NEW NON-STANDARD TOPOLOGIES

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ABSTRACT. In this paper, on a non-standard extension $({}^*X, {}^*d)$ of a metric space (X, d), we construct a chain of new non-standard topologies in terms of convex subrings of ${}^*\mathbb{R}$, its minimal element is the S-topology and its maximal is the Q-topology. Next, we construct $\widehat{X}$, the $\mathscr{F}$-asymptotic hull of X, and we prove that such space is metrizable and complete when $\mathscr{F}$ is generated by an asymptotic scale. Finally, we provide a pseudo-valuation taking integral values, equivalent to the classical Robinson's valuation, on ${}^\rho\mathbb{R}$, the Robinson's field of ρ-asymptotic numbers.

1. INTRODUCTION

There are various complications about topologies of a non-standard extension ${}^*X$ of a topological space X with a topology τ: the space ${}^*X$ does not have a canonical topology, and there are two important topologies on it introduced by Robinson[22]. The first, called the Q-topology and the second important topology, called the S-topology. The Q-topology is finer than the S-topology. We recall the main properties of these well-known topologies. The space ${}^*X$ is not Hausdorff with respect to the S-topology. The restriction of the S-topology to X coincides with τ. The space ${}^*X$ is Hausdorff with respect to the Q-topology, and the restriction of the Q-topology to X normally does not coincide with τ. The shadow map from near-standard point of ${}^*X$ is defined using the S-topology, and hence is not quite compatible with the Q-topology, see [9].

In this paper, on a non-standard extension ${}^*X$ of a metric space X, we construct a family of new non-standard topologies, each topology depends on $\mathscr{F}$ a convex subring of ${}^*\mathbb{R}$ and called the QS-topology generated by $\mathscr{F}$. The S-topology and the Q-topology can be recovered by a suitable choice of convex subrings of ${}^*\mathbb{R}$. In fact, if the convex subring $\mathscr{F}$ is the ring of finite numbers of ${}^*\mathbb{R}$, then the QS-topology induced by $\mathscr{F}$ is the S-topology, and if $\mathscr{F}$ is the field ${}^*\mathbb{R}$, then the induced QS-topology is the Q-topology. Furthermore the set of all QS-topologies on ${}^*X$ is a *totally* ordered set having a minimal element given by the S-topology and a maximal element given by the Q-topology.

These topologies have some common properties of the S-topology and the Q-topology on ${}^*X$. For instance, the space ${}^*X$ is not Hausdorff with respect to all QS-topologies except for the Q-topology and the restriction of a QS-topology to X normally does not coincide with (X, d). In fact, if $\mathscr{F}$ is Archimedean, that is, $\mathscr{F} = {}^b\mathbb{R}$, then the restriction of the QS-topology to X coincides with its metric topology, and if $\mathscr{F}$ is non-Archimedean, then the restriction of the QS-topology to X induces the discrete topology.

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Let (X,d) be a metric space and $\mathscr{F}$ be a *proper* convex subring of ${}^*\mathbb{R}$, we then construct a monad system, called the $\mathscr{F}$-monad system, using the following equivalent relation: for $p,q \in {}^*X$

$$p \approxeq q \text{ if } {}^*d(p,q) \in {}^i\mathscr{F}_+,$$

where ${}^i\mathscr{F}_+ = \{|x| : x \in {}^i\mathscr{F}\}$, and ${}^i\mathscr{F}$ is the unique maximal ideal of $\mathscr{F}$. We remark that the standard or the classical monad system is obtained for $\mathscr{F} = {}^b\mathbb{R}$.

Many topological properties related to the *QS*-topology (QS-open, QS-closed, QS-continuous,..) can be expressed in terms of the $\mathscr{F}$-monad system. Although there are several basic differences between the standard monad system and the $\mathscr{F}$-monad system where $\mathscr{F}$ is a non-Archimedean and a proper subring of ${}^*\mathbb{R}$, that is, ${}^b\mathbb{R} \subsetneqq \mathscr{F} \subsetneqq {}^*\mathbb{R}$:

- Contrary to the classical monad system, where the continuity of a *standard* mapping f is equivalent to the *S*-continuity of ${}^*f$ on X, that is, ${}^*f$ sends monads of standard points to monads, we show that if ${}^*f$ is *QS*-continuous on X, then f is continuous on X, but the converse is false. We prove that if f is a locally Lipschitz continuous mapping, then ${}^*f$ is *QS*-continuous.

- It is well known that the classical monad system is independent of the metric on X and depends only on the topology of X, whereas the $\mathscr{F}$-monad system depends on the metric. We give two equivalent metrics having two different $\mathscr{F}$-monads. But, we show if two distances are Lipschitz equivalent then they provide the same $\mathscr{F}$-monad system.

Next, we construct $(\widehat{X},\widehat{d})$, the $\mathscr{F}$-asymptotic hull of (X,d), as a generalized metric space, that is, $\widehat{d}$ verifies the familiar axioms for metrics but taking its values in $\widehat{\mathscr{F}}$. In particular when $\mathscr{F} = {}^b\mathbb{R}$, $\widehat{d}$ is a metric and the $\mathscr{F}$- asymptotic hull of (X,d) is the usual non-standard hull of (X,d). However, for any *proper* and non-Archimedean subring of ${}^*\mathbb{R}$, we show that $(\widehat{X},\widehat{d})$ is a Hausdorff topological space inducing the discrete topology on X and $\widehat{X}$ is metrizable if the ring $\mathscr{F}$ is generated by an asymptotic scale. For these type of rings we endow $\widehat{X}$, the $\mathscr{F}$-asymptotic hull of X, with an explicit ultrametric δ and we prove that $(\widehat{X},\delta)$ is complete.

Finally, we should note that ${}^\rho\mathbb{R}$, the field of Robinson's real ρ-asymptotic numbers, is exactly the $\mathscr{F}$-asymptotic hull of $(\mathbb{R},|.|)$ where $\mathscr{F}$ is the ring of ρ-moderate non-standard numbers, see example 2.8. In addition, we prove that, on ${}^\rho\mathbb{R}$, the corresponding ultrametric δ is induced by a *pseudo-valuation* taking *integral values*, equivalent to the Robinson's valuation, see Theorem 4.10.

2. Basic Notions

This section of preliminary notions provides a background necessary for the comprehension of the paper.

2.1. **Pseudo-valuations.** Let R be a commutative ring with unit.

Definition 2.1. *By a pseudo-valuation on R, we shall mean a function* $p : R \to \mathbb{R} \cup \{+\infty\}$ *satisfying :*

(1) $p(1) = 0, \; p(0) = +\infty,$
(2) $p(xy) \geq p(x) + p(y) \quad (x,y \in R),$
(3) $p(x-y) \geq \min(p(x),p(y)) \quad (x,y \in R).$

By a valuation we shall mean a pseudo-valuation satisfying the stronger condition:

$$p(xy) = p(x) + p(y). \tag{2.1}$$

Note, that if p is any pseudo-valuation on a *field*, the ideal $p^{-1}(+\infty)$ is necessarily zero.

If p is a pseudo-valuation on a ring R, and α a real number, $0 < \alpha < 1$, then the function

$$m(x) = \alpha^{p(x)}$$

is an ultrametric multiplicative pseudo-valuation on R [20], that is, a nonnegative real-valued function satisfying:

(1') $m(1) = 1, m(0) = 0$,
(2') $m(xy) \leq m(x)m(y)$,
(3') $m(x-y) \leq \max(m(x), m(y))$.

On a field K, a pseudo-valuation generalizes a valuation, and, like a valuation, it defines a topology on K which is compatible with the field structure of K. Two pseudo-valuations m_1, m_2 are called *equivalent* if they define the same topology on K. This notion is analogous to the equivalence of valuations.

Cohn[3] proved the following theorem giving a necessary and sufficient conditions for the topology on a field to be definable by a pseudo-valuation.

Theorem 2.2 ([3], Theorem 7.1)**.** *If K is a topological field, then the topology of K can be defined by a (non-Archimedean) pseudo-valuation if and only if K has a non-empty open bounded subset which is closed under addition and contains only nilpotent elements.*

Recall that an element x of a topological field is said to be *nilpotent* if $x^n \to 0$.

If m is an ultrametric multiplicative pseudo-valuation on K, then the set $\{x \in K : m(x) < 1\}$ is open, closed under addition and contains only nilpotent elements.

2.2. **Convex subrings of $^*\mathbb{R}$.** Let $^*\mathbb{R}$ be a non-standard extension of the field of real numbers $\mathbb{R}$ and $^i\mathbb{R}$, $^b\mathbb{R}$ and $^\infty\mathbb{R}$ stand for the sets of infinitesimals, bounded (or finite) numbers and infinitely large numbers in $^*\mathbb{R}$, respectively. For a comprehensive introduction to nonstandard analysis, the reader is referred to [25], [12], [13] or [16].

Using convex subrings of $^*\mathbb{C}$, a variety of fields $\widehat{\mathscr{F}}$ are constructed by Todorov[28]. These fields are called $\mathscr{F}$-asymptotic hulls and their elements $\mathscr{F}$-asymptotic numbers. This construction can be viewed as a generalization of A. Robinson's theory of asymptotic numbers, see Lightstone-Robinson [17].

