P)$ does not lie in the completion of (K,P), then using a variant of the Theorem on the Continuity of Roots this extension can be transformed into a proper immediate separable-algebraic extension (cf. [K6]). But such an extension cannot exist if we assume that K be separable-algebraically maximal.

If on the other hand $(K(z),P)$ lies in the completion of (K,P), then one can show that if f does not satisfy (3), then $vf(z)=\infty$. But this means that $f(z)=0$, contradicting the assumption that $K(z)|K$ is transcendental.

We deduce the announced result about the computation of $vf(z)$:

Lemma 2.18 *Let $(K(z)|K,P)$ be an immediate transcendental extension such that condition (3) holds. Then for every polynomial $f\in K[z]$ there exist $a,b\in K$ such that the values of the non-zero among the elements $f^{[i]}(a)b^i$ are pairwise distinct and $v\tilde{z}=0$ for $\tilde{z}:=\frac{z-a}{b}$. In particular:*

$$vf(z) \;=\; v(\sum_{i=0}^{\deg f} f^{[i]}(a)(z-a)^i) \;=\; v(\sum_{i=0}^{\deg f} f^{[i]}(a)b^i\tilde{z}^i) \;=\; \min_i \left(v(f^{[i]}(a)b^i)\right). \tag{4}$$

If finitely many polynomials in $K[z]$ are given, then a,b can be chosen such that (4) holds simultaneously for all of them.

Proof: Take finitely many polynomials $f_1,\ldots,f_n\in K[z]$. From Lemma 2.17 we know that for the non-zero among the polynomials $f_j^{[i]}$, $i,j\in\mathbb{N}$, there exist $\alpha_{ij},\beta\in vK$ such that: $\forall a\in B(z,\beta):\; vf_j^{[i]}(a)=\alpha_{ij}$. Since by Lemma 2.5 the set $\{v(z-a)\mid a\in K\}$ has no maximal element, we can choose $\beta\in vK$ so large that for $a\in B(z,\beta)$ and every fixed j, the values of the non-zero elements $f_j^{[i]}(a)(z-a)^i$, $i\in\mathbb{N}$, are pairwise distinct.

Having picked such an element $a\in K$, we choose an element $b\in K$ such that $vb=v(z-a)$. Then (4) holds by the ultrametric triangle law. □

3 Smoothly uniformizable places

In this section we study valued function fields $(F|K,P)$ such that (P,Z) is smoothly $\mathcal{O}_K$-uniformizable for some or all finite sets $Z\subset\mathcal{O}_K$, where $\mathcal{O}_K:=\mathcal{O}_P\cap K$. We provide the basic properties of smooth uniformizability and prove a valuation-theoretic consequence: inertial generation of $F|K$–Theorem 1.6. Moreover we identify two classes of valued function fields whos members $(F|K,P)$ are strongly smoothly $\mathcal{O}_K$-uniformizable. One of these classes consists of the separable, immediate, valued function fields of transcendence degree one over a separably tame field (K,P). The smooth uniformizability of the members of that class is a major building block of the proof of Theorem 1.1.

3.1 Basic properties

Let $(F|K, P)$ be a valued function field. For the problem whether a pair (P, Z) is smoothly R-uniformizable for some subring $R \subseteq \mathcal{O}_K$ it suffices to consider affine R-models $X = \operatorname{Spec} A$ of F, $A \subset F$ being a finitely presented R-algebra. Smoothness of A at the center q of P on X then means that there exists $f \in A \setminus q$ such that A_f is R-flat and the rings $A_f \otimes_R \widetilde{k(p)}$, $\widetilde{k(p)}$ the algebraic closure of $k(p) = \operatorname{Frac}(R/p)$, are regular for all $p \in \operatorname{Spec}(R)$. In the sequel we frequently need to contruct such an algebra A within a given subring of F. The following structure theorem is particularly helpful in that respect–see [EGA IV], (17.11.4) for its proof:

Theorem 3.1 *The R-algebra A is smooth at $q \in \operatorname{Spec} A$ if and only if there exists $u \in A \backslash q$ such that A_u is an étale algebra over a polynomial ring $R[x_1, \ldots, x_d]$.*

Recall that an R-algebra A is called **standard-étale** if it admits a presentation of the form

$$0 \to fR[X]_g \to R[X]_g \xrightarrow{\phi} A \to 0 \tag{5}$$

with $f, g \in R[X]$, f monic and such that $\phi(f') \in A^\times$ for the derivative f' of f. Generalizing this definition we call an R-algebra A **standard-smooth** if for some polynomial ring $T := R[x_1, \ldots, x_d]$ and some $h \in T$ the structure morphism $R \to A$ can be factored as

$$R \to T_h \to A,$$

where $R \to T_h$ is the natural map and $T_h \to A$ is standard-étale. Consequently A admits a presentation

$$0 \to fT_h[X]_g \to T_h[X]_g \xrightarrow{\phi} A \to 0 \tag{6}$$

with $f, g \in T_h[X]$, f monic, $\phi(f') \in A^\times$. If A is a domain, then the polynomial f is prime in $T_h[X]_g$ but not necessarily in $T_h[X]$ itself. However if we assume R to be normal, then T_h is normal too, thus using Gauß' lemma we can choose f to be prime in $T_h[X]$.

Theorem 3.1 and the local structure theorem for étale algebras ([Ray], Ch. V, Thm. 1.) show that an R-algebra A is smooth at $q \in \operatorname{Spec} A$ if and only if there exists some $u \in A \backslash q$ such that A_u is standard-smooth.

Using standard-smooth algebras we prove that smoothness at a prime behaves well with respect to descent and ascent:

Proposition 3.2 *Let $A|S$ be an extension of domains such that S is normal and A is a finitely presented S-algebra that is smooth at $q \in \operatorname{Spec} A$. Let $R \subseteq S$ be a subring of S and let $Z \subset A_q$ be a finite set. Then there exists a finitely generated ring extension $S_0|R$ within S with the property: for every normal domain S' with $S_0 \subseteq S' \subseteq S$, there exists a finitely presented S'-algebra $A' \subseteq A_q$ that is smooth at $q' := qA_q \cap A'$ and satisfies $Z \subset A'_{q'}$. Moreover for $F := \operatorname{Frac} A$, $K := \operatorname{Frac} S$ and $F' := \operatorname{Frac} A'$ the relation $F = F'.K$ holds.*

