REAL HOLOMORPHY RINGS AND THE COMPLETE REAL SPECTRUM

D. GONDARD, M. MARSHALL

Abstract. The complete real spectrum of a commutative ring A with 1 is introduced. Points of the complete real spectrum $\operatorname{Sper}^c A$ are triples $\alpha = (\mathfrak{p}, v, P)$, where $\mathfrak{p}$ is a real prime of A, v is a real valuation of the field $k(\mathfrak{p}) := \mathrm{qf}(A/\mathfrak{p})$ and P is an ordering of the residue field of v. $\operatorname{Sper}^c A$ is shown to have the structure of a spectral space in the sense of Hochster [5]. The specialization relation on $\operatorname{Sper}^c A$ is considered. Special attention is paid to the case where the ring A in question is a real holomorphy ring.

Résumé. Nous introduisons la notion de spectre réel complet d'un anneau A commutatif avec unité. Les points de ce spectre réel complet, noté $\operatorname{Sper}^c A$, sont les triplets $\alpha = (\mathfrak{p}, v, P)$, où $\mathfrak{p}$ est un idéal premier de A, v une valuation réelle du corps $k(\mathfrak{p}) := \mathrm{qf}(A/\mathfrak{p})$ et P un ordre du corps résiduel de v. Nous montrons que $\operatorname{Sper}^c A$ a une structure d'espace spectral au sens de Hochster [5]. On considère aussi la relation de spécialisation sur $\operatorname{Sper}^c A$. Nous nous intéressons particulièrement au cas où l'anneau A est un anneau d'holomorphie réel.

The prime spectrum $\operatorname{Spec} A$ of a ring A (commutative with 1) is a basic object in algebraic geometry. In real algebraic geometry, where one deals with inequalities as well as equations, the prime spectrum, while still an important object, does not contain sufficient information by itself. The prime spectrum of a formally real field K, for example, consists of just a single point, whereas to understand the various 'real' aspects of such a field one needs to consider also the orderings, i.e., the subsets P of K satisfying:

$$P + P \subseteq P,\ PP \subseteq P,\ P \cup -P = K,\ P \cap -P = \{0\}.$$

Similarly, to understand the 'real' aspects of a ring A, one needs to study, not just the primes, but also the pairs $(\mathfrak{p}, P)$ where $\mathfrak{p}$ is a prime

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ideal of A with formally real residue field $k(\mathfrak{p}) := \mathrm{qf}(A/\mathfrak{p})$ and P is an ordering on $k(\mathfrak{p})$. The real spectrum $\operatorname{Sper} A$ consists of all such pairs. For a formally real field K, $\operatorname{Sper} K$ is just the set of all orderings of K. A natural topology is defined, as in the case of the prime spectrum, making $\operatorname{Sper} K$ into a Boolean space and $\operatorname{Sper} A$ into a spectral space in the sense of [5]; see [4] [9] for details.

The real spectrum of a ring A comes equipped with a certain monoid of functions A_{rr}. The pair $(\operatorname{Sper} A, A_{rr})$ is referred to as the space of signs of A. We recall briefly the terminology. See [1], [10] and [12] for more details. Each $a \in A$ determines a sign function $\widetilde{a}$: Sper $A \to \{-1, 0, 1\}$ defined as follows:

$$\widetilde{a}(\mathfrak{p}, P) := \begin{cases} 1 & \text{if } a + \mathfrak{p} > 0 \text{ at } P, \\ 0 & \text{if } a \in \mathfrak{p}, \\ -1 & \text{if } a + \mathfrak{p} < 0 \text{ at } P. \end{cases}$$

Here, $a + \mathfrak{p}$ denotes the canonical image of a in $A/\mathfrak{p} \subseteq k(\mathfrak{p})$. The set of sign functions $\widetilde{a}$, $a \in A$, forms a monoid under pointwise multiplication: $\widetilde{a} \cdot \widetilde{b} = \widetilde{ab}$. The *space of signs* of A is the pair $(\operatorname{Sper} A, A_{rr})$ where $A_{rr} = \{\widetilde{a} : a \in A\}$. If K is a field, K_{rr} decomposes as $K_{rr}^* \cup \{\widetilde{0}\}$ where K_{rr}^* is a group isomorphic to the group $K^*/\sum K^{2*}$. The *space of orderings* of K is the pair $(\operatorname{Sper} K, K_{rr}^*)$.

At the same time, there are indications, e.g., in rigid geometry, and in resolution of singularities via Zariski's method, that the prime spectrum is not large enough, even in the classical context: It does not say enough about the valuation theory of the residue fields. In [6] [7] [13] [14] the valuation spectrum $\operatorname{Spv} A$ is introduced. $\operatorname{Spv} A$ consists of pairs $(\mathfrak{p}, v)$ where $\mathfrak{p}$ is a prime ideal of A and v is a valuation on $k(\mathfrak{p})$.

Valuations also play an important role in the real case, and important use is also made of the orderings on the residue field $B_v/\mathfrak{m}_v$ of the valuation v, where B_v denotes the valuation ring of v, and $\mathfrak{m}_v$ its maximal ideal. One encounters these objects, for example, in understanding specialization in $\operatorname{Sper} A$, in connection to minimal generation of constructible sets in $\operatorname{Sper} A$ and in the (reduced) theory of quadratic forms over formally real fields. This suggests that in the real case one should be considering triples $(\mathfrak{p}, v, P)$ where $\mathfrak{p}$ is a real prime, v is a real valuation on $k(\mathfrak{p})$ and P is an ordering on the residue field $B_v/\mathfrak{m}_v$. (Compare to [13] [14].) We refer to these triples as *residual orderings* of A. In the present paper we examine the set consisting of these objects, with its natural topology, which we call the complete real spectrum of

A. We also define an associated complete space of signs. The motivation for this study comes from properties of the real holomorphy ring of a field.

1. Real holomorphy rings

The real holomorphy ring of a formally real field K is

$$H_K = \{a \in K : \exists \text{ an integer } n \geq 1 \text{ such that } -n \leq a \leq n \text{ on } \operatorname{Sper} K\}.$$

$H = H_K$ can be described in various other ways [2] [16], and carries lots of information about the field K. The space of orderings, the real valuations, the spaces of orderings of the residue fields, the space of real places of K, can all be "read off" from the space of signs of H.

In more detail: Every prime ideal of H is real. If $\mathfrak{p}$ is a prime ideal of H then the local ring $H_\mathfrak{p}$ is a real valuation ring of K. Conversely, every real valuation ring of K is of this form, for some unique $\mathfrak{p}$. The residue field of $H_\mathfrak{p}$ is $k(\mathfrak{p}) :=\ \mathrm{qf}(H/\mathfrak{p})$. The space of orderings of this residue field is identified with the space of support $\mathfrak{p}$ orderings of H. Consequently, every support $\mathfrak{p}$ ordering of H generalizes, via the Baer-Krull Theorem, to a support $\{0\}$ ordering of H. $\operatorname{Sper} K$ is identified with $\operatorname{MinSper} H$. The space of real places of K (places into the field of real numbers) is identified with $\operatorname{MaxSper} H$.