First we recall the definition and some properties of convex subrings of $^*\mathbb{R}$.

Definition 2.3. *Let $\mathscr{F}$ be a subring in $^*\mathbb{R}$. We say that $\mathscr{F}$ is a convex in $^*\mathbb{R}$ if*

$$(\forall x \in {}^*\mathbb{R})(\forall \xi \in \mathscr{F})(|x| \leq |\xi| \Rightarrow x \in \mathscr{F}).$$

It is easy to see that $\mathscr{F}$ contains $^b\mathbb{R}$, the ring of bounded elements of $^*\mathbb{R}$ and $\mathscr{F}$ is an *archimedean* ring if and only if $\mathscr{F} = {}^b\mathbb{R}$. Clearly $\mathscr{F}$ is a valuation ring hence $\mathscr{F}$ is a local ring, its unique maximal ideal $^i\mathscr{F}$ is the set of all non-invertible elements of $\mathscr{F}$.

Throughout this paper, we will consider only convex subrings of $^*\mathbb{R}$ because there is a one-to-one correspondence between convex subrings of $^*\mathbb{C}$ and those of $^*\mathbb{R}$: let $\mathscr{F}'$ be a convex subring of $^*\mathbb{C}$, then $\mathscr{F} = \mathscr{F}' \cap {}^*\mathbb{R}$ is a convex subring of $^*\mathbb{R}$. Conversely let $\mathscr{F}$ be a convex subring of $^*\mathbb{R}$, then

$\mathscr{F}' = \{a \in \mathbb{C} : |a| \in \mathscr{F}\}$ is a convex subring of ${}^*\mathbb{C}$. Hence $\mathscr{F}' \mapsto \mathscr{F} (= \mathscr{F}' \cap {}^*\mathbb{R})$ is one-to-one order preserving correspondence between convex subrings of ${}^*\mathbb{C}$ and those of ${}^*\mathbb{R}$.

We recall the main properties of ${}^i\mathscr{F}$ in the following proposition

Proposition 2.4. *Let $\mathscr{F}$ be a convex subring of ${}^*\mathbb{R}$ and let ${}^i\mathscr{F}$ be the set of the non-invertible elements of $\mathscr{F}$. Then*

(1) ${}^i\mathscr{F} = \{x \in {}^*\mathbb{R} : x = 0 \text{ or } 1/x \notin \mathscr{F}\}$. *Consequently, ${}^*\mathbb{R}$ is the filed of fractions for ${}^*\mathbb{R}$.*
(2) ${}^i\mathscr{F}$ *consists of infinitesimals only i.e.,* ${}^i\mathscr{F} \subset {}^i\mathbb{R}$.
(3) ${}^i\mathscr{F}$ *is a convex ideal in $\mathscr{F}$ i.e., if $x \in \mathscr{F}$ and $\xi \in {}^i\mathscr{F}$, $(|x| \leq |\xi| \Rightarrow x \in {}^i\mathscr{F})$.*
(4) $\mathscr{F}$ *is a field if and only if* $\mathscr{F} = {}^*\mathbb{R}$.

Using the important fact that every convex subring of ${}^*\mathbb{R}$ contains ${}^b\mathbb{R}$, we prove the following

Proposition 2.5. *Let $\mathscr{F}$ be a convex subring in ${}^*\mathbb{R}$. Then every ideal in $\mathscr{F}$ is convex.*

Proof. Let $\mathfrak{I}$ be an ideal in $\mathscr{F}$, $x \in \mathscr{F}$ and $\xi \in \mathfrak{I}$, such that $|x| \leq |\xi|$. If $\xi = 0$, then $x = 0$. Now assume that $\xi \neq 0$ it follows that $\left|\frac{x}{\xi}\right| \leq 1$, which implies $\frac{x}{\xi} \in \mathscr{F}$. Thus $x \in \mathfrak{I}$. □

We give some examples of convex subrings of ${}^*\mathbb{R}$.

2.2.1. *Examples.*

Example 2.6. *(Finite Numbers). The ring of bounded non-standard real numbers ${}^b\mathbb{R}$ is a convex subring of ${}^*\mathbb{R}$. Its maximal ideal is ${}^i\mathbb{R}$, the set of infinitesimals.*

Example 2.7. *(Non-Standard Real Numbers). The field of the real numbers ${}^*\mathbb{R}$ is (trivially) a convex subring of ${}^*\mathbb{R}$. Its maximal ideal is $\{0\}$.*

Example 2.8. *(Robinson Rings). Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$. The ring of the ρ-moderate non-standard numbers is defined by*

$$\mathscr{M}_\rho = \{x \in {}^*\mathbb{R} : |x| \leq \rho^{-n} \text{ for some } n \in \mathbb{N}\}.$$

$\mathscr{M}_\rho$ is a convex subring of ${}^*\mathbb{R}$. For its maximal ideal we have

$$\mathscr{N}_\rho = \{x \in {}^*\mathbb{R} : |x| \leq \rho^n \text{ for all } n \in \mathbb{N}\}.$$

We call numbers in $\mathscr{N}_\rho$ ρ-negligible numbers (or iota numbers). The numbers in ${}^*\mathbb{R} \setminus \mathscr{M}_\rho$ are called mega numbers.

Example 2.9. *(Logarithmic-Exponential Rings) Let ρ be a positive infinitesimal in ${}^*\mathbb{R}$ and let $\mathscr{E}_\rho$ be the smallest convex subring of ${}^*\mathbb{R}$ containing all iterated exponentials of ρ^{-1}, that is,*

$$\mathscr{E}_\rho = \{x \in {}^*\mathbb{R} : |x| \leq \exp_n(\rho^{-1}) \text{ for some } n \in \mathbb{N}\},$$

where $\exp_0(x) = x$ and $\exp_n(x) = \exp(\exp_{n-1}(x))$ for $x \in {}^*\mathbb{R}$ and $n > 0$. The maximal ideal of $\mathscr{E}_\rho$ is

$${}^i\mathscr{E}_\rho = \{x \in {}^*\mathbb{R} : |x| \leq \frac{1}{\exp_n(\rho^{-1})} \text{ for all } n \in \mathbb{N}\}$$

The last two examples can be generalized by introducing the notion of asymptotic scales.

Definition 2.10. *A sequence $(\lambda_n)_{n\in\mathbb{N}}$ of infinitesimal positive numbers (except possibly $n = 0$) is called an asymptotic scale if it satisfies the following conditions:*

(1) *for all $n \in \mathbb{N}$,* $\frac{\lambda_{n+1}}{\lambda_n} \in {}^i\mathbb{R}$,

(2) *for every* $n \in \mathbb{N}$, *there is* $m \in \mathbb{N}$ *such that* $\lambda_m^2 \geq \lambda_n$.

The reader is referred to Astrada and Kanwal[8] for the classical definition of asymptotic sequence of functions, and to Jones [10] and to Van den Berg [29] for the nonstandard treatment of asymptotics. In addition, Aschenbrenner and Van den Dries[1] introduced the asymptotic concepts in the framework of a real closed field.

Finally, we mention that a similar definition of asymptotic scales was introduced by Delcroix and Scarpalézos [6] to endow the space of asymptotic algebras of generalized functions with a topology, called the sharp topology first introduced by Scarpalézos [24] .

Let (λ_n) be an asymptotic scale. The sequence $(\lambda_n)_{n\in\mathbb{N}}$ extends to $(\lambda_n)_{n\in\mathbb{Z}}$ by putting

$$\lambda_{-n} = \frac{1}{\lambda_n} \quad \text{for } n \in \mathbb{N} \setminus \{0\}.$$

By countable comprehensiveness, the sequence $(\lambda_n)_{n\in\mathbb{Z}}$ extends to an internal sequence $(\lambda_n)_{n\in{}^*\mathbb{Z}}$. Furthermore, using Robinson's sequential lemma, see [25] p. 196, we deduce that there exists an infinite integer H, such that

$$\frac{\lambda_{n+1}}{\lambda_n} \in {}^i\mathbb{R} \text{ when } n \in [\![0..H]\!].$$

An asymptotic scale (λ_n) gives rise to a convex ring $\mathscr{F}$ of ${}^*\mathbb{R}$, called the convex ring generated by (λ_n), defined by

$$\mathscr{F} = \{x \in {}^*\mathbb{R} : |x| \leq 1/\lambda_n \text{ for some } n \in \mathbb{N}\}.$$

Its maximal ideal is given by

$${}^i\mathscr{F} = \{x \in {}^*\mathbb{R} : |x| \leq \lambda_n \text{ for all } n \in \mathbb{N}\}.$$

Using spilling principles for proper convex subrings of ${}^*\mathbb{R}$, see Appendix A, it is easy to prove the following characterizations

Proposition 2.11. *Let* $\mathscr{F}$ *be the convex ring generated by an asymptotic scale* $\{\lambda_n\}$ *and* $x \in {}^*\mathbb{R}$, *then*

(1) $x \in \mathscr{F}$ *if and only if* $|x| < \dfrac{1}{\lambda_N}$, *for all* $N \in \mathbb{N}^\infty$,
(2) $x \in {}^i\mathscr{F}$ *if and only if* $|x| \leq \lambda_N$, *for some* $N \in \mathbb{N}^\infty$.

