On finite inertial extensions of a valued field

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Let (K, v) be a valued field, and (L, v) a finite extension of (K, v) within the absolute inertia field of (K, v). Let A denote the valuation ring of (K, v) and B the valuation ring of (L, v).

Proposition. *There is $\eta \in B$ such that*

(1) *$L = K(\eta)$, the (monic) minimum polynomial $h(X)$ of η over K lies in $A[X]$, and η and $h'(\eta)$ are units of B;*
(2) *for each finite $S \subseteq B$ there is $u \in A[\eta]$ such that u is a unit of B and $S \subseteq A[\eta, 1/u]$;*
(3) *if (K, v) has finite rank, then there is $u \in A[\eta]$ such that u is a unit of B and $B = A[\eta, 1/u]$.*

In particular, η is integral over A and is a henselian element over (K, v). The proposition implies (a strong form) of Kuhlmann's Conjecture 1 in [K]. This result seems to be slightly different from other (strong) solutions given in notes by Roquette (May 2000), and Knaf & Kuhlmann (Oct. 2002), so is perhaps worth recording. Roquette shows that each element of B lies in a subring of B of the form $A[\eta, r, s]$ with r, s henselian over (K, v) (and η as in the proposition). Applied to the element $1/u$ in (3), this yields:

Corollary. *If (K, v) has finite rank, then $B = A[\eta, r, s]$ with η the (henselian) element of the proposition, and r, s henselian over (K, v).*

Proof of the Proposition. Let η be the element constructed in the proof of lemma 4 of [K], whose notations we use in what follows. We also use some lemmas from [L]. It is shown in [K] that (1) holds. Let $\mathfrak{m}$ be the maximal ideal of A, let A^* be the integral closure of A in L, and let $\mathfrak{m}^*$ be the maximal ideal of A^* such that $B = A^*_{\mathfrak{m}^*}$. Let $G(X) \in A[X]$ be a monic polynomial with image $g(X)$ in $Kv[X]$ where $Kv = A/\mathfrak{m}$. By lemma 4 in [K] and lemma 12.5.7 in [L] the ring $A[\eta] \subseteq A^*$ has exactly two maximal ideals, namely $(\mathfrak{m}, \eta)A[\eta]$ and $\mathfrak{n} := (\mathfrak{m}, G(\eta))A[\eta]$. Note that $(\mathfrak{m}, \eta)A[\eta]$ is not contained in $\mathfrak{m}^*$ since η is a unit in B, and that $\mathfrak{n}$ is contained in $\mathfrak{m}^*$, so $\mathfrak{m}^* \cap A[\eta] = \mathfrak{n}$. Thus, using Lemma 12.5.17 in [L] for the first inclusion, we have

$$A^* \subseteq \frac{1}{h'(\eta)} A[\eta] \subseteq A[\eta]_{\mathfrak{n}} \subseteq A^*_{\mathfrak{m}^*} = B.$$

Hence by Lemma 12.5.16 in [L] we have

$$B = A[\eta]_{\mathfrak{n}}.$$

This yields (2) by using common denominators.

Assume now that (K,v) has finite rank. Then A has only finitely many prime ideals, so A^* has only finitely many prime ideals, and hence $A[\eta]$ has only finitely many prime ideals. So we can take $u_1 \in A[\eta] \setminus \mathfrak{n}$ such that u_1 lies in each prime ideal of $A[\eta]$ not contained in $\mathfrak{n}$. Put $u := u_1 h'(\eta)$. Then $A[\eta]_u$ is a local ring, because the prime ideals of $A[\eta]_u$ are exactly the $\mathfrak{p}A[\eta]_u$ with $\mathfrak{p}$ a prime ideal of $A[\eta]$ that does not contain u, and each such $\mathfrak{p}$ is contained in $\mathfrak{n}$. From

$$A^* \subseteq \frac{1}{h'(\eta)} A[\eta] \subseteq A[\eta]_u \subseteq A^*_{\mathfrak{m}^*} = B$$

and Lemma 12.5.16 in [L] we obtain $B = A[\eta]_u = A[\eta, 1/u]$ as requested in (3). This finishes the proof.

Remarks

If (K,v) has rank 1, then $(\mathfrak{m},\eta)A[\eta]$ is the only prime ideal of $A[\eta]$ not contained in $\mathfrak{n}$, so we can take $u_1 := \eta$ in that case. Thus $B = A[\eta, 1/(\eta h'(\eta))]$ in the rank 1 case.

Parts (1) and (2) of the proposition go through in a more general setting: Let A be a local domain integrally closed in its field of fractions K, let $I(A)$ be the integral closure of A in K^{sep}, and choose a maximal ideal $\mathfrak{p}$ of $I(A)$. (In the case that A is the valuation ring of a valuation v on K, this is equivalent to choosing an extension of v to a valuation on K^{sep}.) Let

$$G^i := \{\sigma \in Gal(K^{sep}|K) : x - \sigma x \in \mathfrak{p} \text{ for all } x \in I(A)\}$$

be the inertia group of $\mathfrak{p}$, let $L|K$ be a finite extension contained in the fixed field of G^i, let $A^* := I(A) \cap L$ be the integral closure of A in L, put $\mathfrak{m}^* := \mathfrak{p} \cap A^*$ and $B := A^*_{\mathfrak{m}^*}$, so B is a local domain dominating A and integrally closed in its fraction field L. Then lemma 4 in [K] and its proof go through (modulo obvious changes), and yield an element $\eta \in A^*$ with property (1) in the proposition. The proof of the proposition shows that (2) holds as well.

[K] F.-V. Kuhlmann, A conjecture about finite extensions within the absolute inertia field of a valued field (March 2000)

[L] L. van den Dries, Section 12.5, pp. 252–267 in “Homological Questions in Local Algebra”, London Math. Soc. Lecture Note Series 145