Proof: There exists $u \in A \setminus q$ such that $B := A_u$ is a standard-smooth S-algebra. We choose a presentation of the form (6) for $B|S$, where $T = S[x_1, \ldots x_d]$. Let $Z_1 \subset S$ be the finite set of coefficients of $h \in T$ and of the coefficients $c \in T_h$ of f and g. The condition $\phi(f') \in A^\times$ can be rewritten as

$$1 = \phi(f' \frac{t}{g^\ell}), \quad t \in T_h[X], \ell \in \mathbb{N}; \tag{7}$$

let $Z_2 \subset S$ be the finite set of coefficients of the coefficients of t. Every $z \in Z$ can be expressed in the form

$$z = \phi(\frac{p_z}{g^{k_z}})\phi(\frac{q_z}{g^{l_z}})^{-1}, \;\; p_z, q_z \in T_h[X], \; k_z, l_z \in \mathbb{N}, \tag{8}$$

where $\phi(\frac{q_z}{g^{l_z}}) \notin qB$. Let $Z_3 \subset S$ be the finite set of coefficients of the coefficients of the polynomials $\{p_z, q_z : z \in Z\}$. Let $S_0 := R[Z_1 \cup Z_2 \cup Z_3] \subseteq S$ and consider a normal ring $S' \subseteq S$ such that $S_0 \subseteq S'$. In the presentation (6) choosen for $B|S$ we can then replace the ring S by S' thus getting a presentation

$$0 \to fT'_h[X]_g \to T'_h[X]_g \xrightarrow{\phi'} A' \to 0 \tag{9}$$

with $T' := S'[x_1, \ldots, x_d] \subseteq T$. By construction A' is a standard-smooth S'-algebra and the inclusion $T'_h[X]_g \subseteq T_h[X]_g$ induces a homomorphism $A' \to A$. We show that this map is injective: T'_h is normal since S' is so. An application of Gauß' lemma thus yields $fT_h[X] \cap T'_h[X] = fT'_h[X]$ hence $fT_h[X]_g \cap T'_h[X]_g = fT'_h[X]_g$.

By construction $Z \subset A'_{q'}$, $q' := A' \cap A_q$, holds. Eventually with $K' := \operatorname{Frac} S'$ we get $F'.K = K'(x_1, \ldots, x_d, \phi'(X)).K = K(x_1, \ldots, x_d, \phi(X)) = F$ holds. □

As for ascent we obtain:

Proposition 3.3 *Let $A|R$ be an extension of domains such that R is normal and A is a standard-smooth R-algebra. Let $S \supseteq R$ be a normal domain and assume that $F := \operatorname{Frac} A$ and $L := \operatorname{Frac} S$ are subfields of some field Ω such that F, L are algebraically disjoint over $K := \operatorname{Frac} R$. Then the compositum $A.S \subseteq F.L$ is a standard-smooth S-algebra.*

Proof: We choose a presentation of the form (6) for $A|R$; it yields $A = T_h[x, g(x)^{-1}]$ with $T = R[x_1, \ldots, x_d]$ a polynomial ring and $x = \phi(X)$, $g \in T_h[X]$. Consequently we get $A.S = T'_h[x, g(x)^{-1}]$ with $T' = S[x_1, \ldots, x_d]$. The latter is a polynomial ring over S since F and L are assumed to be algebraically disjoint over K. As mentioned earlier the normality of T_h implies that the polynomial $f \in T_h[X]$ appearing in the presentation (6) can be choosen to be the minimal polynomial of x over $K(x_1, \ldots, x_d)$. Let f_1 be the minimal polynomial of x over $L(x_1, \ldots, x_d)$. The normality of T'_h then yields $f_1 \in T'_h[X]$ and thus the exact sequence

$$0 \to f_1 T'_h[X]_g \to T'_h[X]_g \to A.S \to 0.$$

Moreover we have $f = f_1 f_2$ for some $f_2 \in T'_h[X]$. Taking derivatives we obtain $f'_1(x)f_2(x) = f'(x) \in A^\times \subseteq (A.S)^\times$, hence $f'_1(x) \in (A.S)^\times$. □

As an application of Proposition 3.3 we can clarify some properties of smooth uniformizability over a valuation domain in a situation, where the constant field of the valued function field $(F|K, P)$ considered is extended:

Proposition 3.4 *Let $(F|K, P)$ be a finitely generated, valued field extension. Let $L|K$ be a field extension and assume that F and L are subfields of some field Ω such that F and L are algebraically disjoint over K. Let $\mathcal{P}$ be an extension of P to $F.L \subseteq \Omega$.*

1. *If (P, Z) is smoothly $\mathcal{O}_K$-uniformizable, then $(\mathcal{P}, Z)$ is smoothly $\mathcal{O}_L$-uniformizable, where $\mathcal{O}_L := \mathcal{O}_\mathcal{P} \cap L$.*

2. *Assume that $L|K$ is algebraic. If $(\mathcal{P}, Z)$ is smoothly $\mathcal{O}_L$-uniformizable and Z contains a set of generators of $F|K$, then there is a finitely generated subextension $M|K$ of $L|K$ such that $(\mathcal{P}|_{F.M}, Z)$ is smoothly $\mathcal{O}_M$-uniformizable. The field M can be choosen to be algebraically closed in $F.M$.*

3. *Assume that $L|K$ is Galois. If $(\mathcal{P}, Z)$ is smoothly $\mathcal{O}_L$-uniformizable and Z contains a set of generators of $F|K$, then there is a finite Galois subextension $N|K$ of $L|K$ containing $L \cap F$ such that $(\mathcal{P}|_{F.N}, Z)$ is smoothly $\mathcal{O}_N$-uniformizable.*

Proof: **1.** There exists a standard-smooth $\mathcal{O}_K$-algebra $A \subset \mathcal{O}_P$ such that $\operatorname{Frac} A = F$ and $Z \subset A_q$, $q := A \cap \mathcal{M}_P$, hold. By Proposition 3.3 the $\mathcal{O}_L$-algebra $B := A.\mathcal{O}_L \subset \mathcal{O}_\mathcal{P}$ is standard-smooth. It satisfies $\operatorname{Frac} B = F.L$. Moreover for $q_B := B \cap \mathcal{M}_\mathcal{P}$ the inclusion $Z \subset A_q \subseteq B_{q_B}$ holds.

