A study of irreducible polynomials over henselian valued fields via distinguished pairs*

Kamal Aghigh,

Department of Mathematics,

K. N. Toosi University of Technology,

Tehran, Iran

aghigh@kntu.ac.ir,

Anuj Bishnoi, Sanjeev Kumar

Department of Mathematics, Panjab University, Chandigarh-160014, India.

anuj.bshn@gmail.com, sanjeev_kumar_19@yahoo.co.in

and

Sudesh K. Khanduja†

Indian Institute of Science Education and Research (IISER, Mohali)

Sector-81, S. A. S. Nagar-140306, Punjab, India.

skhanduja@iisermohali.ac.in

Abstract

In this paper, we give an introduction of the phenomenon of lifting with respect to residually transcendental extensions, the notion of distinguished pairs and complete distinguished chains which lead to the study of certain invariants associated to irreducible polynomials over valued fields. We give an overview of various results regarding these concepts and their applications.

Keywords : Valued fields; Non-Archimedean valued fields; Irreducible polynomials.

2010 Mathematics Subject Classification : 12J10; 12J25; 12E05.

*The financial support by National Board for Higher Mathematics, Mumbai and CSIR(grant no. 09/135(0598)/2010-EMR-I) is gratefully acknowledged.

†All correspondence may be addressed to this author.

It is an old and interesting problem to study the properties of a given irreducible polynomial with coefficients in a valued field (K, v). Some of the important tools of valuation theory which are used extensively in studying such problems are the notion of lifting of polynomials and distinguished pairs. We first briefly recall these concepts along with a survey of results obtained in this direction and state some results regarding the connection between the two concepts.

In what follows, v is a henselian Krull valuation of arbitrary rank of a field K with valuation ring R_v, maximal ideal M_v, residue field $\overline{K} = R_v/M_v$ and $\tilde{v}$ is its unique prolongation to a fixed algebraic closure $\widetilde{K}$ of K having value group $\widetilde{G}$. For an element ξ belonging to the valuation ring of $\tilde{v}$, $\overline{\xi}$ will denote its $\tilde{v}$-residue, i.e., the image of ξ under the canonical homomorphism from the valuation ring $R_{\tilde{v}}$ of $\tilde{v}$ onto its residue field and for a polynomial $f(x) \in R_{\tilde{v}}[x]$, $\bar{f}(x)$ will stand for the polynomial over the residue field of $\tilde{v}$ obtained by replacing each coefficient of $f(x)$ by its $\tilde{v}$-residue. For any subfield L of $\widetilde{K}$, $\overline{L}$, $G(L)$ will denote respectively the residue field and the value group of the valuation of L which is the restriction of $\tilde{v}$ to L. Any irreducible polynomial $x^n + \overline{a_{n-1}}x^{n-1} + \cdots + \overline{a_0}$ belonging to $\overline{K}[x]$ can be lifted in the usual way to yield irreducible polynomials over R_v of degree n. In 1995, Popescu and Zaharescu [25] extended the notion of usual lifting by introducing lifting with respect to a residually transcendental extension. Recall that a prolongation w of v to a simple transcendental extension $K(x)$ of K is called residually transcendental if the residue field of w is a transcendental extension of the residue field of v. In 1983, Ohm [23] proved the well known Ruled Residue Theorem conjectured by Nagata [22], which says that the residue field of a residually transcendental prolongation w of v to $K(x)$ is a simple transcendental extension of a finite extension of the residue field of v. Popescu et al. in a series of papers that followed, characterized residually transcendental extensions through minimal pairs defined below (cf. [4], [5]).

Definition. Let (K, v), $(\widetilde{K}, \tilde{v})$ be as above and $\widetilde{G}$ be the value group of $\tilde{v}$. A pair $(\alpha, \delta) \in \widetilde{K} \times \widetilde{G}$ will be called a *minimal pair* (more precisely, a (K, v)-minimal pair) if whenever β belongs to $\widetilde{K}$ with $[K(\beta) : K] < [K(\alpha) : K]$, then $\tilde{v}(\alpha - \beta) < \delta$.