Definition 2.12. *The* $\mathscr{F}$*-asymptotic hull is the factor ring* $\widehat{\mathscr{F}} = \mathscr{F}/{}^i\mathscr{F}$.

Let $\widehat{\mathrm{st}} : \mathscr{F} \longrightarrow \widehat{\mathscr{F}}$ stand for the corresponding quotient mapping, called the quasi-standard mapping.

Notation: If $x \in \mathscr{F}$, we shall often write $\widehat{x}$ instead of $\widehat{\mathrm{st}}(x)$ for the quasi-standard part of x.

We can define an order relation in $\widehat{\mathscr{F}}$, inhered from the order in ${}^*\mathbb{R}$, by

$$\hat{x} \leq \hat{y} \text{ if there are representatives } x, y \text{ with } x \leq y.$$

Using the convexity of $\mathscr{F}$, the following proposition is straightforward

Proposition 2.13. $(\widehat{\mathscr{F}}, \leq)$ *is a completely ordered field.*

Proof. We have only to show antisymmetry and transitivity.
If $\hat{x} \leq \hat{y}$ and $\hat{y} \leq \hat{x}$, then there are $r, s \in {}^i\mathscr{F}$ such that $x \leq y + r$ and $y \leq x + s$. Hence $|x - y| \leq \max(|r|, |s|)$. So, $x - y \in {}^i\mathscr{F}$ and $\hat{x} = \hat{y}$.
For transitivity, if $\hat{x} \leq \hat{y}$ and $\hat{y} \leq \hat{z}$. Then $x \leq y + r \leq z + r + s$, so $\hat{x} \leq \hat{z}$. □

We note that Todorov[28] proved a strong form of Proposition 2.13 claiming that $\widehat{\mathscr{F}}$ is a real closed field.

Remark 2.14. *Let A be a lattice-ordered commutative ring and let I be an ideal in A. Convexity of the ideal I is the necessary and sufficient condition that the canonical homomorphism of A into A/I be a lattice homomorphism, see* [11], *Theorem 5.3.*

3. Topologies in ${}^*X$

Let $\mathscr{F}$ be a convex subring in ${}^*\mathbb{R}$. We denote by ${}^a\mathscr{F}_+$ the set of $\mathscr{F}$-appreciable positive elements of $\mathscr{F}$, i.e.,

$$ {}^a\mathscr{F}_+ = \{|x| : x \in \mathscr{F} \setminus {}^i\mathscr{F}\}, $$

and ${}^i\mathscr{F}_+$ the set of non-negative elements of ${}^i\mathscr{F}$, i.e.,

$$ {}^i\mathscr{F}_+ = \{|x| : x \in {}^i\mathscr{F}\}. $$

Let $({}^*X, {}^*d)$ be a non-standard extension of a metric space (X, d). A point $p \in {}^*X$ is called $\mathscr{F}$-bounded if there exists a standard point $q \in X$ such that ${}^*d(p,q) \in \mathscr{F}$. Let $\mathscr{F}({}^*X)$ stand for the set of $\mathscr{F}$- bounded points in ${}^*X$, i.e.,

$$ \mathscr{F}({}^*X) = \{p \in {}^*X : \text{ there exists } q \in X, {}^*d(p,q) \in \mathscr{F}\}. $$

Let p, q be two points in ${}^*X$, using the convexity $\mathscr{F}$, the condition that ${}^*d(p,q) \in \mathscr{F}$ defines an equivalent relation on ${}^*X$. This equivalent relation divides ${}^*X$ into a number of disjoint subsets which will be called the $\mathscr{F}$-galaxies of ${}^*X$, the set $\mathscr{F}({}^*X)$ constitutes its *principal galaxy*.

Definition 3.1. *For any point $p \in {}^*X$, we define the $\mathscr{F}$-halo of p by*

$$ \mu_{\mathscr{F}}(p) = \{q \in {}^*X : {}^*d(p,q) \in {}^i\mathscr{F}\} = \bigcap_{r \in {}^a\mathscr{F}_+} \{q \in {}^*X : {}^*d(p,q) < r\}. $$

We write $p \approxeq q$ if ${}^*d(p,q) \in {}^i\mathscr{F}$ i.e., $q \in \mu_{\mathscr{F}}(p)$.

Proposition 3.2.

(1) *For $\mathscr{F} = {}^b\mathbb{R}$, the $\mathscr{F}$-halo of p is the standard halo of p.*
(2) *The $\mathscr{F}$-halo of p is reduced to p if and only if $\mathscr{F} = {}^*\mathbb{R}$.*
(3) *If (λ_n) is an asymptotic scale and $\mathscr{F}$ is the convex subring generated by (λ_n), then for any point $p \in {}^*X$*

$$ \mu_{\mathscr{F}}(p) = \bigcap_{n \in \mathbb{N}} \{q \in {}^*X : {}^*d(p,q) \leq \lambda_n\} = \bigcup_{n \in \mathbb{N}^\infty} \{q \in {}^*X : {}^*d(p,q) \leq \lambda_n\}. $$

It is not difficult to see that if $p \approxeq p'$ and $q \approxeq q'$ then ${}^*d(p,q) \approxeq {}^*d(p',q')$. Hence, $\widehat{\text{st}}({}^*d(p,q)) = \widehat{\text{st}}({}^*d(p',q'))$ in $\widehat{\mathscr{F}}$, provided one, and hence the other of these quasi-standard parts exist i.e., provided ${}^*d(p,q) \in \mathscr{F}$, that is to say, p, q belong to the same galaxy.

Let $p \in {}^*X$ and let r be an $\mathscr{F}$-*appreciable* positive element i.e., $r \in {}^a\mathscr{F}_+$. An $\mathscr{F}$-appreciable-radius neighbourhood of p is the the hyper-ball

$$B(p,r) = \{q \in {}^*X : {}^*d(p,q) < r\}.$$

The class of $\mathscr{F}$-appreciable-radius neighbourhoods *fails* to be a basis for a topology on ${}^*X$, we can easily construct two $\mathscr{F}$-appreciable-radius balls such that their intersection has a width in ${}^i\mathscr{F}$ and so does not contain any $\mathscr{F}$-appreciable-radius neighbourhoods. This suggests that the overlaps between appreciable-radius neighbourhoods are "too small". One way to remedy this is to modify neighbourhoods by removing members that are close to the boundary and forcing any overlaps to be $\mathscr{F}$-appreciable. To formalism this, for $p \in {}^*X$ and $r \in {}^a\mathscr{F}_+$, put

$$\mathrm{QS}(p,r) = \{q \in {}^*X : \mu_{\mathscr{F}}(q) \subset B(p,r)\} = \{q \in {}^*X : \widehat{\mathrm{st}}({}^*d(p,q)) < \widehat{r}\}.$$

That is, $q \in \mathrm{QS}(p,r)$, if and only if, $r - {}^*d(p,q) \in {}^a\mathscr{F}_+$.

We call $\mathrm{QS}(p,r)$ the QS-ball with center p and radius r.

Theorem 3.3. *The collection* $\{\mathrm{QS}(p,r), p \in {}^*X, r \in {}^a\mathscr{F}_+\}$ *is a basis for a topology on* ${}^*X$ *called the* QS*-topology generated by* $\mathscr{F}$.

The "QS-" prefix is for quasi-standard since the indicated base is constructed using the quasi-standard mapping $\widehat{\mathrm{st}}$.

The proof is similar to Robinson [22] p. 106, where the author proved that S-balls may serve as a base for the S-topology.

Proof. We have only to show that every point in the intersection of two QS-balls is the center of an QS-ball which is included in that intersection.

Consider the intersection of two QS-balls, $\mathrm{QS}(p,r)$ and $\mathrm{QS}(p',r')$, and suppose that it is not empty. Then for any point q in the intersection we have, ${}^*d(p,p') \leq {}^*d(p,q) + {}^*d(q,p')$ and hence $\widehat{\mathrm{st}}({}^*d(p,p')) \leq \widehat{\mathrm{st}}({}^*d(p,q)) + \widehat{\mathrm{st}}({}^*d(q,p'))$ so that

$$\widehat{\mathrm{st}}({}^*d(p,p')) < \widehat{r} + \widehat{r'}.$$

Let $s = {}^*d(p,q)$, $s' = {}^*d(p',q)$. By assumption we have $\widehat{s} < \widehat{r}$, $\widehat{s'} < \widehat{r'}$. Put $\alpha = \min(r-s, r'-s')$ so that $\alpha \in {}^a\mathscr{F}_+$. One can easily verify that

$$QS(q,\alpha) \subset QS(p,r), \quad QS(q,\alpha) \subset QS(p',r').$$

□

Remark 3.4. *The radii of* QS*-balls can be chosen as positive real numbers if the ring* $\mathscr{F}$ *is Archimedean i.e.,* $\mathscr{F} = {}^b\mathbb{R}$ *and in* ${}^i\mathbb{R} \setminus {}^i\mathscr{F}_+$ *if* $\mathscr{F}$ *is non-Archimedean i.e.,* ${}^b\mathbb{R} \subsetneq \mathscr{F}$.