2. Let $A \subseteq \mathcal{O}_\mathcal{P}$ be a finitely presented $\mathcal{O}_L$-algebra that is smooth at $q := \mathcal{M}_\mathcal{P} \cap A$ and satisfies $\operatorname{Frac} A = F.L$ and $Z \subset A_q$. Proposition 3.2 yields a finitely generated extension $S_0 \subseteq \mathcal{O}_L$ of $\mathcal{O}_K$ such that for every valuation ring $\mathcal{O}_{M'} \subseteq \mathcal{O}_L$ containing S_0 there exists a finitely presented $\mathcal{O}_{M'}$-algebra $B \subseteq A_q$ that is smooth at $q_B := qA_q \cap B$ and satisfies $Z \subset B_{q_B}$. We choose $\mathcal{O}_{M'}$ such that $M' = \operatorname{Frac} \mathcal{O}_{M'}$ is a finitely generated extension of K. By assumption about Z the field $E := \operatorname{Frac} B$ contains F. Since $E|M'$ is finitely generated, there exists a finitely generated extension $N|M'$, $N \subseteq L$, such that $E \subseteq F.N = E.N$. By construction $(\mathcal{P}|_E, Z)$ is smoothly $\mathcal{O}_{M'}$-uniformizable, hence applying (1) $(\mathcal{P}|_{F.N}, Z)$ is smoothly $\mathcal{O}_N$-uniformizable. For the algebraic closure M of N in $F.N$ we have $F.N = F.M$, thus we can apply (1) again to conclude that $(\mathcal{P}|_{F.M}, Z)$ is smoothly $\mathcal{O}_M$-uniformizable.

3. Similarly to the first part of the proof of (2) we choose an $\mathcal{O}_M$-algebra $B \subset A_q$ smooth at the center q_B of $\mathcal{P}$ on B and such that $Z \subset B_{q_B}$ holds. Since $(L \cap F)|K$ is finite we can assume that $M|K$ is a finite extension and $(L \cap F) \subseteq M$ holds. By assumption about Z the field $E := \operatorname{Frac} B$ contains F, thus the isomorphism of Galois groups

$$\operatorname{Gal}(F.L|F) \to \operatorname{Gal}(L|L \cap F),\ \sigma \mapsto \sigma|_L$$

yields $E = F.M'$ for some finite extension $M'|(L \cap F)$ such that $M' \subseteq L$ and $M \subseteq M'$. Let N be the normal hull of $M'|K$. Since $(\mathcal{P}|_E, Z)$ is smoothly $\mathcal{O}_M$-uniformizable by construction an application of (1) yields that $(\mathcal{P}|_{E.N}, Z)$ is smoothly $\mathcal{O}_N$-uniformizable. The equation $E.N = F.M'.N = F.N$ concludes the proof. □

Let B be a smooth R-algebra and C be a smooth B-algebra, then C is a smooth R-algebra–[EGA IV],(17.3.3). Similarly if the R-algebra B is regular at the prime q_B and q_C is a prime of the smooth B-algebra C lying above q_B, then C is regular at q_C–[EGA IV],(6.5.1). We next prove similar properties for (smooth) uniformizability.

Proposition 3.5 *Let $(F|L, P)$ be a finitely generated, valued field extension and assume that (P, Z), $Z \subset \mathcal{O}_P$ finite, is smoothly $\mathcal{O}_L$-uniformizable. Let R be a subring of $\mathcal{O}_L$ and let $Z' \subset \mathcal{O}_L$ be a finite set. Consider the following two cases:*

case 1: *$P|_L$ is strongly smoothly R-uniformizable,*

case 2: *R is noetherian and $P|_L$ is strongly R-uniformizable.*

Then there exists a tower $R \subseteq B \subseteq C \subset \mathcal{O}_P$ of domains with fields of fractions $\operatorname{Frac} B = L$ *and* $\operatorname{Frac} C = F$ *such that:*

- *the R-algebra B is finitely presented in both cases, smooth in case 1, and has the property that B_{q_B} is regular in case 2,*
- *the B-algebra C is finitely presented in both cases, smooth in case 1, and has the property that C_{q_B} is a smooth B_{q_B}-algebra in case 2,*
- *in both cases $Z' \subset B_{q_B}$ and $Z \subset C_{q_C}$ hold.*

Consequently the pair (P, Z) is smoothly R-uniformizable in case 1 and R-uniformizable in case 2.

Proof: Take a standard-smooth $\mathcal{O}_L$-algebra $A \subset \mathcal{O}_P$ with the properties $\operatorname{Frac} A = F$ and $Z \subset A_{q_A}$, where $q_A := \mathcal{M}_P \cap A$. After choosing a presentation of $A|\mathcal{O}_L$ of the form (6), we can define the finite sets $Z_1, Z_2, Z_3 \subset \mathcal{O}_L$ as in the proof of Proposition 3.2. By assumption there then exists a finitely presented R-algebra $B \subseteq \mathcal{O}_L$ with the properties $\operatorname{Frac} B = L$ and $Z' \cup Z_1 \cup Z_2 \cup Z_3 \subset B_{q_B}$, where $q_B := \mathcal{M}_P \cap B$. Furthermore $B|R$ is smooth at q_B in case 1 and B_{q_B} is regular in case 2.

In case 1 by passing from B to a suitable localization B_u, $u \notin q_B$, we can assume that $Z' \cup Z_1 \cup Z_2 \cup Z_3 \subset B$ and that B is a smooth R-algebra. In particular B is normal, hence as in the proof of Proposition 3.2 we can construct a standard-smooth B-algebra $C \subseteq A$ such that $\operatorname{Frac} C = \operatorname{Frac} A$ and $Z \subset C_{q_C}$ hold. Since C then is a smooth R-algebra the proposition is proved in case 1.

In case 2 since B_{q_B} is normal as in the proof of Proposition 3.2 we can construct a standard-smooth B_{q_B}-algebra $C' \subseteq A$ such that $\operatorname{Frac} C' = \operatorname{Frac} A$ and $Z \subset C'_{q_{C'}}$ hold, where $q_{C'} := \mathcal{M}_P \cap C'$. For $C' = B_{q_B}[x_1, \ldots, x_r]$ we set $C := B[x_1, \ldots, x_r]$; then $C_{q_B} = C'$ and $C_{q_C} = C'_{q_{C'}}$. The smoothness of $C'|B_{q_B}$ and the regularity of B_{q_B} thus imply the regularity of C_{q_C} and the proof is complete in the case 2. □

Corollary 3.6 *Let $(F|K, P)$ be a finitely generated, valued field extension and L an intermediate field of $F|K$. If $P|_L$ is strongly smoothly $\mathcal{O}_K$-uniformizable and P is strongly smoothly $\mathcal{O}_L$-uniformizable, then P is strongly smoothly $\mathcal{O}_K$-uniformizable.*

3.2 Inertially generated function fields

Applying the results of the previous subsection we are now able to provide the proof of Theorem 1.6.