For any prime $\mathfrak{q}$ of H, the real holomorphy ring of $k(\mathfrak{q})$ is precisely $H_{k(\mathfrak{q})} = H/\mathfrak{q}$. Consequently, everything said above works equally well with H replaced by $H/\mathfrak{q}$. In particular, if $\mathfrak{p}$, $\mathfrak{q}$ are any prime ideals of H with $\mathfrak{q} \subseteq \mathfrak{p}$, then support $\mathfrak{p}$ orderings of H generalize to support $\mathfrak{q}$ orderings of H via the Baer-Krull theorem applied to the valuation ring $(H/\mathfrak{q})_{(\mathfrak{p}/\mathfrak{q})}$ in the field $k(\mathfrak{q})$ with residue field equal to $k(\mathfrak{p})$.

This suggests that to have more of this structure available, one should perhaps be studying spaces of signs of real holomorphy rings of fields rather than spaces of orderings of fields.

One can also argue a step further, thinking of rings instead of fields. All rings considered here are commutative with 1. The space of signs of a ring A carries information about the spaces of orderings of each of the residue fields $k(\mathfrak{p})$, $\mathfrak{p}$ a real prime of A, but typically does not carry enough information concerning the real holomorphy rings of these residue fields.

The real holomorphy ring H_A of an arbitrary commutative ring A [3] [17] has been introduced and has proved to be a useful object, e.g., in studying the Moment Problem from functional analysis. Actually, there are two versions in the ring case, the *geometric* version:

$$H_A = \{a \in A : \exists \text{ an integer } n \geq 1 \text{ such that } -n \leq a \leq n \text{ on } \operatorname{Sper} A\},$$

and the generally smaller *arithmetic* version:

$$H'_A = \{a \in A : \exists \text{ an integer } n \geq 1 \text{ such that } n \pm a \in \sum A^2\}.$$

If A is a finitely generated $\mathbb{R}$-algebra with $\operatorname{Hom}(A, \mathbb{R})$ compact then, by Schmüdgen's Theorem [15], both versions coincide. The same is true if the elements of $1 + \sum A^2$ are invertible in A (not such a drastic assumption from the point of view of what we are trying to do here). But even with special assumptions of this sort, H_A is not big enough, in general, to carry good information concerning the valuations on the residue fields of A.

The image of $\operatorname{Sper} A$ in $\operatorname{Sper} H_A$ under the restriction map $\operatorname{Sper} A \to \operatorname{Sper} H_A$ is often not dense in $\operatorname{Sper} H_A$. Consequently, $H_{(H_A)}$ is often strictly smaller than H_A. Define H^n_A recursively by $H^n_A = H_{(H^{n-1}_A)}$ and set $H^\infty_A = \cap_{n\geq 1} H^n_A$. H^∞_A is the largest subring B of A satisfying $H_B = B$. H'_A is better behaved in this regard, since $H'_{(H'_A)} = H_{(H'_A)} = H'_A$ (so, in particular, $H'_A \subseteq H^\infty_A$), but H^∞_A and H'_A are generally not equal. In [11] an example of an $\mathbb{R}$-algebra A is given where $H^\infty_A = A$, $H'_A = \mathbb{R}$. If A is an $\mathbb{R}$-algebra of finite transcendence degree d (in particular, if A is finitely generated), then $H^\infty_A = H^d_A = H'_A$. The first equality is proved in [3]. The second (the so-called Monnier Conjecture) is proved in [17].

We say a ring A is a *real holomorphy ring* if the following equivalent conditions hold:

1) $H_A = A$.

2) $A/\mathfrak{p} \subseteq H_{k(\mathfrak{p})}$ holds for each real prime $\mathfrak{p}$ of A.

3) $A/\mathfrak{p} \subseteq B_v$ holds for each real prime $\mathfrak{p}$ of A and each real valuation v of $k(\mathfrak{p})$.

We say a real holomorphy ring A is *complete* if, in addition,

4) $A/\mathfrak{p} = H_{k(\mathfrak{p})}$ holds for each real prime $\mathfrak{p}$ of A.

Remark 1.1. 1) The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial. The implication (3) $\Rightarrow$ (1) uses the compactness of the real spectrum.

2) In [1] real holomorphy rings are referred to as totally Archimedean rings.

3) If $A/\mathfrak{p} = H_{k(\mathfrak{p})}$ then, for any prime $\mathfrak{q}$ lying over $\mathfrak{p}$, $A/\mathfrak{q} = H_{k(\mathfrak{q})}$. Consequently, to check that a real holomorphy ring is complete, it suffices to check that $A/\mathfrak{p} = H_{k(\mathfrak{p})}$ for each minimal real prime of A.

Example 1.2. 1) Real holomorphy rings: For any ring A, $B = H^\infty_A$ and $C = H'_A$ are real holomorphy rings. For any ring A such that each element of $1 + \sum A^2$ is a unit in A, H_A is a holomorphy ring (since $H_A = H'_A$ in this case).

2) Complete real holomorphy rings: The real holomorphy ring of a formally real field and the ring of continuous functions $\mathrm{Cont}(X, \mathbb{R})$ for any compact space X. The next theorem provides additional examples.

Theorem 1.3. *Suppose A is real (i.e., the real nilradical of A is zero).*

1) *If A is zero dimensional then $H = H_A$ is a real holomorphy ring which is complete.*
2) *If the space of minimal primes of A is compact and $S^{-1}A$ denotes the complete ring of quotients of A, then $H = H_{S^{-1}A}$ is a real holomorphy ring which is complete.*

Remark 1.4. 1) The hypotheses of (1) and (2) of Theorem 1.3 are pretty restrictive. Still, there are many examples.

2) A finite space is compact. Consequently (2) applies when A has only finitely many minimal primes, e.g., when A is Noetherian. In this case,

$$S^{-1}A = k(\mathfrak{p}_1) \times \cdots \times k(\mathfrak{p}_n)$$

where the $\mathfrak{p}_i$ are the minimal primes of A, and

$$H = H_{k(\mathfrak{p}_1)} \times \cdots \times H_{k(\mathfrak{p}_n)}.$$

3) In general, for complete real holomorphy rings, it would seem that the space of minimal real primes can be pretty complicated.

Proof. Since A is real, the minimal primes of A are all real. Let $\mathfrak{p}_\lambda$, $\lambda \in \Lambda$, be the set of minimal primes of A. By definition, $S = A \backslash \cup_{\lambda\in\Lambda}\mathfrak{p}_\lambda$. Suppose $\mathfrak{q}$ is a prime ideal of A with $\mathfrak{q} \subseteq \cup_{\lambda\in\Lambda}\mathfrak{p}_\lambda$. If $\mathfrak{q} \not\subseteq \mathfrak{p}_\lambda$ for all λ, then there exists $a_\lambda \in \mathfrak{q}$, $a_\lambda \notin \mathfrak{p}_\lambda$, i.e. the open sets $D(a_\lambda)$ cover the space of minimal primes, where $D(a) := \{\mathfrak{p} \in \operatorname{Spec} A : a \notin \mathfrak{p}\}$. By compactness, we have finitely many elements a_i of A with $a_i \in \mathfrak{q}$ and, for each λ, $a_i \notin \mathfrak{p}_\lambda$ for some i. Then $\sum a_i^2 \in \mathfrak{q}$, $\sum a_i^2 \notin \mathfrak{p}_\lambda$. This contradicts our assumption. Thus the hypothesis of (2) implies that the only primes of $S^{-1}A$ are those coming from the $\mathfrak{p}_\lambda$, so $S^{-1}A$ is zero dimensional. Thus it only remains to prove (1).