We describe the QS-topology for some examples of the convex subrings of ${}^*\mathbb{R}$.

Example 3.5.

(1) *The* QS*-topology generated by* ${}^b\mathbb{R}$ *is the* S*-topology.*
(2) *The* QS*-topology generated* ${}^*\mathbb{R}$ *is the* Q*-topology.*
(3) *Let* $(\lambda_n)_{n\in\mathbb{N}}$ *be an asymptotic scale, (see Definition 2.2.1). Then the collection*

$$\{y \in {}^*X : \widehat{\mathrm{st}}({}^*d(x,y)) < \widehat{\lambda_n}\},$$

where $x \in {}^*X$ *and* $n \in \mathbb{N}$, *is a basis for the QS-topology on* ${}^*X$ *generated by* $(\lambda_n)_{n\in\mathbb{N}}$.

It is clear that every open in the S-topology is open also in the QS-topology and every open in the QS-topology is open the Q-topology. Thus, the QS-topology is finer than the S-topology and coarser than the Q-topology.

More generally, we have

Proposition 3.6. *Let $\mathscr{F}$ and $\mathscr{G}$ be two convex rings such that $\mathscr{F} \subset \mathscr{G}$. Then the* QS*-topology generated by $\mathscr{F}$ is coarser than the* QS*-topology generated by $\mathscr{G}$.*

Proof. Since $\mathscr{F} \subset \mathscr{G}$, we have ${}^i\mathscr{F} \supset {}^i\mathscr{G}$. Hence, ${}^a\mathscr{F}_+ \subset {}^a\mathscr{G}_+$. Consequently, every QS-ball for the QS-topology generated by $\mathscr{F}$ is a QS-ball for the QS-topology generated by $\mathscr{G}$. □

The latter condition on $\mathscr{F}$ and $\mathscr{G}$ in not restrictive : if $\mathscr{F}$ and $\mathscr{G}$ be two convex subrings of ${}^*\mathbb{R}$, then ${}^i\mathscr{F}$ and ${}^i\mathscr{G}$ are ideals in the valuation ring ${}^b\mathbb{R}$. Hence either ${}^i\mathscr{F} \subset {}^i\mathscr{G}$ or ${}^i\mathscr{G} \subset {}^i\mathscr{F}$. Using ${}^{ii}\mathscr{F} = \mathscr{F}$ and ${}^{ii}\mathscr{G} = \mathscr{G}$, we deduce that either $\mathscr{G} \subset \mathscr{F}$ or $\mathscr{F} \subset \mathscr{G}$.

Theorem 3.7. *All QS-topologies, generated by convex rings on ${}^*X$, are totally ordered; its minimal element is the S-topology and its maximal element is the Q-topology.*

Furthermore, we have

Proposition 3.8. *The* QS*-topology on ${}^*X$ generated by $\mathscr{F}$ is Hausdorff if and only if $\mathscr{F} = {}^*\mathbb{R}$.*

Proof. If $\mathscr{F} = {}^*\mathbb{R}$ then the QS-topology is the Q-topology which is Hausdorff by transfer. Conversely, if $\mathscr{F} \neq {}^*\mathbb{R}$, then for any $x \in {}^*X$, the $\mathscr{F}$-halo of x is not reduced to the point x. Hence, there is $y \in {}^*X$ such that $y \neq x$ and $y \approxeq x$. So, $QS(x,r) = QS(y,r)$ for any $r \in {}^a\mathscr{F}_+$. Thus, the QS-topology is not Hausdorff. □

We show that the restriction of the QS-topology to X coincides with (X,d) if the $\mathscr{F}$ is Archimedean whereas the restriction of the QS-topology generated by $\mathscr{F}$ to X is the discrete topology if the ring $\mathscr{F}$ is non-Archimedean.

Proposition 3.9. *Let $\mathscr{F}$ be a convex subring in ${}^*\mathbb{R}$.*

(1) *If $\mathscr{F}$ is Archimedean, that is $\mathscr{F} = {}^b\mathbb{R}$, then the restriction of the* QS*-topology to X coincide with its topology.*
(2) *If $\mathscr{F}$ is non-Archimedean, the the restriction of the* QS*-topology to X is the discrete topology.*

So, if the distance d is not inducing the discrete topology on X, then the S-topology is the only QS-topology whose its restriction to X coincides with the topology of (X,d).

Proof.
If $\mathscr{F} = {}^b\mathbb{R}$, then the QS-topology is the S-topology and its well known that its restriction to X is the topology of X. If $\mathscr{F}$ in non-Archimedean, then ${}^i\mathscr{F} \subsetneq {}^i\mathbb{R}$ and radii of QS-balls can be chosen in ${}^i\mathbb{R}_+ \setminus {}^i\mathscr{F}$. For $x \in X$ and $r \in {}^i\mathbb{R}_+ \setminus {}^i\mathscr{F}$, we have

$$QS(x,r) \cap X \subset B(x,r) \cap X = \{x\}.$$

Consequently, the restriction of the QS-topology to X is the discrete topology. □

We shall indicate the notions of the QS-topology by prefixing ”QS-” to the appropriate term, e.g., QS-open set, QS-interior QS-closure.

Let $\mathscr{F}$ be a *proper* convex subring of ${}^*\mathbb{R}$.

Theorem 3.10. *Let A be any internal set in ${}^*X$ endowed with the QS-topology generated by $\mathscr{F}$. Then the following properties are equivalent:*

(1) *the point p belongs to the QS-interior of A,*
(2) $\mu_{\mathscr{F}}(p) \subset A$,
(3) $d(p,A^c) \notin {}^i\mathscr{F}$.

Where A^c stands for the complement of A to X and $d(p,A^c)$ denotes the distance between p and the internal set A^c.

Proof. $(1 \Rightarrow 2)$ If p belongs to the QS-interior of A then $QS(p,r) \subset A$ for some $r \in {}^a\mathscr{F}_+$. So we deduce that $\mu_{\mathscr{F}}(p) \subset A$ since $\mu_{\mathscr{F}}(p) \subset QS(p,r)$.

$(2 \Rightarrow 3)$ Let $\mathscr{A}$ be the internal set ${}^*\mathbb{R}$ defined by

$$\mathscr{A} = \{r \in {}^*\mathbb{R} : B(p,|r|) \subset A\}.$$

By hypothesis the internal set $\mathscr{A}$ contains arbitrarily large numbers in ${}^i\mathscr{F}$, then by the overflow of ${}^i\mathscr{F}$, $\mathscr{A}$ contains arbitrarily small numbers in $\mathscr{F} \setminus {}^i\mathscr{F}$ i.e., there is $r \in {}^a\mathscr{F}_+$ such that $B(p,r) \subset A$, we conclude that $d(p,A^c) \geq r$, hence $d(p,A^c) \notin {}^i\mathscr{F}$.

$(3 \Rightarrow 1)$ The condition $d(p,A^c) \notin {}^i\mathscr{F}$ implies that there exists $r \in {}^a\mathscr{F}_+$ such that $d(p,A^c) \geq r$. Hence $QS(p,r/2) \subset B(p,r/2) \subset A$ and the point p belongs to the QS-interior of A. □

Similarly, we prove

Theorem 3.11. *Let A be any internal set in ${}^*X$ endowed with the QS-topology generated by $\mathscr{F}$. Then the following properties are equivalent*

(1) *the point p belongs to the QS-closure of A,*
(2) $\mu_{\mathscr{F}}(p) \cap A \neq \emptyset$,
(3) $d(p,A) \in {}^i\mathscr{F}$.

It is well known that the monads of a topological space encode its topology and many topological properties as being open, closed can be characterized using monads. We show that $\mathscr{F}$-monads can also be used to describe these properties.

Proposition 3.12. *Let $A \subset X$, then*

(1) *A is open if and only if $\forall x \in A$ $\mu_{\mathscr{F}}(x) \subset {}^*A$.*
(2) *A is closed if and only if $\forall x \in X$ $((\mu_{\mathscr{F}}(x) \cap {}^*A) \neq \emptyset \Rightarrow x \in A)$.*
(3) *For $x \in X$, $x \in \overline{A}$ if and only if $\mu_F(x) \cap {}^*A \neq \emptyset$.*

Since the proof is similar to [13] p.112 or [12] Theorem 10.1.1, we give only the proof of (1).

Proof. 1) Assume that A is open. Let $x \in A$, then there exists a positive number r such that $B(x,r) \subset A$. Thus $\mu_{\mathscr{F}}(x) \subset {}^*B(x,r) \subset {}^*A$. Conversely, assume that $\mu_{\mathscr{F}}(x) \subset {}^*A$, then the sentence

$$(\exists r \in {}^*\mathbb{R}_{>0})(\forall x \in {}^*X)({}^*d(x,p) < r \Rightarrow x \in {}^*A)$$

is seen to be true by interpreting r as any positive element in ${}^i\mathscr{F}_+$. But then by transfer there is some real $r > 0$ for which $B(x,r) \subset A$ and hence A is open.