Lemma 3.7 *Let $(F|L, P)$ be a finite valued field extension and let $R \subseteq \mathcal{O}_L$ be a subring of L with* $\operatorname{Frac} R = L = \operatorname{Frac} \mathcal{O}_L$. *Then the following statements hold:*

1. *If P is smoothly R-uniformizable, then $\mathcal{O}_P|\mathcal{O}_L$ is local-étale.*
2. *P is strongly smoothly $\mathcal{O}_L$-uniformizable if and only if $\mathcal{O}_P|\mathcal{O}_L$ is local-étale.*
3. *$\mathcal{O}_P|\mathcal{O}_L$ is local-étale if and only if (F, P) lies in the absolute inertia field of (L, P).*

Proof: **1.:** Let $A \subset \mathcal{O}_P$ be a finitely presented R-algebra that is smooth at $q := \mathcal{M}_P \cap A$ and satisfies $\operatorname{Frac} A = F$. Since $F|L$ is algebraic Theorem 3.1 shows that we can assume $A|R$ to be standard-étale. An application of Proposition 3.3 yields that the $\mathcal{O}_L$-algebra $B := A.\mathcal{O}_L \subseteq \mathcal{O}_P$ is standard-étale too. Hence it suffices to prove $B_{q_B} = \mathcal{O}_P$ for $q_B := \mathcal{M}_P \cap B$. Indeed as an étale extension of the normal domain $\mathcal{O}_L$ the domain B is normal too [Ray], Ch. VII, Prop. 2. Hence B_{q_B} is a normal local extension of $\mathcal{O}_L$ in the finite extension $F|L$ and thus a valuation domain contained in $\mathcal{O}_P$. Since the valuation rings of F that are local extensions of $\mathcal{O}_L$ are pairwise incomparable with respect to inclusion we get $B_{q_B} = \mathcal{O}_P$ as desired.

2.: The remaining implication $\Leftarrow$ is obvious.

3.: See [Ray], Ch. X., Thm. 1. □

Proof of Theorem 1.6: let $X = \operatorname{Spec} A$ be an affine $\mathcal{O}_K$-model of the valued function field $(F|K, P)$, that is smooth at the center $q := \mathcal{M}_P \cap A$ of P on X. By Theorem 3.1 we can assume that A is an étale extension of a polynomial ring $\mathcal{O}_K[x_1, \ldots, x_d] \subseteq A$. In particular the set $T := \{x_1, \ldots, x_d\}$ forms a transcendence basis of $F|K$ and $(P, \emptyset)$ is smoothly $\mathcal{O}_K[x_1, \ldots, x_d]$-uniformizable. An application of Lemma 3.7, (1) yields that $\mathcal{O}_P|\mathcal{O}_{K(T)}$ is local-étale. Thus F lies in the absolute inertia field of $(K(T), P)$ by (3) of the same lemma. Finally assume that $FP = KP$ holds. Then F is an extension of $K(T)$ within the inertia field of $(K(T), P)$ such that $FP = K(T)P$. Thus F must lie in the henselization of $(K(T), P)$. □

3.3 Immediate function fields of transcendence degree one

It is tempting to try to prove the reversed implication in Theorem 1.6. This however amounts to proving the $\mathcal{O}_K$-uniformizability of all valued, rational function fields $(K(T), P)$. In the sequel we present a case, where rational function fields of one variable are strongly smoothly $\mathcal{O}_K$-uniformizable and draw some conclusions utilizing a structure theorem for immediate function fields over a separably tame field:

Theorem 3.8 *Let $(F|K, P)$ be an immediate, valued function field of transcendence degree 1 and assume that (K, P) is separably tame. If $F|K$ is separable, then there exists $x \in F$ such that (F, P) lies in the henselization $(K(x)^h, P^h)$, that is $(F|K, P)$ is henselian generated.*

For the case $\operatorname{char} KP = 0$ the assertion is a direct consequence of the fact that every such field is defectless–in fact every $x \in F \setminus K$ will then do the job. In contrast to this, the case $\operatorname{char} KP \neq 0$ requires a much deeper structure theory of immediate algebraic extensions of henselian fields, in order to find suitable elements x. For the proof of the theorem in this case see [K8].

Lemma 3.9 *Let $(K(x)|K, P)$ be an immediate, transcendental extension possessing the property (3) stated in Lemma 2.17, then P is strongly smoothly $\mathcal{O}_K$-uniformizable.*

Proof: Let $z_1, \ldots, z_m \in \mathcal{O}_P$ and write $z_j = f_j(x)/g_j(x)$ with polynomials $f_j, g_j \in K[x]$. We apply Lemma 2.18 to these finitely many polynomials and choose $\tilde{x} = \frac{x-a}{b}$, $a, b \in K$, according to this lemma. Then by (4), for every j we can find i_j, k_j such that

$$vf_j(x) = vf_j^{[i_j]}(a)\, b^{i_j} = \min_i vf_j^{[i]}(a)\, b^i \text{ and } vg_j(x) = vg_j^{[k_j]}(a)\, b^{k_j} = \min_i vg_j^{[i]}(a)\, b^i.$$

Thus we can write

$$z_j \;=\; \frac{f_j^{[i_j]}(a)\, b^{i_j}}{g_j^{[k_j]}(a)\, b^{k_j}} \cdot \frac{\tilde{f}_j(\tilde{x})}{\tilde{g}_j(\tilde{x})}, \tag{10}$$

where $\tilde{f}_j, \tilde{g}_j \in \mathcal{O}_K[\tilde{x}]$ and $v\tilde{f}_j(\tilde{x}) = 0 = v\tilde{g}_j(\tilde{x})$. Consequently the first factor in the representation (10) is an element of $\mathcal{O}_K$ and we have shown that $z_1, \ldots, z_m \in \mathcal{O}_K[\tilde{x}]_q$ for the prime $q := \mathcal{O}_K[\tilde{x}] \cap \mathcal{M}_P$. □

Proposition 3.10 *Let $(F|K, P)$ be an immediate, valued function field of transcendence degree 1 and assume that $F|K$ is separable and that (K, P) is separably tame. Then P is strongly smoothly $\mathcal{O}_K$-uniformizable.*

Proof: By Theorem 3.8 there exists some $x \in F$ such that $(F, P) \subset (K(x)^h, P^h)$. Since (K, P) is separably tame and hence separable-algebraically maximal, Lemma 2.17 shows that condition (3) holds in $(K(x)|K, P)$. Therefore $P|_{K(x)}$ is strongly smoothly $\mathcal{O}_K$-uniformizable by Lemma 3.9. Lemma 3.7, (2) and (3) yield that P is strongly smoothly $\mathcal{O}_{K(x)}$-uniformizable. The assertion now follows from Corollary 3.6. □

3.4 Extensions within the completion

In this subsection the proof of the main result Theorem 1.5 is provided. The subsequent two facts are main ingredients of that proof.