Thus we assume now that A is zero dimensional. Note that $S =$ the group of units of A in this case, so $S^{-1}A = A$. Since all the primes of A are real, $1 + \sum A^2 \subseteq S$. It follows that H is a real holomorphy ring. Since A is zero dimensional, $\operatorname{Spec} A$ is a Boolean space and A is the ring of global sections of a sheaf on $\operatorname{Spec} A$ with the $A/\mathfrak{p} = k(\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Spec} A$, as the stalks. If $A = A_1 \times A_2$ then clearly $H = H_1 \times H_2$, where $H_i = H_{A_i}$. Consequently, H is also the ring of global sections of a sheaf on $\operatorname{Spec} A$ with the $H/(H \cap \mathfrak{p})$ as stalks. If $\mathfrak{q}$ is a prime ideal of H, a compactness argument shows that $H \cap \mathfrak{p} \subseteq \mathfrak{q}$ for some prime $\mathfrak{p}$ of A. (Otherwise we would have finitely many $b_i \in H$ with $b_i \notin \mathfrak{q}$ but for each prime $\mathfrak{p}$ of A, $b_i \in \mathfrak{p}$ for some i. Then $\prod b_i \notin \mathfrak{q}$, but $\prod b_i$ lies in all primes of A so is zero.) Note that $k(H \cap \mathfrak{p}) = k(\mathfrak{p})$. (Use the identity $a/b = \frac{a}{1+a^2+b^2} / \frac{b}{1+a^2+b^2}$.)

We are reduced to showing that $H/(H \cap \mathfrak{p}) = H_{k(\mathfrak{p})}$. One inclusion is clear. For the other, suppose $a, b \in A$, $b \notin \mathfrak{p}$, and $\frac{a+\mathfrak{p}}{b+\mathfrak{p}} \in k(\mathfrak{p})$ is bounded. By Lemma 2.1 below we can assume $nb^2 - a^2 \in \sum A^2$ for some integer $n \geq 1$. $\mathfrak{p}$ lies in the clopen set $C := D(b)$ in $\operatorname{Spec} A$. Let $D = Z(b) := \{\mathfrak{q} \in \operatorname{Spec} A : b \in \mathfrak{q}\}$, the complement of C in $\operatorname{Spec} A$. The decomposition $\operatorname{Spec} A = C \cup D$ allows us to produce an element $e \in H$ which agrees with a/b at $\mathfrak{p}$, namely $e = a/b$ on C, $e = 0$ on D. □

What are the special properties of the space of signs of a complete real holomorphy ring A which distinguishes it from the space of signs of an arbitrary ring? Part of the answer is that, for these rings, specialization is very well-behaved. The following result is clear from properties of the real holomorphy ring of a field noted at the beginning of the section.

Theorem 1.5. *Suppose A is a complete real holomorphy ring and $\mathfrak{p}$, $\mathfrak{q}$ are real primes of A with $\mathfrak{p} \subseteq \mathfrak{q}$. Then:*

1) *$(A/\mathfrak{p})_{(\mathfrak{q}/\mathfrak{p})}$ is a real valuation ring of $k(\mathfrak{p})$.*
2) *The orderings with support $\mathfrak{p}$ which specialize to an ordering with support $\mathfrak{q}$ are those compatible with the valuation ring $(A/\mathfrak{p})_{(\mathfrak{q}/\mathfrak{p})}$. The specialization of any such ordering is determined by the Baer-Krull Theorem.*
3) *Every ordering with support $\mathfrak{q}$ generalizes to an ordering with support $\mathfrak{p}$. Its set of generalizations is determined by the Baer-Krull Theorem.*
4) *The real primes contained in a given real prime form a chain with respect to inclusion.*
5) *If the real radical of A is zero (so the minimal primes of A are real) then all primes of A are real.*

It is important to note, again since we are assuming that A is complete, that every real valuation v of $k(\mathfrak{p})$ arises via this process: $\mathfrak{m}_v \cap A/\mathfrak{p}$ is of the form $\mathfrak{q}/\mathfrak{p}$ for some (real) prime $\mathfrak{q}$. Consequently $B_v = (A/\mathfrak{p})_{(\mathfrak{q}/\mathfrak{p})}$, and the residue field of B_v is $k(\mathfrak{q})$.

The space of maximal specializations of orderings with support $\mathfrak{p}$ is homeomorphic to $M_{k(\mathfrak{p})}$:= the space of real places of $k(\mathfrak{p})$. If $\mathfrak{p} \subseteq \mathfrak{q}$, then $M_{k(\mathfrak{q})}$ is naturally identified with a subspace in $M_{k(\mathfrak{p})}$. $\bigcup_{\mathfrak{p}} M_{k(\mathfrak{p})}$ (modulo these identifications) is identified with $\operatorname{MaxSper} A = \operatorname{Hom}(A, \mathbb{R})$.

By the Stone-Weierstrass approximation property, it is possible to separate disjoint closed sets in the maximal real spectrum using elements of A: Given disjoint closed C, D in MaxSper A, there exists $\widetilde{a} \in A_{rr}$ such that $\widetilde{a} = 1$ on C, $\widetilde{a} = -1$ on D.

In the case $A = H_K$, the real holomorphy ring of a field K, there is a unique minimal real prime. In the general case, a complete real holomorphy ring may have a large number of minimal real primes. Is it possible to say anything about the space of minimal real primes?

It would be nice to have a list of abstract properties of the space of signs of a complete real holomorphy ring.

2. The complete real spectrum of a ring

Is it possible to think of the space of signs of A as being part of some big "super object" associated to A which takes into account all real valuations on the $k(\mathfrak{p})$ and all the orderings on the corresponding residue fields of $k(\mathfrak{p})$ into account?

It seems that, in a certain sense at least, this is in fact the case. We define a big object $\mathrm{Sper}^c A$ which we call the *complete real spectrum* of A. There are various connections between this and the valuation spectra considered in [6] [7] [13] [14]. Roughly speaking, the complete real spectrum is related to the valuation spectrum in the same way that the real spectrum is related to the prime spectrum. A is any commutative ring with 1. We define a topology on $\mathrm{Sper}^c A$ and prove that $\mathrm{Sper}^c A$, with this topology, is a spectral space. $\mathrm{Sper}^c A$ gives rise to a *complete space of signs* $(\mathrm{Sper}^c A, A^c_{rr})$, but this is not a space of signs in the usual sense. This should not be viewed as a drawback. Rather, $\mathrm{Sper}^c A$ should be viewed as a new sort of structure, interesting in its own right.

In Sect. 8.6 of [10] another sort of attempt is made to overcome shortcomings of the real spectrum of A by introducing the space of real places of A, which we denote here by M_A. By definition, M_A consists of pairs $(\mathfrak{p}, \lambda)$ where $\mathfrak{p}$ is a real prime of A and λ is a place from the residue field $k(\mathfrak{p})$ into the field of real numbers. This takes care of the real places in a satisfactory way but does not keep track of all real valuations on the $k(\mathfrak{p})$ and all the orderings on the corresponding residue fields of $k(\mathfrak{p})$. Still, the M_A construction in [10] is closely related to the complete real spectrum construction described below.