□

Theorem 3.13. *Every $\mathscr{F}$-galaxy in ${}^*X$ is QS-open and QS-closed.*

Proof. Let G be a galaxy in ${}^*X$. Then $G = \cup_{p\in G}\text{QS}(\text{p},1)$. Hence, G is open. The complement of G is union of all the other galaxies, which is also open. Thus, G is closed. $\square$.

Let (p_n) be a sequence in ${}^*X$. We shall say that the point p in ${}^*X$ is a QS-limit for (p_n), if the sequence $(p_n)_{n\in N}$ converges to p for the QS-topology i.e, for every $\varepsilon \in {}^a\mathscr{F}_+$, there exists a finite number ν such that $p_n \in \text{QS}(p,\varepsilon)$ for all finite $n > \nu$. which is clearly equivalent to: for every $\varepsilon \in {}^a\mathscr{F}_+$, there exists a finite number ν such that ${}^*d(p,p_n) < \varepsilon$ for all finite $n > \nu$.

Theorem 3.14. *Let (p_n) be an internal sequence and let p be a point in ${}^*X$. If there exists an infinite number H such that $p_N \in \mu_{\mathscr{F}}(p)$ for all infinite $N \leq H$ then p is a QS-limit of the sequence (p_n). The converse is true if $\mathscr{F}$ is generated by an asymptotic scale.*

Proof. Let $\varepsilon \in {}^a\mathscr{F}_+$. Consider the internal set defined by

$$\mathscr{A}_\varepsilon = \{n \in {}^*\mathbb{N} : {}^*d(p_n,p) < \varepsilon\}.$$

If every infinite number $N \leq H$ belongs to $\mathscr{A}_\varepsilon$, then, by the underflow principle, there is some $k_\varepsilon \in \mathbb{N}$ such that every finite $n \geq k_\varepsilon$ belongs to $\mathscr{A}_\varepsilon$. i.e., for every $n \geq k_\varepsilon$, we have ${}^*d(p_n,p) < \varepsilon$.

Conversely, let $k \in \mathbb{N}$ and consider the internal set

$$\mathscr{B}_k = \{n \in {}^*\mathbb{N} : {}^*d(p_n,p) < \lambda_k\}.$$

If the sequence (p_n) converges to p, then there is a finite number ν_k such that $n \in \mathscr{A}_k$ for all $n \in \mathbb{N}$ with $n > \nu_k$, we conclude, by the overflow principle, that there is an infinite number $H_k \in {}^*\mathbb{N}$ such that $n \in \mathscr{A}_k$ for all $n \in {}^*\mathbb{N}$ with $\nu_k \leq n \leq H_k$. By sequential comprehensiveness, there some infinite number H that is smaller than every H_k (cf. [12] Theorem 15.4.3). Thus for all infinite numbers $N \leq H$, we have ${}^*d(p_N,p) < \lambda_k$, for all $k \in \mathbb{N}$ i.e., $p_N \in \mu_{\mathscr{F}}(p)$ for all infinite numbers N smaller than H. $\square$.

Now, let (X,d), (Y,d') be two metric spaces and f be a mapping defined on set of points of ${}^*X$ and to take values in ${}^*Y$.

Definition 3.15. *We say that the function f is QS-bounded on a set D, if there exist a point $p \in {}^*Y$ and a number $m \in {}^a\mathscr{F}_+$ such that $f(D) \subset B(p,m)$.*

Theorem 3.16. *Let f be an internal function defined on the internal set D. Then the function f is QS-bounded on D if and only if $f(D)$ belongs to the same galaxy in ${}^*Y$.*

Proof. The condition in clearly necessary. The condition is also sufficient, assume that f in not QS-bounded on D. Let $\mathscr{A}$ be set defined by

$$\mathscr{A} = \{r \in {}^*\mathbb{R} : (\forall q \in {}^*Y)(\exists p \in D)\, {}^*d'(f(p),q) > |r|\}.$$

$\mathscr{A}$ is an internal set containing $\mathscr{F} \setminus {}^i\mathscr{F}$. It follows, by the overflow of $\mathscr{F}$, that $\mathscr{A}$ contains a number $\nu \in {}^*\mathbb{R} \setminus \mathscr{F}$. Let p_0 be a point in D, there is some $p \in D$, such that ${}^*d'(f(p),f(p_0)) > \nu$. Thus, $f(p)$ and $f(p_0)$ belong to different galaxies. $\square$

Let $\mathscr{F}$ be a proper convex subring of ${}^*\mathbb{R}$. The spaces ${}^*X$ and ${}^*Y$ are equipped with the respective quasi-standard topologies generated by $\mathscr{F}$.

Let $D \subset {}^*X$, $f : D \longrightarrow {}^*Y$ be a function defined on D and let p be a point which belongs to the QS-closure of D.

Definition 3.17. *We say that the point $s \in {}^*Y$ is a QS-limit of f as q approaches p in D if for every $\varepsilon \in {}^a\mathscr{F}_+$ there is $\eta \in {}^a\mathscr{F}_+$ such that ${}^*d'(f(q),s) < \varepsilon$ for all q in $D \setminus \{p\}$ for which ${}^*d(p,q) < \eta$.*

Theorem 3.18. *Suppose that D is an internal set, p is a point of the QS-closure of D, and f is an internal function defined on D. Then the point s is a QS-limit of f as x approaches p if and only if*

$$f(\mu_{\mathscr{F}}(p) \cap (D \setminus \{p\})) \subset \mu_{\mathscr{F}}(s).$$

Proof. Suppose s is a QS-limit of f as x approaches p in D. By the definition of the $\mathscr{F}$-monad of p, every $q \in \mu_{\mathscr{F}}(p)$ satisfies $d(p,q) < \eta$ for all $\eta \in {}^a\mathscr{F}_+$. Hence, if q is at the same time in $D \setminus \{p\}$ then ${}^*d'(f(q),s) < \varepsilon$ for arbitrary $\varepsilon \in {}^a\mathscr{F}_+$ i.e., $f(q) \in \mu_{\mathscr{F}}(s)$.
Suppose that the condition is satisfied, and let $\varepsilon \in {}^a\mathscr{F}_+$. Consider the set

$$\mathscr{A} = \{r \in {}^*\mathbb{R} : {}^*d'(f(q),s) < \varepsilon \text{ for every } q \in D \setminus \{p\} \cap B(p,|r|)\}.$$

$\mathscr{A}$ is an internal set containing ${}^i\mathscr{F}$. By the overflow of ${}^i\mathscr{F}$, the set $\mathscr{A}$ contains $r \in \mathscr{F} \setminus {}^i\mathscr{F}$. Setting $\eta = |r|$, we see that the condition of the theorem is also sufficient.

3.1. **QS-continuity.** Let f be defined on D and $p \in D$. We say that f is QS-continuous at p if the function f is continuous at p from D to ${}^*Y$ and both of these spaces are equipped with the respective QS-topologies generated by $\mathscr{F}$. That is, for every $\varepsilon \in {}^a\mathscr{F}_+$ there exists $\eta \in {}^a\mathscr{F}_+$ such that ${}^*d'(f(q),f(q)) < \varepsilon$ for all q in D such that ${}^*d(p,q) < \eta$.

Using Theorem 3.18, we deduce the following characterization of the continuity in terms of $\mathscr{F}$-monads.

Theorem 3.19. *Let f be an internal mapping from an internal set D into ${}^*Y$. Then the function f is QS-continuous at a point p in D if and only if $f(\mu_{\mathscr{F}}(p) \cap D) \subset \mu_{\mathscr{F}}(f(p))$, that is,*

$$f(x) \approxeq f(p) \text{ for all } x \in D \text{ such that } x \approxeq p.$$

Given a standard mapping $f : (X,d) \longrightarrow (Y,d')$ and $p \in X$. By transfer, one shows that f is continuous at p if and only if the extension mapping ${}^*f$ is Q-continuous at p. On the other hand, by the non-standard characterization of the continuity, the mapping f is continuous at p if and only if ${}^*f$ is S-continuous at p. A natural question arises, if the continuity of a standard mapping f is equivalent to the QS-continuity of f?
We prove that if ${}^*f$ is QS-continuous then f is continuous but the converse is false in general.

Proposition 3.20. *Let $f : (X,d) \longrightarrow (Y,d')$ be a mapping and $p \in X$. If ${}^*f$ is QS-continuous at p then f is continuous at p.*

Proof. Let ε be a positive real number. The QS-continuity of ${}^*f$ at p implies that the following assertion

$$\exists(\eta \in {}^*\mathbb{R}_{>0})(\forall x \in {}^*X)({}^*d(x,p) < \eta) \Rightarrow ({}^*d'(f(x),f(p)) < \varepsilon)$$

is true by interpreting η as any positive element in ${}^i\mathscr{F}$. Therefore, by transfer there exists a positive standard number η, such that

$$(\forall x \in X)(d(x,p) < \eta) \Rightarrow (d'(f(x),f(p)) < \varepsilon).$$

So we conclude that f is continuous at p.