Proposition 3.11 *Let $(L|K, P)$ be a finitely generated, separable extension within the completion of (K, P). Then P is strongly smoothly $\mathcal{O}_K$-uniformizable.*

Proof: By assumption there exists a transcendence basis T of $L|K$ such that $L|K(T)$ is separable-algebraic. By induction on the transcendence degree, using the Lemmata 2.17 and 3.9 and Corollary 3.6 we find that $P|_{K(T)}$ is strongly smoothly $\mathcal{O}_K$-uniformizable.

Since $L|K(T)$ is separable-algebraic and L lies in the completion of K which is also the completion of $K(T)$, L must lie within the henselization of $K(T)$. Hence by Lemma 3.7 P is strongly smoothly $\mathcal{O}_{K(T)}$-uniformizable and thus again by Corollary 3.6 strongly smoothly $\mathcal{O}_K$-uniformizable. □

Lemma 3.12 *Every immediate extension of a defectless field is separable.*

Proof: Let $(L|K, P)$ be an immediate extension and assume that (K, P) is defectless. It suffices to show that every finite, purely inseparable extension M of K is linearly disjoint to L over K. Let e_M and f_M be the ramification index and the residue degree of the unique extension of P to M and define similarly the ramification index $e_{L.M}$ and the residue degree $f_{L.M}$ in the extension $L.M|L$. The fundamental (in)equality then yields:

$$[M : K] \geq [L.M : L] \geq e_{L.M} f_{L.M} \geq e_M f_M = [M : K]$$

and thus the assertion $[M : K] = [L.M : L]$. □

Proof of Theorem 1.5: let F_0 be an intermediate field of $F|K$ such that $P|_{F_0}$ is an Abhyankar place and (F, P) lies in the completion of (F_0, P).

The valued field (K, P) is defectless by assumption respectively because $P|_K = \mathrm{id}_K$. Hence the Generalized Stability Theorem [K7], Thm. 1 yields that (F_0, P) is defectless. The extension $F|F_0$ is immediate hence separable due to Lemma 3.12. Proposition 3.11 thus yields that the place P is strongly smoothly $\mathcal{O}_{F_0}$-uniformizable.

Let $Z \subset \mathcal{O}_P$ be a finite set. For every $z \in Z \cap \mathcal{M}_P$ we choose a representation $z = uz'$ such that $u \in \mathcal{O}_P^\times$ and $z' \in \mathcal{O}_{F_0}$ holds. Let $U \subset \mathcal{O}_P^\times$ and $Z' \subset \mathcal{O}_{F_0}$ be the finite sets consisting of all of the elements u and z' appearing in these representations.

Case 1 of the theorem: we apply Theorem 1.1 of [K–K] to obtain that $P|_{F_0}$ is strongly smoothly K-uniformizable. Corollary 3.6 then already yields that P is strongly smoothly K-uniformizable. However to prove the existence of the morphism $f : X \to X_0$ we use case 1 of the Proposition 3.5: it yields the existence of a morphism $f : \operatorname{Spec} C \to \operatorname{Spec} B$ between affine K-models of F and F_0 such that:

- the K-algebra B is smooth at $q_B := \mathcal{M}_P \cap B$ and $Z' \subset B_{q_B}$,
- f is smooth at $q_C := \mathcal{M}_P \cap C$ and $U \subset C_{q_C}$.

Theorem 1.1 of [K–K] yields the existence of a regular system of parameters $(a_1, \ldots, a_m)$ of B_{q_B} such that every $z' \in Z'$ is a B_{q_B}-monomial in these parameters. Since the ring extension $C_{q_C}|B_{q_B}$ is flat, the elements $a_1, \ldots, a_m$ remain a part of a regular system of parameters of C_{q_C}. Thus by construction every element $z = uz'$ of $Z \subset C_{q_C}$ is a C_{q_C}-monomial in a regular system of parameters of C_{q_C} as required. Using Theorem 1.1 of [K–K] a last time and the equation $vF_0 = vF$ we get:

$$\dim C_{q_C} \geq \dim B_{q_B} = \dim(vF_0 \otimes \mathbb{Q}) = \dim(vF \otimes \mathbb{Q}).$$

Case 2 of the theorem: we apply Theorem 1.2 of [K–K] which yields that $P|_{F_0}$ is strongly R-uniformizable. Next we invoke case 2 of Proposition 3.5 to obtain that (P, Z) is R-uniformizable and the existence of a morphism $f : \operatorname{Spec} C \to \operatorname{Spec} B$ such that:

- the R-algebra B is regular at q_B and $Z' \subset B_{q_B}$,
- the B_{q_B}-algebra C_{q_B} is smooth and $U \subset C_{q_C}$.

The arguments used in case 1 to prove the assertions of the theorem carry over to case 2 just repacing Theorem 1.1 of [K–K] by Theorem 1.2. □

4 Local uniformization by finite extension

This section is devoted to the proofs of the main results Theorem 1.1, 1.2 and 1.4. In each of the three theorems local uniformization is achieved only after a finite extension of the function field in consideration. This finite extension can be choosen to be either Galois or an extension within a given separably tame extension of the function field. Although the proofs for the two cases are similar we present them separately to keep the exposition well-accessible.

4.1 The Proof of Theorem 1.1

Uniformization after a Galois extension

We proceed by induction on the transcendence degree $n := \operatorname{trdeg} E|K$ starting with the case $n = 1$. Since by assumption vE/vK is torsion and $EP|KP$ is algebraic Lemma 2.1 implies that the extension $(E^{\mathrm{sep}}|K^{\mathrm{sep}}, \mathcal{P})$ and hence also its subextension $(E.K^{\mathrm{sep}}|K^{\mathrm{sep}}, \mathcal{P})$ are immediate. Since $(K^{\mathrm{sep}}, \mathcal{P})$ is a separably tame field, we can apply Proposition 3.10 to see that $\mathcal{P}|_{E.K^{\mathrm{sep}}}$ is strongly smoothly $\mathcal{O}_{K^{\mathrm{sep}}}$-uniformizable.

We express every $z \in Z$ in the form $z = uz'$, $u \in \mathcal{O}^{\times}_{E.K^{\mathrm{sep}}}$ and $z' \in \mathcal{O}_{K^{\mathrm{sep}}}$: let U and Z' be the finite sets of elements u and z' appearing in these expressions. Moreover let $Z_g \subset \mathcal{O}_P$ be a finite set of generators of $E|K$. An application of Proposition 3.4 (3) yields the existence of a finite Galois extension $\mathcal{K}|K$ with the following properties:

- $(\mathcal{P}|_{\mathcal{E}}, U \cup Z' \cup Z_g)$ is smoothly $\mathcal{O}_{\mathcal{K}}$-uniformizable, where $\mathcal{E} := E.\mathcal{K}$,
- $\mathcal{K}$ contains $K^{\mathrm{sep}} \cap E$.