The elements of $\mathrm{Sper}^c A$, which we refer to as *residual orderings* of A, are triples $(\mathfrak{p}, v, P)$ where $\mathfrak{p}$ is a real prime of A, v is a real valuation (more precisely, an equivalence class of real valuations) on the residue field $k(\mathfrak{p})$ and P is an ordering on the residue field $B_v/\mathfrak{m}_v$ of v. Here,

$B_v \subseteq k(\mathfrak{p})$ denotes the valuation ring of v and $\mathfrak{m}_v$ its maximal ideal. Equivalently, elements of $\mathrm{Sper}^c A$ are pairs $(\mathfrak{p}, Q)$ where $\mathfrak{p}$ is a real prime of A and Q is an element of $\mathrm{Sper}\, H_{k(\mathfrak{p})}$. The pair $(\mathfrak{p}, v)$ will be referred to as the *support* of $(\mathfrak{p}, v, P)$.

There are natural maps

$$(\mathfrak{p}, v, P) \mapsto \mathfrak{p}, \quad (\mathfrak{p}, v, P) \mapsto (\mathfrak{p}, v)$$

from $\mathrm{Sper}^c A$ into $\mathrm{Spec}\, A$ (the prime spectrum of A) and from $\mathrm{Sper}^c A$ into $\mathrm{Spv}\, A$ (the valuation spectrum of A [6] [7]), and a natural map

$$(\mathfrak{p}, P) \mapsto (\mathfrak{p}, 0, P)$$

(where 0 denotes the trivial valuation on $k(\mathfrak{p})$) from $\mathrm{Sper}\, A$ into $\mathrm{Sper}^c A$. There is also the specialization map

$$(\mathfrak{p}, Q) \mapsto (\mathfrak{p}, Q')$$

from $\mathrm{Sper}^c A$ onto the space of real places $M = M_A$ defined in [10]. Here, Q' denotes the unique maximal specialization of Q in $\mathrm{Sper}\, H_{k(\mathfrak{p})}$; also see [2]. The composite map $\mathrm{Sper}\, A \to M_A$ is just the P-structure map Λ considered in [10].

Note: The complete real spectrum of a formally real field K is naturally identified with the real spectrum of its real holomorphy ring H_K. More generally, if A is real and zero dimensional, then $\mathrm{Sper}^c A$ is naturally identified with $\mathrm{Sper}\, H_A$.

Subbasic open sets in $\mathrm{Sper}^c A$ are defined using pairs of elements of A. For $(a, b) \in A \times A$, we define:

$$U(a, b) = \{(\mathfrak{p}, v, P) \in \ \mathrm{Sper}^c A : \ v(a) = v(b) \neq \infty, \ \frac{a + \mathfrak{p}}{b + \mathfrak{p}} + \mathfrak{m}_v > 0 \text{ at } P\}.$$

Here, $v(a)$ is standard shorthand notation for $v(a + \mathfrak{p})$.

For the alternate description of elements of $\mathrm{Sper}^c A$ as pairs $(\mathfrak{p}, Q)$, Q an ordering of $H_{k(\mathfrak{p})}$, it is convenient to consider the set

$$S_A := \{(a, b) \in A \times A : \exists \text{ an integer } n \geq 1 \text{ such that } nb^2 - a^2 \in \sum A^2\}.$$

For any $(a, b) \in A \times A$, $(ab, a^2 + b^2) \in S_A$ and $U(a, b) = U(ab, a^2 + b^2)$. For $(a, b) \in S_A$,

$$U(a, b) = \{(\mathfrak{p}, Q) \in \ \mathrm{Sper}^c A : b \notin \mathfrak{p}, \ \frac{a + \mathfrak{p}}{b + \mathfrak{p}} > 0 \text{ at } Q\}.$$

Note that $(a, b) \in S_A$, $b \notin \mathfrak{p} \Rightarrow \frac{a+\mathfrak{p}}{b+\mathfrak{p}} \in H_{k(\mathfrak{p})}$. For our purposes it is important to know that *every* element of $H_{k(\mathfrak{p})}$ is represented this way.

Lemma 2.1. *For each* $s \in H_{k(\mathfrak{p})}$, *there exists* $(a, b) \in S_A$ *with* $b \notin \mathfrak{p}$ *and* $\frac{a+\mathfrak{p}}{b+\mathfrak{p}} = s$.

Note: An essentially equivalent statement is: The natural surjection $A_\mathfrak{p} \to k(\mathfrak{p})$ maps $H_{A_\mathfrak{p}}$ onto $H_{k(\mathfrak{p})}$. This holds for any local ring with formally real residue field.

Proof. Choose a positive integer n so that $n - s^2 = \sum s_i^2$, $s_i \in k(\mathfrak{p})$. Choose a common denominator $b + \mathfrak{p}$ for s and the s_i. Clearing denominators yields

$$nb^2 - a^2 = \sum a_i^2 + r, \quad r \in \mathfrak{p}, \quad s = \frac{a+\mathfrak{p}}{b+\mathfrak{p}}, \quad s_i = \frac{a_i + \mathfrak{p}}{b + \mathfrak{p}}.$$

Multiplying each side by b^2, subtracting rb^2 from each side, and adding $\frac{r^2}{4n}$ to each side yields

$$n(b^2 - \frac{r}{2n})^2 - a^2b^2 = \sum a_i^2 b^2 + \frac{r^2}{4n}.$$

Take $a_1 = ab$, $b_1 = b^2 - \frac{r}{2n}$. Then $(a_1, b_1) \in S_A$ and $\frac{a_1+\mathfrak{p}}{b_1+\mathfrak{p}} = s$. □

We also use the following elementary facts:

Lemma 2.2. *Suppose $(a, b), (c, d) \in S_A$. Then $(ac, bd) \in S_A$ and $(ad+bc, bd) \in S_A$.*

Proof. Suppose $nb^2 - a^2 = s$, $md^2 - c^2 = t$, $s, t \in \sum A^2$. Then

$$mnb^2d^2 - a^2c^2 = (a^2 + s)(c^2 + t) - a^2c^2 = a^2t + c^2s + st \in \sum A^2.$$

Also

$$\begin{aligned}(1 + n)(1 + m)b^2d^2 &= b^2d^2 + nb^2d^2 + mb^2d^2 + mnb^2d^2 \\ &= b^2d^2 + (a^2 + s)d^2 + (c^2 + t)b^2 + (a^2 + s)(c^2 + t) \\ &= (ad + bc)^2 + (ac - bd)^2 + b^2t + d^2s + a^2t + c^2s + st,\end{aligned}$$

so $(1 + n)(1 + m)b^2d^2 - (ad + bc)^2 \in \sum A^2$ as required. □

Theorem 2.3. *$\mathrm{Sper}^c A$ is a spectral space.*

Proof. The method of proof is standard [9] [7]. One must show that $\mathrm{Sper}^c A$, endowed with the (finer) patch topology, is a Boolean space. One is reduced to showing that the map

$$\Psi : \ \mathrm{Sper}^c A \to \{0, 1\}^{S_A}, \ \ x \mapsto f_x, \ \text{where } f_x(a, b) := \begin{cases} 1 \text{ if } x \in U(a, b), \\ 0 \text{ otherwise.} \end{cases}$$

is injective, and that the image of Ψ is closed, where $\{0, 1\}^{S_A}$ is endowed with the product topology.