□

The following shows that the converse of Proposition 3.20 is false in general.

Example 3.21. *Let f be the standard function defined on $\mathbb{R}$ by:*

$$f(t) = \begin{cases} -1/\log|t| & \text{if } t \neq 0, \\ 0 & \text{if } t = 0\,. \end{cases}$$

The function f is continuous at 0. But ${}^*f$ is not QS-continuous at 0 with respect to the QS-topology generated by the scale (ρ^n), where ρ is a positive infinitesimal number. Since the number $x = e^{-1/\rho}$ is an iota but $f(x) = \rho$ is not an iota.

If the function is internal not necessarily standard, in general, there is no relationship between Q-continuity, QS-continuity and S-continuity:

Example 3.22. *$\mathscr{F}$ be the convex subring generated by an asymptotic scale (λ_n) and let $f(t) = \omega t$, where $\omega \in {}^*\mathbb{R} \setminus \{0\}$. Clearly, the internal function f is Q-continuous at 0.*

1. *If ω is bounded, i.e., $\omega \in {}^b\mathbb{R}$, then f is S-continuous and QS-continuous at 0.*
2. *If ω is moderate but infinite e.g. $\omega = \dfrac{1}{\lambda_n}$ for some $n \in \mathbb{N}$, then f is QS-continuous at 0 but not S-continuous at 0.*
3. *If ω is not moderate, i.e. $\omega = \dfrac{1}{\lambda_N}$ for some $n \in \mathbb{N}^\infty$, then f is neither S-continuous at 0 nor QS-continuous at 0.*

The Example 3.21 shows that being continuous is not a sufficient condition for the function to send $\mathscr{F}$-monads to $\mathscr{F}$-monads. This shows that $\mathscr{F}$-monads depend on the metric and not only on the topology. We provide an example of two equivalent metrics but having different $\mathscr{F}$-monads.

Example 3.23.
Let $\mathscr{F}$ be the convex ring generated by the scale (ρ^n) and f be the function defined in Example 3.21. Consider on $\mathbb{R}$ the following metric $\delta(x,y) = |x-y| + |f(x) - f(y)|$ which is equivalent to the absolute value $|.|$. One can easily verify that $\mu^{\delta}_{\mathscr{F}}(0) \subsetneq \mu^{|.|}_{\mathscr{F}}(0)$, (see Example 3.21).

The following is straightforward

Proposition 3.24. *If $f : X \to Y$ is a locally Lipschitz continuous mapping then ${}^*f$ is QS-continuous on $\mathrm{ns}({}^*X)$ that is for every pair of points p,q in $\mathrm{ns}({}^*X)$, $p \approxeq q$ implies $f(p) \approxeq f(q)$.*

Consequently any continuously differentiable function is QS-continuous and since all norms on a finite-dimensional vector space are equivalent, it follows

Corollary 3.25. *On a finite-dimensional vector space X, the $\mathscr{F}$-monad system depends only on the topology of X.*

The following gives a sufficient condition on the metrics to induce the same $\mathscr{F}$-monad system.

Corollary 3.26. *Two Lipschitz equivalent metrics d and δ on X provide the same $\mathscr{F}$-monad system, i.e., for all $x \in {}^*X$, $\mu^{d}_{\mathscr{F}}(x) = \mu^{\delta}_{\mathscr{F}}(x)$.*

Recall that two metrics d and δ on X are Lipschitz equivalent if there exist α, β two positive numbers such that

$$\alpha d(x,y) \leq \delta(x,y) \leq \beta d(x,y) \quad \text{for all } x,y \in X.$$

4. $\mathscr{F}$-ASYMPTOTIC HULLS OF METRIC SPACES

The concept of nonstandard hull of a metric space was introduced by Luxemburg [18] and has proved to be powerful tool in nonstandard analysis of Banach spaces. Later Luxemburg [19] extended this construction to the non-Archimedean case, by defining ${}^{\rho}\widehat{E}$ a normed linear space over ${}^{\rho}\mathbb{R}$, the Robinson field of asymptotic numbers, see [17],[27],[21].

In this section, we generalize these constructions by considering the $\mathscr{F}$-nonstandard hull of a metric space where $\mathscr{F}$ is a proper convex subring of ${}^*\mathbb{R}$. We note that the mentioned constructions follow as a particular case for $\mathscr{F} = {}^b\mathbb{R}$ and $\mathscr{F} = \mathscr{M}_\rho$, see Example 2.8.

Let (X,d) be a metric space and let $\mathscr{F}({}^*X)$ denote the principal galaxy in ${}^*X$ with respect to $\mathscr{F}$ i.e., $\mathscr{F}({}^*X)$ is the set of $\mathscr{F}$-bounded points in ${}^*X$.

Let
$$\widehat{X} := \mathscr{F}({}^*X)/\approxeq .$$
The space $\widehat{X}$ is a "generalized metric space" that is $\widehat{X}$ is equipped with $\widehat{d} : \widehat{X} \times \widehat{X} \longrightarrow \widehat{\mathscr{F}}$ defined as follows
$$\widehat{d}(\widehat{x},\widehat{y}) := \widehat{\mathrm{st}}({}^*d(x,y)) \;\text{ for }\; \widehat{x},\widehat{y} \in \widehat{X},$$
and $\widehat{d}$ verifies the usual properties of a metric.

The space $\widehat{X}$ is equipped with the quotient topology of the restriction of QS-topology of ${}^*X$ to $\mathscr{F}({}^*X)$. The collection of balls
$$\widehat{B}(\widehat{x},R) := \{\widehat{y} \in \widehat{X} \,:\, \widehat{d}(\widehat{x},\widehat{y})) < R\},$$
where $\widehat{x} \in \widehat{X}$ and $R \in \widehat{\mathscr{F}}^+$ is a basis for the the quotient topology on $\widehat{X}$.

The following is immediate

Proposition 4.1. *The space $\widehat{X}$ is Hausdorff and the quotient mapping $\widehat{\mathrm{st}} : \mathscr{F}({}^*X) \longrightarrow \widehat{X}$ is continuous, open.*

Theorem 4.2. *Let $f : {}^*X \longrightarrow {}^*Y$ be an internal map and the restriction of f to $\mathscr{F}({}^*X)$ is QS-continuous and f sends $\mathscr{F}({}^*X)$ to $\mathscr{F}({}^*Y)$. Then f gives rise to a continuous mapping $\widehat{f} : \widehat{X} \longrightarrow \widehat{Y}$ defined by*
$$\widehat{f}(\widehat{x}) = \widehat{f(x)} \quad \textit{for all } \widehat{x} \in \widehat{X}.$$

Proof. The mapping $\widehat{\mathrm{st}} \circ f|_{\mathscr{F}({}^*X)} : \mathscr{F}({}^*X) \xrightarrow{f|_{\mathscr{F}({}^*X)}} \mathscr{F}({}^*Y) \xrightarrow{\widehat{\mathrm{st}}} \widehat{Y}$ is continuous. By Theorem 3.19, the condition on QS-continuity implies that for all $x,y \in \mathscr{F}({}^*X)$, we have $f(x) \approxeq f(y)$ whenever $x \approxeq y$. Therefore the mapping $\widehat{\mathrm{st}} \circ f|_{\mathscr{F}({}^*X)}$ descends to a continuous quotient mapping $\widehat{f} : \widehat{X} \longrightarrow \widehat{Y}$ and the following diagram commutes

$$\begin{array}{ccc} \mathscr{F}({}^*X) & \xrightarrow{f|_{\mathscr{F}({}^*X)}} & \mathscr{F}({}^*Y) \\ \widehat{\mathrm{st}}\Big\downarrow & & \Big\downarrow\widehat{\mathrm{st}} \\ \widehat{X} & \xrightarrow{\widehat{f}} & \widehat{Y} \end{array}$$

$\square$.

The latter Theorem can be reformulated as

Theorem 4.3. *Let $f : {}^*X \longrightarrow {}^*Y$ be an internal map such that*

(1) *f sends $\mathscr{F}({}^*X)$ to $\mathscr{F}({}^*Y)$,*
(2) *for all $x,x' \in \mathscr{F}({}^*X)$, if $x \approxeq x'$, then $f(x) \approxeq f(x')$.*

Then f gives rise to a continuous mapping $\widehat{f} : \widehat{X} \longrightarrow \widehat{Y}$ defined by
$$\widehat{f}(\widehat{x}) = \widehat{f(x)}.$$

In other words, the conditions to ensure that the mapping f descends to the quotient mapping are sufficient to induce the continuity of $\widehat{f}$. A variant of such result was proved by Vernaeve [30] and Delcroix [7] in the context of Colombeau's generalized functions, see Colombeau [4, 5]. In fact it is not surprising to get the continuity of $\widehat{f}$, since the QS-continuity of the function f is expressed in terms of $\mathscr{F}$-monads.