$\mathcal{K}$ is algebraically closed in $\mathcal{E}$: $E|K$ is assumed to be separable, hence $K^{\mathrm{sep}} \cap E$ is the algebraic closure of K in E. We conclude that E and K^{sep} are linearly disjoint over $K^{\mathrm{sep}} \cap E$, thus $\mathcal{E}$ and K^{sep} are linearly disjoint over $\mathcal{K}$, which yields the assertion.

Let X be an $\mathcal{O}_{\mathcal{K}}$-model of $\mathcal{E}|\mathcal{K}$ that is smooth at the center x of $\mathcal{P}$ and such that $U \cup Z' \subset \mathcal{O}_{X,x}$ holds. Then $U \subset \mathcal{O}^{\times}_{X,x}$ and the factorizations $uz' = z \in \mathcal{O}_{X,x}$ hold. Moreover $z' \in \mathcal{O}_{X,x} \cap \mathcal{O}_{K^{\mathrm{sep}}} = \mathcal{O}_{\mathcal{K}}$, where the last equality holds because $\mathcal{K}$ is algebraically closed in $\mathcal{E}$.

Finally let $E_0|K$ be an arbitrary subextension of $E|K$ of transcendence degree $n-1 = 0$. Then $E_0|K$ is a finite separable extension, hence $E_0 \subseteq \mathcal{K}$ and the assertion is proved for $n = 1$.

Let us now assume that $n > 1$. We choose a subextension $E_0|K$ of $E|K$ of transcendence degree $n - 1$ such that $E|E_0$ is separable. Such a subextension always exists: choose a separating transcendence basis T of $E|K$ and a subset $T_0 \subset T$ such that $\operatorname{trdeg} E|K(T_0) = 1$. Set $E_0 := K(T_0) \subset E$, then $E|E_0$ is separable.

Since vE/vK is a torsion group and $EP|KP$ is algebraic, the same holds for vE/vE_0 and $EP|E_0P$. Hence by what we have already shown for the case $n = 1$ and by the remarks on standard-smooth algebras following Theorem 3.1, there exists a finite Galois extension $\mathcal{E}_0|E_0$ and an affine $\mathcal{O}_{\mathcal{E}_0}$-model $\operatorname{Spec} A$ of $E.\mathcal{E}_0|\mathcal{E}_0$, $A \subset \mathcal{O}_{\mathcal{P}}$, with the following properties:

- $A|\mathcal{O}_{\mathcal{E}_0}$ is standard-smooth,
- $\forall z \in Z :\ \exists u \in A^{\times}_{q_A}, z' \in \mathcal{O}_{\mathcal{E}_0} :\ \ z = uz'$, (11)

where $q_A := A \cap \mathcal{M}_{\mathcal{P}}$. Let $U \subset A^{\times}_{q_A}$ and $Z' \subset \mathcal{O}_{\mathcal{E}_0}$ be the finite sets of elements u and z' appearing in the factorizations (11).

Next we invoke Proposition 3.2 which yields a finitely generated $\mathcal{O}_K$-algebra

$$S_0 = \mathcal{O}_K[x_1, \ldots, x_r] \subseteq \mathcal{O}_{\mathcal{E}_0} \tag{12}$$

such that for every integrally closed domain $S' \subseteq \mathcal{O}_{\mathcal{E}_0}$ with $S_0 \subseteq S'$ there exists a standard-smooth S'-algebra $A' \subseteq A_{q_A}$ with the property $U \subset (A')_{q'}$, $q' := A' \cap q_A$.

Choose a valuation of E^{sep} associated to $\mathcal{P}$; it is an extension of the valuation v and we will denote it by v too. As a direct consequence of the fact that vE/vK is torsion and $EP|KP$ is algebraic we have that $v\mathcal{E}_0/vK$ is torsion and $\mathcal{E}_0\mathcal{P}|KP$ is algebraic.

Let $E_1|K$ be a subextension of $E_0|K$ such that $E_0|E_1$ is separable and of transcendence degree 1, then $\mathcal{E}_0|E_1$ is separable and of transcendence degree 1 too. We apply the induction hypothesis to the valued function field $(\mathcal{E}_0|K, \mathcal{P})$ and the subfunction field $E_1|K$: there exists a finite Galois extension $\mathcal{E}_1|E_1$ and a finite Galois extension $\mathcal{K}|K$ within $\mathcal{E}_0.\mathcal{E}_1$ such that $\mathcal{E}_0.\mathcal{E}_1|\mathcal{K}$ possesses an affine $\mathcal{O}_\mathcal{K}$-model $\operatorname{Spec} B$, $B \subset \mathcal{O}_\mathcal{P}$, with the following properties:

- $B|\mathcal{O}_\mathcal{K}$ is smooth, (13)
- $\{x_1, \ldots, x_r\} \subset B$ (see (12)), (14)
- B contains a finite set of generators of $\mathcal{E}_0|K$, (15)
- $\forall z' \in Z' : \exists u' \in B_{q_B}^\times, z'' \in \mathcal{O}_\mathcal{K} : \quad z' = u'z''$, (16)

where $q_B := \mathcal{M}_\mathcal{P} \cap B$. Since $B_0 := B \cap \mathcal{E}_0$ is normal and contains S_0 (14) there exists a standard-smooth B_0-algebra $A_0 \subseteq A_{q_A}$ with the property

$$U \subset (A_0)_{q_0}^\times, \tag{17}$$

where $q_0 := A_0 \cap q_A$. The requirement (15) implies $\operatorname{Frac} B_0 = \mathcal{E}_0$ hence $\operatorname{Frac} A_0 = \operatorname{Frac} A = E.\mathcal{E}_0$ by Proposition 3.2.

We next consider the B-algebra $C := A_0.B \subseteq \mathcal{O}_\mathcal{P}$: by Proposition 3.3 it is standard-smooth, consequently C is a smooth $\mathcal{O}_\mathcal{K}$-algebra due to (13). Furthermore we have $\operatorname{Frac} C = E.\mathcal{E}_0.\mathcal{E}_1$ and since $\mathcal{E}_0$ and $E_0.\mathcal{E}_1$ are finite Galois extensions of E_0 , so is $\mathcal{E}_0.\mathcal{E}_1$.

Let $q_C := \mathcal{M}_\mathcal{P} \cap C$, the localization C_{q_C} is a local extension of the ring $(A_0)_{q_0}$, hence $U \subset C_{q_C}^\times$ by (17). Similarly C_{q_C} is a local extension of B_{q_B} so that (16) yields

$$\forall z' \in Z' : \exists u' \in C_{q_C}^\times, z'' \in \mathcal{O}_\mathcal{K} : \quad z' = u'z''.$$

Combined with (11) this shows that every $z \in Z$ can be factored in the form

$$z = uu'z''$$

with $u, u' \in C_{q_C}^\times$ and $z'' \in \mathcal{O}_\mathcal{K}$.