Injectivity of Ψ: Let $x = (\mathfrak{p}, Q)$. Observe that for $a \in A$,

$$f_x(a, a) = \begin{cases} 0 & \text{if } a \in \mathfrak{p}, \\ 1 & \text{if } a \notin \mathfrak{p}. \end{cases}$$

Thus $\mathfrak{p} = \{a \in A : f_x(a,a) = 0\}$. Now suppose $(a,b) \in S_A$, $b \notin \mathfrak{p}$, so $\frac{a+\mathfrak{p}}{b+\mathfrak{p}} \in H_{k(\mathfrak{p})}$. Then

$$\frac{a+\mathfrak{p}}{b+\mathfrak{p}} \in Q \quad \Leftrightarrow \quad f_x(-a,b) = 0.$$

It follows, using Lemma 2.1, that

$$Q = \{\frac{a+\mathfrak{p}}{b+\mathfrak{p}} : \ (a,b) \in S_A, \ f_x(b,b) = 1, \ f_x(-a,b) = 0\}.$$

Im(Ψ) is closed: Let $f : S_A \to \{0,1\}$ be in the closure of Im(Ψ). Thus, for each finite set of points in S_A, there exists $g \in$ Im(Ψ) agreeing with f on this finite set. Let

$$\mathfrak{p} = \{a \in A : f(a,a) = 0\}.$$

On checks that $0 \in \mathfrak{p}$, $a,b \in \mathfrak{p} \Rightarrow a+b \in \mathfrak{p}$, $a \in \mathfrak{p}$, $b \in A \Rightarrow ab \in \mathfrak{p}$, $1 \notin \mathfrak{p}$, and $ab \in \mathfrak{p} \Rightarrow a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. The argument in each case is the same. E.g., to show closure under addition, pick $a,b \in \mathfrak{p}$. Pick $g \in$ Im(Ψ) which agrees with f at (a,a), (b,b), and $(a+b,a+b)$. Thus $g(a,a) = f(a,a) = 0$, $g(b,b) = f(b,b) = 0$, so $f(a+b,a+b) = g(a+b,a+b) = 0$. This proves that $\mathfrak{p}$ is a prime ideal. Now let

$$Q = \{\frac{a+\mathfrak{p}}{b+\mathfrak{p}} : f(b,b) = 1, f(-a,b) = 0\}.$$

We show that Q is an ordering of $H_{k(\mathfrak{p})}$. The argument is similar to the above. We show closure of Q under multiplication: Pick $(a_i,b_i) \in S_A$, $b_i \notin \mathfrak{p}$, and suppose $\frac{a_i+\mathfrak{p}}{b_i+\mathfrak{p}} \in Q$, $i = 1,2$. Pick $g \in$ Im(Ψ), say $g = \Psi(\mathfrak{p}',Q')$, agreeing with f at (b_i,b_i) and $(-a_i,b_i)$, $i = 1,2$ and at (a_1a_2, b_1b_2). Then $b_i \notin \mathfrak{p}'$, $g(-a_i,b_i) = f(-a_i,b_i) = 0$ so $\frac{a_i+\mathfrak{p}}{b_i+\mathfrak{p}} \in Q'$, $i = 1,2$. Since Q' is closed under multiplication, this implies $f(-a_1a_2, b_1b_2) = g(-a_1a_2, b_1b_2) = 0$. The other properties of an ordering are checked in a similar way. Finally, one checks that $f = \Psi(\mathfrak{p},Q)$. $\square$

It is natural to mimic the space of signs construction outlined in the introduction: Consider the sign functions

$$\widetilde{(a,b)} : \mathrm{Sper}^c A \to \{-1,0,1\},$$

$(a,b) \in S_A$, defined by

$$\widetilde{(a,b)}(\mathfrak{p},Q) := \begin{cases} 1 \text{ if } \frac{a+\mathfrak{p}}{b+\mathfrak{p}} > 0 \text{ at } Q, \\ -1 \text{ if } \frac{a+\mathfrak{p}}{b+\mathfrak{p}} < 0 \text{ at } Q, \\ 0 \text{ if } \frac{a+\mathfrak{p}}{b+\mathfrak{p}} = 0 \text{ at } Q. \end{cases}$$

We leave $\widetilde{(a,b)}$ undefined at $\alpha = (\mathfrak{p}, Q)$ if $b \in \mathfrak{p}$. Note that $\widetilde{(a,b)} \cdot \widetilde{(c,d)} = \widetilde{(ac,bd)}$ at all points where both sides are defined. The sign functions $\widetilde{(a,b)}$, $(a,b) \in S_A$, form a semigroup. We denote this semigroup by A^c_{rr}, i.e.

$$A^c_{rr} = \{\widetilde{(a,b)} : (a,b) \in S_A\}.$$

We refer to the pair $(\mathrm{Sper}^c A, A^c_{rr})$ as the *complete space of signs* of A. It is not a space of signs in the usual sense, but has interesting structure which needs to be investigated further.

Note: The complete space of signs of a formally real field K is precisely the regular space of signs of H_K.

For future use, we also define $\widetilde{(a,b)}$ for $(a,b) \in A \times A$, $(a,b) \notin S_A$. For (a,b) arbitrary in $A \times A$, we define:

$$\widetilde{(a,b)}(\mathfrak{p}, v, P) := \begin{cases} 1 & \text{if } v(a) = v(b) \neq \infty \text{ and } \frac{a+\mathfrak{p}}{b+\mathfrak{p}} + \mathfrak{m}_v > 0 \text{ at } P \\ -1 & \text{if } v(a) = v(b) \neq \infty \text{ and } \frac{a+\mathfrak{p}}{b+\mathfrak{p}} + \mathfrak{m}_v < 0 \text{ at } P \\ 0 & \text{if } v(a) > v(b) \neq \infty. \end{cases}$$

Note: For $(a,b) \in S_A$, this coincides with the previous definition.

3. Specialization and separation

As is typical for a spectral space, $\mathrm{Sper}^c A$ is usually not Hausdorff (although it is Hausdorff in its patch topology). It has a natural specialization relation: For $\alpha, \beta \in \mathrm{Sper}^c A$, α *specializes* β (equivalently, that β *generalizes* α), denoted $\beta \succeq \alpha$ (equivalently $\alpha \preceq \beta$), if α lies in the topological closure of the singleton set $\{\beta\}$. In view of the definition of the topology on $\mathrm{Sper}^c A$, this is equivalent to saying:

$$\forall (a,b) \in A \times A, \quad \widetilde{(a,b)}(\alpha) > 0 \ \Rightarrow \ \widetilde{(a,b)}(\beta) > 0.$$

We distinguish two basic types. (Compare to [6] [7]):

Type I. Suppose $\alpha = (\mathfrak{p}, v, P)$ and $\mathfrak{q}$ is a prime ideal of A containing $\mathfrak{p}$ such that

$$\forall a, b \in A \ \ a \in \mathfrak{q}, b \notin \mathfrak{q} \ \Rightarrow \ v(a) > v(b). \tag{1}$$