Let $\mathscr{F}$ be a *proper and non-Archimedean* ring and $p \in X$, then the standard monad of p, $\mu(p)$ is QS-open. Since there exists $r \in {}^i\mathbb{R}^+ \setminus {}^i\mathscr{F}$ such that $QS(p,r) \subset B(p,r) \subset \mu(p)$. Hence $\mathrm{ns}({}^*X)$ is QS-open, where $\mathrm{ns}({}^*X)$ denotes the set of near-standard elements of ${}^*X$, that is, $\mathrm{ns}({}^*X) = \cup_{p\in X}\mu(p)$.

In general, the set of near-standard points in not S-open, however, if we assume that X is locally compact then $\mathrm{ns}({}^*X) = \mathrm{cpt}({}^*X)$ where

$$\mathrm{cpt}({}^*X) = \bigcup_{K \Subset X} {}^*K.$$

Therefore $\mathrm{ns}({}^*X)$ is S-open.

Furthermore, if X is a *proper* metric space or a Heine-Borel metric space, that is, a metric space in which every closed ball is compact then $\mathrm{ns}({}^*X) = \mathrm{cpt}({}^*X) = \mathrm{bd}({}^*X)$, where $\mathrm{bd}({}^*X)$ denotes the set of bounded points of ${}^*X$.

Corollary 4.4. *Let $\mathscr{F}$ be a non-Archimedean ring and $f : {}^*X \longrightarrow {}^*Y$ be an internal map such that*

(1) *f sends $\mathrm{ns}({}^*X)$ to $\mathscr{F}({}^*Y)$,*
(2) *for all $x, x' \in \mathrm{ns}({}^*X)$, if $x \approxeq x'$ then $f(x) \approxeq f(x')$.*

Then f gives rise to a continuous mapping $\widehat{f} : \widehat{\mathrm{ns}({}^*X)} \longrightarrow \widehat{Y}$ defined by

$$\widehat{f}(\widehat{x}) = \widehat{f(x)}, \quad \textit{for all } \widehat{x} \in \widehat{\mathrm{ns}({}^*X)}.$$

When X is an open set of $\mathbb{R}^d$, $\widehat{\mathrm{ns}({}^*X)}$ is used to construct a generalized pointvalues of an asymptotic function, see [28] and $\widehat{\mathrm{ns}({}^*X)}$ is called the set of generalized compactly supported points. A similar construction in Colombeau's theory can be found in [15].

We recall that if an internal function $f : {}^*X \longrightarrow {}^*Y$ is S-continuous on $\mathrm{ns}({}^*X)$ then f sends $\mathrm{ns}({}^*X)$ to $\mathrm{ns}({}^*Y)$.

Corollary 4.5. *Let $\mathscr{F}$ be a non-Archimedean ring and $f : {}^*X \longrightarrow {}^*Y$ be an internal S-continuous mapping such that for all $x, x' \in \mathrm{ns}({}^*X)$, if $x \approxeq x'$ then $f(x) \approxeq f(x')$.*

Then f gives rise to a continuous mapping $\widehat{f} : \widehat{\mathrm{ns}({}^*X)} \longrightarrow \widehat{\mathrm{ns}({}^*Y)}$ defined by

$$\widehat{f}(\widehat{x}) = \widehat{f(x)}, \quad \textit{for all } \widehat{x} \in \widehat{\mathrm{ns}({}^*X)}.$$

Recall that $\widehat{X}$ is equipped with the generalized metric $\widehat{d}$, hence it induces a uniformity on $\widehat{X}$ and the collection

$$\{(\widehat{x},\widehat{y}) \in \widehat{X}^2 : \widehat{d}(\widehat{x},\widehat{y})) < R\}, \quad (R \in \widehat{\mathscr{F}^+})$$

is a fundamental system of entourages for a uniformity on $\widehat{X}$ which induces the quotient topology on $\widehat{X}$. Kalsich [14] proved that the converse is also true, that is, any uniformity is induced by a generalized metric. Now, it is natural to ask when the topology of $\widehat{X}$ is metrizable.

Theorem 4.6 (Bourbaki [2], Theorem 1 p.152)**.** *A uniformity is metrizable if and only if it is Hausdorff and the filter of entourages of the uniformity has a countable base*

It is shown that a uniform structure that admits a countable fundamental system of entourages (and hence in particular a uniformity defined by a countable family of pseudometrics) can be defined by a single pseudometric which turns to be a metric if in addition the space is Hausdorff.

Since $\widehat{X}$ is Hausdorff, we deduce that $\widehat{X}$ is metrizable if and only if the uniformity has a countable base.

Let (λ_n) be an asymptotic scale, then the collection

$$U_n := \{(\widehat{x},\widehat{y}) \in \widehat{X}^2 : \widehat{d}(\widehat{x},\widehat{y})) < \widehat{\lambda_n}\},$$

is a (countable) fundamental system of entourages for the uniformity on $\widehat{X}$ induced by the generalized metric $\widehat{d}$.

Theorem 4.7. *Let (λ_n) be an asymptotic scale. Then the quotient topology on $\widehat{X}$ generated by (λ_n) is metrizable. Furthermore, the following function*

$$\delta(\widehat{x},\widehat{y}) = \exp\left(-\sup\left\{ n \in \mathbb{Z} : \frac{{}^*d(x,y)}{\lambda_n} \in {}^b\mathbb{R} \right\}\right)$$

is an ultra-metric generating the uniformity on $\widehat{X}$ defined by $\widehat{d}$.

Before giving the proof, let us provide some properties of the sequence

$$\mu_n(x,y) := \frac{{}^*d(x,y)}{\lambda_n}.$$

The sequence $(\mu_n(x,y))_{n\in\mathbb{Z}}$ is increasing and for $n,m \in \mathbb{Z}$, we have $\mu_n(x,y) = \mu_m(x,y)\dfrac{\lambda_m}{\lambda_n}$. Define

$$q(x,y) = \sup\left\{ n \in \mathbb{Z} : \mu_n(x,y) \in {}^b\mathbb{R} \right\} \in \mathbb{Z} \cup \{+\infty\}.$$

If $q(x,y) \in \mathbb{Z}$, one can easily check that

$$\mu_n(x,y) \in \begin{cases} {}^i\mathbb{R} & \text{if } n < q(x,y), \\ {}^b\mathbb{R} & \text{if } n = q(x,y)\ , \\ {}^\infty\mathbb{R} & \text{if } n > q(x,y). \end{cases} \tag{4.2}$$

If $q(x,y) = +\infty$, i.e, for all $n \in \mathbb{Z}$, $\mu_n(x,y) \in {}^b\mathbb{R}$ which is equivalent to for each $n \in \mathbb{N}$, ${}^*d(x,y) \leq \lambda_n$, that is, ${}^*d(x,y) \approxeq 0$.

Proof. First, one can verify that if $x_1 \approxeq x_2$ and $y_1 \approxeq y_2$, then

$$\sup\left\{ n \in \mathbb{Z} : \frac{{}^*d(x_1,y_1)}{\lambda_n} \in {}^b\mathbb{R} \right\} = \sup\left\{ n \in \mathbb{Z} : \frac{{}^*d(x_2,y_2)}{\lambda_n} \in {}^b\mathbb{R} \right\}. \tag{4.3}$$

Hence δ is well defined. Now, we have to check that δ is an ultra-metric on $\widehat{X}$.

Clearly $\delta(\widehat{x},\widehat{y}) = 0$ if and only if ${}^*d(x,y) \leq \lambda_n$ for all $n \in \mathbb{N}$, that is, $\widehat{x} = \widehat{y}$. It is obvious that δ is symmetric. It remains to prove the stronger triangle inequality

$$\delta(\widehat{x},\widehat{z}) \leq \max\left(\delta(\widehat{x},\widehat{y}), \delta(\widehat{y},\widehat{z})\right), \quad \text{for all } \widehat{x},\widehat{y},\widehat{z} \in \widehat{X}.$$

Assume without loss of generality that $0 < \delta(\widehat{x},\widehat{y}) \leq \delta(\widehat{y},\widehat{z})$. Hence there are two integers $N, M \in \mathbb{Z}$, such that $M \leq N$ and $\delta(\widehat{x},\widehat{y}) = \mathrm{e}^{-N}$, $\delta(\widehat{y},\widehat{z}) = \mathrm{e}^{-M}$. The following inequality

$$\frac{{}^*d(x,z)}{\lambda_M} \leq \frac{{}^*d(x,y)}{\lambda_M} + \frac{{}^*d(y,z)}{\lambda_M}$$

combined with (4.2) imply that $\dfrac{{}^*d(x,z)}{\lambda_M} \in {}^b\mathbb{R}$ as $\dfrac{{}^*d(y,z)}{\lambda_M} \in {}^b\mathbb{R}$ and $\dfrac{{}^*d(x,y)}{\lambda_M} \in {}^i\mathbb{R}$.

Finally, we have to show that the uniformity induced by the metric δ coincides with the uniformity induced by the generalized metric $\hat{d}$.

Let

$$V_r = \{(\widehat{x},\widehat{y}) \in \widehat{X}^2 \, : \, \delta(\widehat{x},\widehat{y}) < r\}, \ r \in \mathbb{R}^+$$

stand for a fundamental system of entourages for the uniformity induced by the metric δ.