Altogether we have shown that the $\mathcal{O}_\mathcal{K}$-model $\operatorname{Spec} C$ of $E.\mathcal{E}_0.\mathcal{E}_1|\mathcal{K}$ fullfills the requirements stated in the assertion of Theorem 1.1. $\square$

Uniformization after an extension within a separably tame field

The proof is similar to the one in the Galois case. We therefore put the focus on the differences between the two.

We proceed by induction on the transcendence degree $n := \operatorname{trdeg} E|K$ and start with the case $n = 1$. Let K' be the algebraic closure of K within E^{st}. Since by assumption vE/vK is torsion and $EP|KP$ is algebraic Proposition 2.16 implies that $(K', \mathcal{P})$ is a separably tame field and that the extension $(E^{\text{st}}|K', \mathcal{P})$ and hence also its subextension $(E.K'|K', \mathcal{P})$ are immediate. Proposition 3.10 now yields that $\mathcal{P}|_{E.K'}$ is strongly smoothly $\mathcal{O}_{K'}$-uniformizable.

We define the finite sets $U \subset \mathcal{O}_{E.K'}^{\times}$, $Z' \subset \mathcal{O}_{K'}$ and $Z_g \subset \mathcal{O}_P$ as in the proof of the Galois case. An application of Proposition 3.4 (2) yields the existence of a finite extension $\mathcal{K}|K$ within K' such that:

- $(\mathcal{P}|_{\mathcal{E}}, U \cup Z' \cup Z_g)$ is smoothly $\mathcal{O}_{\mathcal{K}}$-uniformizable, where $\mathcal{E} := E.\mathcal{K}$,
- $\mathcal{K}$ is algebraically closed in $\mathcal{E}$.

The assertions of the theorem in the case $n = 1$ now follow as in the Galois case.

Let us now assume that $n > 1$ and that the assertion is true for transcendence degrees smaller than n. We take an arbitrary subextension $E_0|K$ of $E|K$ of transcendence degree $n-1$ such that $E|E_0$ is separable.

As in the Galois case we deduce from what we have shown in the case $n = 1$ the existence of a finite extension $\mathcal{E}_0|E_0$ within E^{st} such that $E.\mathcal{E}_0|\mathcal{E}_0$ possesses an affine $\mathcal{O}_{\mathcal{E}_0}$-model $\operatorname{Spec} A$, $A \subset \mathcal{O}_{\mathcal{P}}$, with the properties

- $A|\mathcal{O}_{\mathcal{E}_0}$ is standard-smooth,
- $\forall z \in Z:\ \exists u \in A_{q_A}^{\times}, z' \in \mathcal{O}_{\mathcal{E}_0}:\ \ z = uz',$ (18)

where $q_A := A \cap \mathcal{M}_{\mathcal{P}}$. Let U and Z' be the finite sets of elements u and z' appearing in the factorizations (18).

Again choose a finitely generated $\mathcal{O}_K$-algebra

$$S_0 = \mathcal{O}_K[x_1, \ldots, x_r] \subseteq \mathcal{O}_{\mathcal{E}_0} \tag{19}$$

as described in Proposition 3.2.

Let $E_1|K$ be a subextension of $E_0|K$ such that $E_0|E_1$ is separable and of transcendence degree 1. Since the extension $E^{\mathrm{st}}|E$ is assumed to be separable-algebraic, the extension $\mathcal{E}_0|E_0$ is separable too. Hence $\mathcal{E}_0|E_1$ is a separable extension of transcendence degree 1.

Let $\mathcal{E}_0^{\mathrm{st}}$ be the separable closure of $\mathcal{E}_0$ in E^{st}. Since $E^{\mathrm{st}}\mathcal{P}|\mathcal{E}_0^{\mathrm{st}}\mathcal{P}$ is an algebraic extension, Lemma 2.16 shows that $\mathcal{E}_0^{\mathrm{st}}$ is a separably tame field.

We apply the induction hypothesis to the valued function field $(\mathcal{E}_0|K, \mathcal{P})$, the subfunction field $E_1|K$ and the separably tame extension field $\mathcal{E}_0^{\mathrm{st}} \subseteq \mathcal{E}_0^{\mathrm{sep}}$: we obtain a finite extension $\mathcal{E}_1|E_1$ within $\mathcal{E}_0^{\mathrm{st}}$ and a finite extension $\mathcal{K}|K$ within $\mathcal{E}_0.\mathcal{E}_1 \subseteq \mathcal{E}_0^{\mathrm{st}}$ such that $\mathcal{E}_0.\mathcal{E}_1|\mathcal{K}$ possesses an affine $\mathcal{O}_{\mathcal{K}}$-model $\operatorname{Spec} B$, $B \subset \mathcal{O}_{\mathcal{P}}$, with the following properties:

- $B|\mathcal{O}_{\mathcal{K}}$ is smooth, (20)
- $\{x_1, \ldots, x_r\} \subset B$ (see (19)),
- B contains a finite set of generators of $\mathcal{E}_0|K$,
- $\forall z' \in Z':\ \exists u' \in B_{q_B}^{\times}, z'' \in \mathcal{O}_{\mathcal{K}}:\ \ z' = u'z'',$ (21)

where $q_B := \mathcal{M}_{\mathcal{P}} \cap B$.

Next a standard-smooth B-algebra $C \subset \mathcal{O}_{\mathcal{P}}$ is constructed as in the Galois case. It satisfies $\operatorname{Frac} C = E.\mathcal{E}_0.\mathcal{E}_1$ and due to (20) yields a smooth $\mathcal{O}_{\mathcal{K}}$-model X of the function field $E.\mathcal{E}_0.\mathcal{E}_1|\mathcal{K}$.

Using (18) and (21) as in the Galois case the required factorization of the elements $z \in Z$ in the local ring C_{q_C}, $q_C := \mathcal{M}_{\mathcal{P}} \cap C$, can be verified. □

4.2 The proofs of the Theorems 1.2 and 1.4

We start by proving the existence of an intermediate field F_0 of $F|K$ as required in both Theorem 1.2 and Theorem 1.4. We therefore do not assume that the place P be trivial on K: by Proposition 2.3 we can choose a separating transcendence basis of $F|K$, which contains elements $x_1, \ldots, x_\rho, y_1, \ldots, y_\tau$ such that $\{vx_1 + vK, \ldots, vx_\rho + vK\}$ is a maximal set of rationally independent elements in vF/vK, and $\{y_1P, \ldots, y_\tau P\}$ forms a transcendence basis of $FP|KP$; let $F_0 := K(x_1, \ldots, x_\rho, y_1, \ldots, y_\tau)$. The extension $F|F_0$ then is separable, vF/vF_0 is a torsion group and $FP|F_0P$ is algebraic. Moreover $P|_{F_0}$ is an Abhyankar place by construction.