Consider the local ring $(A/\mathfrak{p})_{(\mathfrak{q}/\mathfrak{p})}$ in $k(\mathfrak{p})$ with residue field $k(\mathfrak{q})$. Denote by v' the valuation on $k(\mathfrak{q})$ defined by

$$B_{v'} = \{\frac{a+\mathfrak{q}}{b+\mathfrak{q}} : \ b \notin \mathfrak{q}, \ v(a) \geq v(b)\},$$

i.e, $B_{v'}$ is the image of $(A/\mathfrak{p})_{(\mathfrak{q}/\mathfrak{p})} \cap B_v$ under the natural map $\frac{a+\mathfrak{p}}{b+\mathfrak{p}} \mapsto \frac{a+\mathfrak{q}}{b+\mathfrak{q}}$ from $(A/\mathfrak{p})_{(\mathfrak{q}/\mathfrak{p})}$ to $k(\mathfrak{q})$. $B_{v'}/\mathfrak{m}_{v'}$ is naturally embedded in $B_v/\mathfrak{m}_v$ via

$$\frac{a+\mathfrak{q}}{b+\mathfrak{q}} + \mathfrak{m}_{v'} \mapsto \frac{a+\mathfrak{p}}{b+\mathfrak{p}} + \mathfrak{m}_v.$$

Denote by P' the ordering on $B_{v'}/\mathfrak{m}_{v'}$ obtained by restricting P to $B_{v'}/\mathfrak{m}_{v'}$. Then $\beta = (\mathfrak{q}, v', P')$ is a specialization of α, called a *type I specialization* of α. Each prime ideal $\mathfrak{q}_\lambda$ of A containing $\mathfrak{p}$ and satisfying (1) determines an upper cut $\{v(a) : a \in \mathfrak{q}_\lambda\}$ in $v(A/\mathfrak{p})$. The set of such upper cuts is totally ordered by inclusion. Consequently, the set of such prime ideals is also totally ordered by inclusion with largest element $\mathfrak{q} = \cup_\lambda \mathfrak{q}_\lambda$.

Type II. Suppose $\alpha = (\mathfrak{p}, v, P)$, w is a valuation on the field $k(\mathfrak{p})$ with $B_w \subseteq B_v$ and such that P is compatible with the valuation ring $B_w/\mathfrak{m}_v$ in $B_v/\mathfrak{m}_v$ and Q is the pushdown of P to $B_w/\mathfrak{m}_w$. Then $\gamma = (\mathfrak{p}, w, Q)$ is a specialization of α, called a *type II specialization* of α. Type II specializations of α form a chain with maximal element. The maximal type II specialization of α is obtained by taking w so that $B_w/\mathfrak{m}_v$ is the convex hull of $\mathbb{Z}$ with respect to the ordering P in the field $B_v/\mathfrak{m}_v$.

We analyze specialization in more detail. Suppose $\alpha_1 \succeq \alpha_2$, $\alpha_i = (\mathfrak{p}_i, v_i, P_i)$, $i = 1, 2$. Note that:

1) $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$. (If $a \in \mathfrak{p}_1$, $a \notin \mathfrak{p}_2$, then $\widetilde{(a,a)}$ is positive at α_2, zero at α_1.)

2) $v_2(a) \geq v_2(b) \neq \infty \Rightarrow v_1(a) \geq v_1(b)$. (If $v_2(a) > v_2(b) \neq \infty$ and $v_1(a) < v_1(b)$ then $\widetilde{(b+a, b-a)}$ is positive at α_2, negative at α_1. If $v_2(a) = v_2(b) \neq \infty$ and $v_1(a) < v_1(b)$ then, replacing a by $-a$ if necessary, $\widetilde{(a,b)}$ is positive at α_2, zero at α_1.)

Conditions (1) and (2) combined are just saying that $(\mathfrak{p}_1, v_1) \succeq (\mathfrak{p}_2, v_2)$ as elements of $\operatorname{Spv} A$. From (2) it follows that:

3) If $a \in \mathfrak{p}_2$, $b \notin \mathfrak{p}_2$, then $v_1(a) \geq v_1(b)$.

Theorem 3.1. *Suppose $\alpha_1 \succeq \alpha_2$. Then*

1) *There is a unique intermediate β such that $\alpha_1 \succeq \beta$ is maximal type II. $\beta \succeq \alpha_2$ is type I.*
2) *There is a unique intermediate γ such that $\alpha_1 \succeq \gamma$ is maximal type I. Either $\gamma \succeq \alpha_2$ is of type II or the valuation of γ is trivial.*
3) *There is a unique intermediate δ with $\alpha_1 \succeq \delta$ minimal of type II such that there exists ζ, $\delta \succeq \zeta \succeq \alpha_2$, with $\delta \succeq \zeta$ of type I and $\zeta \succeq \alpha_2$ of type II. If $\delta \neq \alpha_1$, then the valuation of ζ is trivial.*

Proof. Let $\alpha_i = (\mathfrak{p}_i, v_i, P_i)$, $i = 1, 2$.

(1) Choose v_0 so that B_{v_0} is the smallest valuation ring of B_{v_1} such that $B_{v_0}/\mathfrak{m}_{v_1}$ is compatible with P_1. Choose v so that $B_v = B_{v_0} \cdot \lambda^{-1}(B_{v_2})$ where $\lambda : (A/\mathfrak{p}_1)_{(\mathfrak{p}_2/\mathfrak{p}_1)} \to k(\mathfrak{p}_2)$ denotes the natural homomorphism. $B_v \subseteq B_{v_1}$. $\beta = (\mathfrak{p}_1, v, P)$ where P denotes the pushdown of P_1 to $B_v/\mathfrak{m}_v$. $\beta \succeq \alpha_2$. It remains to show that $\beta \succeq \alpha_2$ is type I. If this is not the case, then we would have

$$\frac{a+\mathfrak{p}_1}{b+\mathfrak{p}_1} = \frac{c+\mathfrak{p}_1}{d+\mathfrak{p}_1} \cdot x$$

with $b, d \notin \mathfrak{p}_2$, $v(a) \geq v(b)$, $v_2(a) < v_2(b)$, $v_2(c) \geq v_2(d)$, $v_0(x) \geq 0$. Thus $v(a) = v(b)$ and $v(c) \geq v(d)$. Also, $v_0(bc) \leq v_0(ad)$. We can assume $\frac{a}{b}$, $\frac{c}{d}$ and x are positive at P. Thus there exists a positive integer n such that $\frac{nbc-ad}{bc} = n - \frac{ad}{bc}$ and $\frac{nbc+ad}{bc} = n + \frac{ad}{bc}$ are strictly positive at P. Consequently, $\frac{nbc-ad}{nbc+ad}$ is strictly positive at P and strictly negative at P_2, a contradiction.