Consider an entourage V_r. Take any $n > -\ln(r/2)$, then we easily prove that $U_n \subset V_r$. Conversely, if we take an entourage U_n element in the base of the uniformity induced by $\widehat{d}$, it suffices to consider any positive real number r, such that $\ln r < -n$ to have $V_r \subset U_n$. □

We should notice that the metric δ induces on X the discrete metric defined by

$$\delta_{|X}(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y\,. \end{cases}$$

Theorem 4.8. *The metric space $(\widehat{X},\delta)$ and the generalized metric space $(\widehat{X},\widehat{d})$ are Cauchy complete.*

Proof. Since the uniformity generated by δ is the same uniformity generated by $\widehat{d}$, we have only to prove that $(\widehat{X},\widehat{d})$ is Cauchy complete.

Let $\{\widehat{x_n} : n \in \mathbb{N}\}$ be a Cauchy sequence in $(\widehat{X},\widehat{d})$. The sequence $\{x_n : n \in \mathbb{N}\}$ of points in $\mathscr{F}({}^*X)$ extends to an internal hyper-sequence $\{x_n : n \in {}^*\mathbb{N}\}$ in ${}^*X$.

For each $n \in \mathbb{N}$, by the Cauchy property there exists $k_n \in \mathbb{N}$ such that for all standard $m \geq k_n$,

$$ {}^*d(x_m, x_{k_n}) < \lambda_{n+1}. \tag{4.4}$$

But the set $\{m \in {}^*\mathbb{N} : {}^*d(x_m, x_{k_n}) < \lambda_{n+1}\}$ is an internal set, so we conclude that there is some infinite $\omega_n \in {}^*\mathbb{N}$ such that (4.4) holds for all $n \in {}^*\mathbb{N}$ with $k_n \leq m \leq \omega_n$. By sequential comprehensiveness, there is some infinite $\omega \in {}^*\mathbb{N}$ that is smaller than every ω_n and for all $m \in \mathbb{N}$, with $m \geq k_n$, we have

$${}^*d(x_m, x_\omega) \leq {}^*d(x_m, x_{k_n}) + {}^*d(x_{k_n}, x_\omega) \leq 2\lambda_{n+1} < \lambda_n.$$

It follows that x_ω is $\mathscr{F}$-bounded and $\{\widehat{x_n} : n \in \mathbb{N}\}$ converges to $\widehat{x_\omega}$.

□

Finally, we will compare on ${}^\rho\mathbb{R}$, the field of Robinson's real ρ-asymptotic numbers, the classical valuation v introduced by Robinson and defined by $v(\widehat{x}) = \mathrm{st}(\ln_\rho |x|)$, where st is the standard part mapping in ${}^*\mathbb{R}$ and the function $p : {}^\rho\mathbb{R} \to \mathbb{Z} \cup \{+\infty\}$ defined by

$$p(\widehat{x}) = q(x,0) = \sup\left\{ n \in \mathbb{Z} \, : \, \frac{|x|}{\rho^n} \in {}^b\mathbb{R} \right\}.$$

Using the equality (4.3), we deduce that p is well defined.

Proposition 4.9. *The function p is a pseudo-valuation on ${}^\rho\mathbb{R}$ compatible with its order, trivial on $\mathbb{R}$ and satisfying:*

(i) $p(\lambda\widehat{x}) = p(\widehat{x})$, *for any* $\widehat{x} \in {}^\rho\mathbb{R}$ *and* $\lambda \in \mathbb{R} \setminus \{0\}$,
(ii) $p(\widehat{x}) \leq v(\widehat{x})$, *for any* $\widehat{x} \in {}^\rho\mathbb{R}$.

Furthermore p is not a valuation on ${}^{\rho}\mathbb{R}$.

Proof. We leave the verification to the reader that p is a pseudo-valuation satisfying (i) and compatible with the order of ${}^{\rho}\mathbb{R}$. Now, we shall prove the property (ii).

$$p(\widehat{x}) \leq v(\widehat{x}). \tag{4.5}$$

Indeed, let $n \in \mathbb{Z}$ such that $\dfrac{|x|}{\rho^n}$ is bounded, i.e, there is a positive real number M such that $\dfrac{|x|}{\rho^n} \leq M$. It follows that $\ln_\rho(|x|) \geq \ln_\rho(M) + n$. Hence $\mathrm{st}(\ln_\rho |x|) \geq n$ and $v(\widehat{x}) \geq N(\widehat{x})$. Finally, to prove that p is not a valuation, that is, the condition (2.1) is not satisfied, it suffices to remark that $p(\widehat{\ln\rho^2}) = -1$ and $p(\widehat{\ln\rho}) = -1$. ☐

Theorem 4.10. *The pseudo-valuation p and the valuation v induce the same topology on ${}^{\rho}\mathbb{R}$ which is the order topology and all the topologies are compatible with the field structure of ${}^{\rho}\mathbb{R}$.*

Proof. It suffices to prove that for all $n \in \mathbb{N}^+$, we have

$$\{\widehat{x} \in {}^{\rho}\mathbb{R} : p(\widehat{x}) > n\} \subset \{\widehat{x} \in {}^{\rho}\mathbb{R} : v(\widehat{x}) > n\} \subset \{\widehat{x} \in {}^{\rho}\mathbb{R} : p(\widehat{x}) \geq n\}.$$

The first inclusion follows from the inequality (4.5). For the second, let $\widehat{x} \in {}^{\rho}\mathbb{R}$ and $n \in \mathbb{N}^+$ such that $v(\widehat{x}) > n$, that is, $\mathrm{st}(\ln_\rho |x|) > n$. It follows that $\ln_\rho |x| > n$. Therefore $|x| < \rho^n$ and $p(\widehat{x}) \geq n$. ☐

Remark 4.11. *It is not surprising that the order topology on ${}^{\rho}\mathbb{R}$ can be defined by a pseudo-valuation not a necessary a valuation. The set $H = \{\widehat{x} \in {}^{\rho}\mathbb{R} :$ there exist $M \in \mathbb{R}^+, n \in \mathbb{N}^+$ and $|x| \leq M\rho^n\}$ satisfies all the requirements in Cohn's Theorem 2.2, that is, H is open, closed under addition and contains only nilpotent elements. Indeed, one can easily show that $H = \{\widehat{x} \in {}^{\rho}\mathbb{R} : p(\widehat{x}) > 0\}$.*

Appendix A. Spilling Principles

We recall several spilling principles in terms of a *proper* convex subring $\mathscr{F}$ of ${}^*\mathbb{R}$. We should note that the familiar underflow and overflow principles in non-standard analysis follow as a particular case for $\mathscr{F} = {}^b\mathbb{R}$.

Theorem A.1 (Spilling Principles)**.** [28] *Let $\mathscr{F}$ be a proper convex subring of ${}^*\mathbb{R}$ and $\mathscr{A} \subset {}^*\mathbb{R}$ be an internal set. Then:*

(i) *Overflow of $\mathscr{F}$: If $\mathscr{A}$ contains arbitrarily large numbers in $\mathscr{F}$, then $\mathscr{A}$ contains arbitrarily small numbers in ${}^*\mathbb{R} \setminus \mathscr{F}$. In particular,*

$$\mathscr{F} \setminus {}^i\mathscr{F} \subset \mathscr{A} \Rightarrow \mathscr{A} \cap ({}^*\mathbb{R} \setminus \mathscr{F}) \neq \emptyset$$

(ii) *Underflow of $\mathscr{F} \setminus {}^i\mathscr{F}$: If $\mathscr{A}$ contains arbitrarily small numbers in $\mathscr{F} \setminus {}^i\mathscr{F}$, then $\mathscr{A}$ contains arbitrarily large numbers in ${}^i\mathscr{F}$. In particular,*

$$\mathscr{F} \setminus {}^i\mathscr{F} \subset \mathscr{A} \Rightarrow \mathscr{A} \cap {}^i\mathscr{F} \neq \emptyset$$

(iii) *Overflow of ${}^i\mathscr{F}$: If $\mathscr{A}$ contains arbitrarily large numbers in ${}^i\mathscr{F}$, then $\mathscr{A}$ contains arbitrarily small numbers in $\mathscr{F} \setminus {}^i\mathscr{F}$. In particular,*

$${}^i\mathscr{F} \subset \mathscr{A} \Rightarrow \mathscr{A} \cap (\mathscr{F} \setminus {}^i\mathscr{F}) \neq \emptyset$$

(iv) *Underflow of ${}^*\mathbb{R} \setminus \mathscr{F}$: If $\mathscr{A}$ contains arbitrarily small numbers in ${}^*\mathbb{R} \setminus \mathscr{F}$, then A contains arbitrarily large numbers in $\mathscr{F}$. In particular,*

$${}^*\mathbb{R} \setminus \mathscr{F} \subset \mathscr{A} \Rightarrow \mathscr{A} \cap (\mathscr{F} \setminus {}^i\mathscr{F}) \neq \emptyset$$

We should mention that these spilling principles *fail* if $\mathscr{F} = {}^*\mathbb{R}$ and ${}^i\mathscr{F} = \{0\}$.

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