Proof of Theorem 1.2

Let F_0 be an arbitrary intermediate field of $F|K$ with the properties required in Theorem 1.2. Let $\mathcal{P}$ be an extension of P to the separable closure F^{sep} of F and let v be a valuation associated to $\mathcal{P}$.

By Theorem 1.1 there exists a finite extension $\mathcal{F}|F$ within F^{sep} and a finite extension $\mathcal{F}_0|F_0$ within $\mathcal{F}$ such that the function field $\mathcal{F}|\mathcal{F}_0$ possesses an affine $\mathcal{O}_{\mathcal{F}_0}$-model $\operatorname{Spec} A$ with the following properties:

- A is smooth at $q_A := A \cap \mathcal{M}_\mathcal{P}$,
- $\forall z \in Z : \exists u \in A_{q_A}^\times, z' \in \mathcal{O}_{\mathcal{F}_0} : \quad z = uz'$. (22)

Let $U \subset A_{q_A}^\times$ and $Z' \subset \mathcal{O}_{\mathcal{F}_0}$ be the finite sets of elements appearing in these factorizations.

Next we choose a finitely generated K-algebra

$$S_0 = K[x_1, \ldots, x_r] \subseteq \mathcal{O}_{\mathcal{F}_0} \tag{23}$$

according to Proposition 3.2 applied to the $\mathcal{O}_{\mathcal{F}_0}$-algebra A and the finite set U.

By the choice of F_0 the place $\mathcal{P}|_{\mathcal{F}_0}$ is an Abhyankar place of $\mathcal{F}_0|K$. In particular the extension $\mathcal{F}_0\mathcal{P}|K$ is finitely generated. Thus there exists a finite purely inseparable extension $\mathcal{K}|K$ such that the extension $\mathcal{F}_0\mathcal{P}.\mathcal{K}|\mathcal{K}$ is separable. Consequently, replacing K by $\mathcal{K}$, F_0 by $F_0.\mathcal{K}$ and F by $F.\mathcal{K}$ we can assume that already the residue field extension $\mathcal{F}_0\mathcal{P}|K$ is separable.

We can thus apply Theorem 1.1 of [K–K] to the valued function field $(\mathcal{F}_0|K, \mathcal{P})$: there exists an affine, smooth K-model $X_0 = \operatorname{Spec} B$ of $\mathcal{F}_0|K$, $B \subset \mathcal{O}_{\mathcal{F}_0}$, and a regular parameter system $(a_1, \ldots, a_d)$ of B_{q_B}, $q_B := \mathcal{M}_{\mathcal{F}_0} \cap B$, with the properties:

- $Z' \cup \{x_1, \ldots, x_r\} \subset B$ (see (23)), (24)
- every $z' \in Z'$ is a B_{q_B}-monomial in $\{a_1, \ldots, a_d\}$, (25)
- $\dim B_{q_B} = \dim(v\mathcal{F}_0 \otimes \mathbb{Q})$. (26)

Property (24) implies $S_0 \subseteq B$, therefore due to Proposition 3.2 and since B is normal, there exists a finitely presented B-algebra $C \subseteq A_{q_A}$ with the properties:

- C is smooth at $q_C := C \cap \mathcal{M}_\mathcal{P}$ and $\operatorname{Frac} C = \mathcal{F}$,
- $U \subset C_{q_C}$. (27)

Note that since $A_{q_A}|C_{q_C}$ is a local extension (27) and the definition of U yield $U \subset C_{q_C}^\times$. Consequently (22) and (25) show that every $z \in Z$ is a C_{q_C}-monomial in $\{a_1, \ldots, a_d\}$.

Since the extension $C|B$ is smooth the local extension $C_{q_C}|B_{q_B}$ is flat, hence $\{a_1, \ldots, a_d\}$ is part of a regular parameter system of C_{q_C}, [M], Thm. 23.7. Moreover using (26) we get

$$\dim C_{q_C} \geq \dim B_{q_B} = \dim(v\mathcal{F}_0 \otimes \mathbb{Q}) = \dim(vF \otimes \mathbb{Q}),$$

where the last equality holds because $v\mathcal{F}_0/vF_0$ and vF/vF_0 are torsion groups.

Altogether we have shown that the smooth K-morphism $f : \operatorname{Spec} C =: X \to X_0$ induced by the extension $C|B$ fullfills the requirements stated in the assertions of Theorem 1.2. □

Proof of Theorem 1.4

The proof is very similar to that of Theorem 1.2. Using the same notation we therefore only point out the differences between the two.

- In (23) S_0 is choosen to be a finitely generated R-algebra $R[x_1, \ldots, x_r] \subseteq \mathcal{O}_{\mathcal{F}_0}$.
- Theorem 1.2 of [K–K] is used to obtain an affine R-model $X_0 = \operatorname{Spec} B$ of the function field $\mathcal{F}_0|K$ that is regular at the center q_B of $\mathcal{P}$ on X_0. The requirements for an application of this theorem are satisfied by assumption except for the separability of $\mathcal{F}_0\mathcal{P}|KP$ that follows from the assumed perfectness of KP.
- The dimension formula (26) has to be replaced with

$$\dim B_{q_B} = \begin{cases} \dim(v\mathcal{F}_0/vK \otimes \mathbb{Q}) + 1 & \text{if } \dim R = 1 \text{ or } \operatorname{trdeg}(KP|R/M) > 0 \\ \dim(v\mathcal{F}_0/vK \otimes \mathbb{Q}) + 2 & \text{in the remaining cases} \end{cases}$$

- Note that B is not necessarily normal so that in the construction of the B-algebra C we have to add an intermediate step: using Proposition 3.2 and (24) we obtain a finitely presented B_{q_B}-algebra $C' \subseteq A_{q_A}$ that is smooth at $q_{C'} = \mathcal{M}_P \cap C'$ and satisfies $U \subset C'_{q_{C'}}$. For $C' = B_{q_B}[x_1, \ldots, x_r]$ we then set $C := B[x_1, \ldots, x_r]$ and get $C_{q_B} = C'$ and $C_{q_C} = C'_{q_{C'}}$. The smoothness of $C'|B_{q_B}$ at $q_{C'}$ and the regularity of B_{q_B} imply the regularity of C_{q_C}

Altogether it is shown that the R-morphism $f : \operatorname{Spec} C =: X \to X_0 := \operatorname{Spec} B$ induced by the extension $C|B$ fullfills the requirements stated in the assertions of Theorem 1.4. □

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