(2) Take $\mathfrak{q} = \cup_\lambda \mathfrak{q}_\lambda$ where $\{\mathfrak{q}_\lambda\}$ is the set (chain) of intermediate primes $\mathfrak{p}_1 \subseteq \mathfrak{q}_\lambda \subseteq \mathfrak{p}_2$ satisfying $v_1(a) > v_1(b)$ for all $a \in \mathfrak{q}_\lambda$, $b \notin \mathfrak{q}_\lambda$. $\mathfrak{q}$ is prime and $v_1(a) > v_1(b)$ holds for all $a \in \mathfrak{q}$, $b \notin \mathfrak{q}$. $\gamma = (\mathfrak{q}, w, Q)$ is the type I specialization of α_1 determined by $\mathfrak{q}$. If $\mathfrak{q} = \mathfrak{p}_2$ then the specialization $\gamma \succeq \alpha_2$ is type II. Suppose $\mathfrak{q} \neq \mathfrak{p}_2$. We know that $w(a) \geq w(b)$ holds for any $a \in \mathfrak{p}_2$, $b \notin \mathfrak{p}_2$. Thus $w(a_0) = w(b_0)$ holds for some $a_0 \in \mathfrak{p}_2$, $b_0 \notin \mathfrak{p}_2$. Then, for any $c \in A$, $a_0c \in \mathfrak{p}_2$, so $w(a_0c) \geq w(b_0) = w(a_0)$, so $w(c) \geq 0$. If $w(b_0) > 0$ then $w(b_0^2) > w(b_0) = w(a_0)$. since $b_0^2 \notin \mathfrak{p}_2$, $a_0 \in \mathfrak{p}_2$, this is a contradiction. Thus $w(b_0) = 0$. Thus, for any $c \in A$, if $c \notin \mathfrak{p}_2$, then $w(c) \leq w(a_0) = w(b_0) = 0$. This proves that $w(c) = 0$ for all $c \in A$, $c \notin \mathfrak{p}_2$. It follows that $(A/\mathfrak{q})_{(\mathfrak{p}_2/\mathfrak{q})} \subseteq B_w$ so the prime ideal $\mathfrak{q} \subseteq \mathfrak{q}' \subseteq \mathfrak{p}_2$ defined by $\mathfrak{q}'/\mathfrak{q} = \mathfrak{m}_w \cap (A/\mathfrak{q})$ defines a type I specialization of γ. Thus $\mathfrak{q}' = \mathfrak{q}$. Thus w is the trivial valuation in this case.

(3) If $v_1(a) > v_1(b)$ holds for all $a \in \mathfrak{p}_1$, $b \notin \mathfrak{p}_1$, then $\delta = \alpha_1$. Otherwise, arguing as above, $(A/\mathfrak{p}_1)_{(\mathfrak{p}_2/\mathfrak{p}_1)} \subseteq B_{v_1}$. In this case take $\delta = (\mathfrak{p}_1, v', P')$ where $B_{v'} = (A/\mathfrak{p}_1)_{(\mathfrak{p}_2/\mathfrak{p}_1)} \cdot B_{v_0}$ and P' is the pushdown of P_1 to $B_{v'}/\mathfrak{m}_{v'}$. We claim $v'(a) > v'(b)$ holds for all $a \in \mathfrak{p}_2$, $b \notin \mathfrak{p}_2$. Otherwise $v'(a) = v'(b)$ for some such a, b. Then

$$\frac{b}{a} = \frac{c}{d} \cdot x$$

for some $d \notin \mathfrak{p}_2$, $v_0(x) \geq 0$. Thus $v_0(\frac{bd}{ac}) \geq 0$. We can assume $\frac{bd}{ac}$ is positive at P_1. There exists a positive integer n such that $n - \frac{bd}{ac}$ is positive at P_1. Then $\frac{nac-bd}{nac+bd}$ is positive at P_1, and negative at P_2, a contradiction. This proves the claim. It follows that there is a type I specialization ζ of δ such that α_2 is a type II specialization of ζ.

At the same time, the valuation of ζ is trivial, so ζ has no type II generalization. It follows that δ is minimal with the desired property. $\square$

As a consequence of the above argument we also see:

Corollary 3.2. *If α_1 specializes to $\alpha_2 = (\mathfrak{p}_2, v_2, P_2)$ and also to $\alpha_2' = (\mathfrak{p}_2, v_2, P_2')$, then $P_2 = P_2'$, i.e., $\alpha_2 = \alpha_2'$.*

Given α_1, α_2 in $\operatorname{Sper} A$, with $\alpha_1 \not\succeq \alpha_2$, $\alpha_2 \not\succeq \alpha_1$, it is well-known that there exists $a \in A$ which separates α_1 and α_2 in the sense that $\widetilde{a}$ is strictly positive at one of α_1, α_2 and strictly negative at the other [9]. This fails in $\operatorname{Sper}^c A$, but only when α_1 and α_2 are related in a particular way:

Theorem 3.3. *Suppose $\alpha_i \in \operatorname{Sper}^c A$, $i = 1, 2$, $\alpha_1 \not\succeq \alpha_2$, $\alpha_2 \not\succeq \alpha_1$. The following are equivalent:*

1) *There is no $(a, b) \in A \times A$ such that $\widetilde{(a, b)}(\alpha_1) > 0$ and $\widetilde{(a, b)}(\alpha_2) < 0$.*
2) *Either $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ or $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$. If $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ then there exists a type I specialization $\alpha = (\mathfrak{p}_2, v, P)$ of α_1 which is at the same time a type II specialization of α_2. If $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$ the same holds, but with the roles of α_1, α_2 interchanged.*

Proof.
(1) $\Rightarrow$ (2). Let $\alpha_i = (\mathfrak{p}_i, v_i, P_i)$, $i = 1, 2$. Suppose $a \in \mathfrak{p}_1$, $a \notin \mathfrak{p}_2$, $b \in \mathfrak{p}_2$, $b \notin \mathfrak{p}_1$. Then $(a + b, a - b)$ is strictly positive at α_1 and strictly negative at α_2 contradicting our assumption. Thus either $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ or $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$, say $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$. Since $\alpha_1 \not\succeq \alpha_2$ there exists (a, b) with $\widetilde{(a, b)}(\alpha_2) \neq 0$, $\widetilde{(a, b)}(\alpha_1) = 0$. Since $\alpha_2 \not\succeq \alpha_1$ there exists (c, d) with $\widetilde{(c, d)}(\alpha_1) \neq 0$, $\widetilde{(c, d)}(\alpha_2) = 0$. Thus $v_2(a) = v_2(b) \neq \infty$, $ab \notin \mathfrak{p}_1$ (since $ab \notin \mathfrak{p}_2$) and $v_1(a) \neq v_1(b)$. Also $v_1(c) = v_1(d) \neq \infty$ and either $cd \in \mathfrak{p}_2$ or $cd \notin \mathfrak{p}_2$ and $v_2(c) \neq v_2(d)$. Suppose $cd \notin \mathfrak{p}_2$. Interchanging a and b and c and d if necessary, we can assume $v_1(a) < v_1(b)$ and $v_2(c) < v_2(d)$. Then $(ad - bc, ad + bc)$ separates α_1, α_2. Thus $cd \in \mathfrak{p}_2$. In particular, $\mathfrak{p}_1$ is properly contained in $\mathfrak{p}_2$. This exact same argument can be used to prove the following:

Claim 1: If c, d are not in $\mathfrak{p}_2$ and $v_1(c) = v_1(d)$ then $v_2(c) = v_2(d)$.