For a (K, v)- minimal pair (α, δ), we shall denote by $\widetilde{w}_{\alpha,\delta}$ the valuation of

$\widetilde{K}(x)$ defined by

$$\widetilde{w}_{\alpha,\delta}\Big(\sum_i c_i(x-\alpha)^i\Big) = \min_i \{\tilde{v}(c_i) + i\delta\},\ c_i \in \widetilde{K};$$

its restriction to $K(x)$ will be denoted by $w_{\alpha,\delta}$. It is known that a prolongation w of v to $K(x)$ is residually transcendental if and only if $w = w_{\alpha,\delta}$ for some (K,v)-minimal pair (α,δ) (cf. [4, Theorem 2.1]). It is also known that for two (K,v)-minimal pairs (α,δ), (α_1,δ_1) $w_{\alpha,\delta} = w_{\alpha_1,\delta_1}$ if and only if $\delta = \delta_1$ and $\tilde{v}(\alpha_1 - \alpha') \geq \delta$ for some K-conjugate α' of α (see [3] ,[5], [21, Theorem 2.1]). So the valuation $w_{\alpha,\delta}$ is determined by the minimal polynomial $f(x)$ (say) of α over K and δ. A simple description of $w_{\alpha,\delta}$ and its residue field is given by the result stated below proved in [4] using $f(x)$-expansion[1] of any given polynomial $g(x) \in K[x]$.

Theorem 1. *Let $(K,v),(\widetilde{K},\tilde{v})$ be as above and (α,δ) be a (K,v)-minimal pair. Let $f(x)$ be the minimal polynomial of α over K of degree m and let λ stand for $w_{\alpha,\delta}(f(x))$. Then the following hold:*

(i) For any polynomial $g(x)$ belonging to $K[x]$ with $f(x)$-expansion $\sum_i g_i(x)f(x)^i$, $\deg g_i(x) < m$, one has $w_{\alpha,\delta}(g(x)) = \min_i \{\tilde{v}(g_i(\alpha)) + i\lambda\}$.

(ii) Let e be the smallest positive integer such that $e\lambda \in G(K(\alpha))$ and $h(x)$ belonging to $K[x]$ be a polynomial of degree less than m with $\tilde{v}(h(\alpha)) = e\lambda$, then the $w_{\alpha,\delta}$-residue $\Big(\overline{\dfrac{f(x)^e}{h(x)}}\Big)$ of $\dfrac{f(x)^e}{h(x)}$ is transcendental over $\overline{K(\alpha)}$ and the residue field of $w_{\alpha,\delta}$ is canonically isomorphic to $\overline{K(\alpha)}\Big(\Big(\overline{\dfrac{f(x)^e}{h(x)}}\Big)\Big)$.

The above theorem proved in 1988 led Popescu and Zaharescu [25] to generalize the notion of usual lifting of polynomials from $\overline{K}[x]$ to $K[x]$. In this attempt, they introduced the concept of lifting of a polynomial belonging to $\overline{K(\alpha)}[Y]$ (Y an indeterminate) with respect to a (K,v)-minimal pair (α,δ) as follows:

Definition. For a (K,v)-minimal pair (α,δ), let $f(x), m, \lambda, e$ and $h(x)$ be as in Theorem 1. A monic polynomial $F(x)$ belonging to $K[x]$ is said to be a *lifting* of

---

[1] The expansion of a polynomial $g(x) \in K[x]$ obtained on dividing it by successive powers of $f(x)$ of the type $\sum_i g_i(x)f(x)^i, g_i(x) \in K[x], \deg g_i(x) < \deg f(x)$, is called the $f(x)$-expansion of $g(x)$.

a monic polynomial $T(Y)$ belonging to $\overline{K(\alpha)}[Y]$ having degree $t \geq 1$ with respect to (α, δ) if the following three conditions are satisfied:

(i) $\deg F(x) = etm$,

(ii) $w_{\alpha,\delta}(F(x)) = w_{\alpha,\delta}(h(x)^t) = et\lambda$,

(iii) the $w_{\alpha,\delta}$-residue of $\dfrac{F(x)}{h(x)^t}$ is $T\Big(\overline{\dfrac{f(x)^e}{h(x)}}\Big)$.