Claim 2: If $c, d \in A$, $d \notin \mathfrak{p}_2$, and $v_1(c) \geq v_1(d)$ then $v_2(c) \geq v_2(d)$. This is clear by Claim 1 if $v_1(c) = v_1(d)$. Suppose $v_1(c) > v_1(d)$. Then $v_1(c \pm d) = v_1(d)$. By Claim 1 this implies $v_2(c \pm d) = v_2(d)$ which, in turn, implies $v_2(c) \geq v_2(d)$.

Claim 3: If $c \in \mathfrak{p}_2$, $d \notin \mathfrak{p}_2$, then $v_1(c) > v_1(d)$. Otherwise $v_1(c) \leq v_1(d)$ and consequently $v_1(ac) < v_1(bd)$. Then the pair $(ac+bd, ac-bd)$ separates α_1 and α_2.

Claim 3 allows us to construct the type I specialization $\alpha = (\mathfrak{p}_2, v, Q)$ of α_1. Suppose now that $c, d \in A$, $d \notin \mathfrak{p}_2$ satisfy $v(c) \geq v(d)$. Using Claim 2 together with the definition of v this yields $v_2(c) \geq v_2(d)$. This proves $B_v \subseteq B_{v_2}$. Using the fact that α_1 and α_2 cannot be separated, we see that P_2 is compatible with $B_v/\mathfrak{m}_{v_2}$ and pushes down to the ordering Q in $B_v/\mathfrak{m}_v$. Thus α is a type II specialization of α_2.

(2) $\Rightarrow$ (1). Suppose $\widetilde{(a,b)}(\alpha_1) > 0$, $\widetilde{(a,b)}(\alpha_2) < 0$. Then $a, b \notin \mathfrak{p}_2$, $v_2(a) = v_2(b)$, and $v_1(a) = v_1(b)$, $\frac{a+\mathfrak{p}_2}{b+\mathfrak{p}_2} + \mathfrak{m}_{v_2} < 0$ at P_2, and $\frac{a+\mathfrak{p}_1}{b+\mathfrak{p}_1} + \mathfrak{m}_{v_1} > 0$ at P_1. Thus $v(a) = v(b)$ and $\frac{a+\mathfrak{p}}{b+\mathfrak{p}} > 0$ at Q. Since P_2 pushes down to Q, this contradicts $\frac{a+\mathfrak{p}_2}{b+\mathfrak{p}_2} > 0$ at P_2. $\square$

Motivated by this and the case of the real holomorphy ring of a formally real field, we examine a certain subspace of $\mathrm{Sper}^c A$. We define $\widetilde{\mathrm{Sper}}^c A :=$ the set of elements of $\mathrm{Sper}^c A$ which are maximal with respect to type I specialization. We also denote by $\mathrm{s} : \mathrm{Sper}^c A \to \widetilde{\mathrm{Sper}}^c A$ the natural map associating to each α in $\mathrm{Sper}^c A$, its unique maximal type I specialization.

Example: Suppose $A = K$, a formally real field. Then $\widetilde{\mathrm{Sper}}^c K = \mathrm{Sper}^c K$ which is naturally identified with $\mathrm{Sper}\, H$, where $H = H_K$.

Corollary 3.4. *If $\alpha_1, \alpha_2 \in \widetilde{Sper}^c A$ satisfy $\alpha_1 \not\preceq \alpha_2$, $\alpha_2 \not\preceq \alpha_1$, then there exists (a,b) in $A \times A$ separating α_1 and α_2.*

Corollary 3.5. *$\widetilde{Sper}^c A$ is completely normal. For each $\alpha \in \widetilde{Sper}^c A$, the specializations of α in $\widetilde{Sper}^c A$ form a chain.*

Note: The map $(\mathfrak{p}, P) \mapsto (\mathfrak{p}, 0, P)$ identifies $\mathrm{Sper}\, A$ with the elements of $\mathrm{Sper}^c A$ which are minimal with respect to type II generalization. Observe that each such $(\mathfrak{p}, 0, P)$ is also maximal with respect to type I specialization.

Theorem 3.6. *The following are equivalent:*

1) *A is a real holomorphy ring.*
2) *The natural embedding $(\mathfrak{p}, P) \mapsto (\mathfrak{p}, 0, P)$ of $Sper\, A$ into $\widetilde{Sper}^c A$ is surjective, i.e., a homeomorphism.*

Proof.

(1) $\Rightarrow$ (2). Let $\alpha = (\mathfrak{p}, v, P)$ be an arbitrary element of $\mathrm{Sper}^c A$. Since $A/\mathfrak{p} \subseteq B_v$ one checks easily that the maximal type I extension of α has the form $\alpha' = (\mathfrak{q}, v', P')$ where $\mathfrak{q}$ is defined by $\mathfrak{q}/\mathfrak{p} := \mathfrak{m}_v \cap A/\mathfrak{p}$. But then $(A/\mathfrak{p})_{(\mathfrak{q}/\mathfrak{p})} \cap B_v = (A/\mathfrak{p})_{(\mathfrak{q}/\mathfrak{p})}$, i.e., $B_{v'} = k(\mathfrak{q})$, i.e., $v' = 0$.

(2) $\Rightarrow$ (1). Let $\mathfrak{p}$ be a real prime and let v be a real valuation ring of $k(\mathfrak{p})$. Let $\alpha = (\mathfrak{p}, v, P)$ where P is some fixed ordering on $B_v/\mathfrak{m}_v$. By (2), α has a type I specialization of the form $(\mathfrak{q}, v', P')$ with $v' = 0$. We have the natural ring homomorphism from $(A/\mathfrak{p})_{(\mathfrak{q}/\mathfrak{p})} \cap B_v$ onto $B_{v'} = k(\mathfrak{q})$ with kernel $(\mathfrak{q}/\mathfrak{p})_{(\mathfrak{q}/\mathfrak{p})}$. A standard diagram chase shows that $(A/\mathfrak{p})_{(\mathfrak{q}/\mathfrak{p})} \cap B_v = (A/\mathfrak{p})_{(\mathfrak{q}/\mathfrak{p})}$. This proves $A/\mathfrak{p} \subseteq B_v$. $\square$

Remark 3.7. Consider the pair $(\widetilde{\operatorname{Sper}}^c A, \widetilde{A}^c_{rr})$ where $\widetilde{A}^c_{rr}$ denotes the set of restrictions of elements of A^c_{rr} to $\widetilde{\operatorname{Sper}}^c A$. Is $\widetilde{\operatorname{Sper}}^c A$ a spectral space? Is $(\widetilde{\operatorname{Sper}}^c A, \widetilde{A}^c_{rr})$ a space of signs? When $H_A = A$, $(\widetilde{\operatorname{Sper}}^c A, \widetilde{A}^c_{rr})$ is identified with $(\operatorname{Sper} A, A_{rr})$, so, in this case at least, the answer to both questions is 'yes'.

One can also consider the subspace of $\operatorname{Sper}^c A$ consisting of elements which are maximal with respect to type II specialization. This is precisely the space of real places M_A considered in [10].

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Institut de Mathématiques de Jussieu, Université Paris 6, 4 Place Jussieu, 75252 Paris Cedex 05, France

E-mail address: `gondard@math.jussieu.fr`

Department of Mathematics & Statistics, University of Saskatchewan, 106 Wiggins Road, Saskatoon, SK Canada, S7N 5E6

E-mail address: `marshall@math.usask.ca`