To be more precise, the above lifting will be referred to as the one with respect to (α, δ) and $h(x)$. This notion of lifting extends the usual one because a usual lifting $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ of a polynomial $x^n + \overline{a_{n-1}}x^{n-1} + \cdots + \overline{a_0} \in \overline{K}[x]$ is lifting with respect to the minimal pair (0,0) and $h = 1$. In 1997, Khanduja and Saha proved that a polynomial belonging to $K[x]$, which is a lifting of a monic irreducible polynomial $T(Y) \neq Y$ belonging to $\overline{K(\alpha)}[Y]$ with respect to a (K, v)-minimal pair (α, δ), is irreducible over K (see [19, Theorem 2.2]). As a consequence of this result, they also extended Schönemann Irreducibility Criterion [26, Chapter 3, D] and Eisenstein Irreducibility Criterion for polynomials over arbitrary valued fields; in fact polynomials satisfying the hypothesis of these criteria were shown to be liftings of linear polynomials with respect to some suitable (K, v)-minimal pairs (cf. [19]).

Clearly there are many liftings of a given polynomial with respect to a minimal pair (α, δ). This leads to the following natural question:

> *Let (K, v) be a henselian valued field of any rank and (α, δ) be a (K, v)-minimal pair. Let $F(x)$ and $F_1(x)$ be two liftings of a monic irreducible polynomial $T(Y) \neq Y$ belonging to $\overline{K(\alpha)}[Y]$ with respect to (α, δ). Then given any root θ of $F(x)$, does there exist a root η of $F_1(x)$ such that $K(\theta) = K(\eta)$?*

In 2002, Bhatia and Khanduja [6] showed that the answer to the above question is '*no*' in general and proved that it is '*yes*' if each finite extension of (K, v) is tamely ramified.[2] Indeed they proved the following result in this direction.

[2]As in [16], a finite extension (K', v') of a henselian valued field (K, v) is said to be *defectless* if $[K' : K] = ef$, where e, f are the index of ramification and the residual degree of the extension v'/v. A defectless extension $(K', v')/(K, v)$ is said to be *tamely ramified* if the residue field of v' is a separable extension of the residue field of v and the index of ramification of v'/v is not divisible by the characteristic of the residue field of v.

Theorem 2. *Let (K, v) be a henselian valued field of any rank and (α, δ) be a (K, v)-minimal pair. Let $F(x)$ and $F_1(x)$ be two liftings of a monic irreducible polynomial $T(Y) \neq Y$ belonging to $\overline{K(\alpha)}[Y]$ with respect to (α, δ). Suppose that ξ is a root of $F(x)$ and η is a root of $F_1(x)$. Then $G(K(\xi)) = G(K(\eta))$ and $\overline{K(\xi)}$ is $\overline{K}$-isomorphic to $\overline{K(\eta)}$.*

Theorem 3. *Let (K, v) be as in the above theorem. Assume that each finite extension of (K, v) is tamely ramified and (α, δ) be a (K, v)-minimal pair. Let $F(x)$ and $F_1(x)$ be two liftings of a monic irreducible polynomial $T(Y) \neq Y$ belonging to $\overline{K(\alpha)}[Y]$ with respect to (α, δ) and $h(x)$. Then for any root ξ of $F(x)$, there exists a root η of $F_1(x)$ such that $K(\xi) = K(\eta)$.*

It is immediate from the definition of a minimal pair that for any α in K and δ in $\widetilde{G}$, (α, δ) is a (K, v)-minimal pair; however as can be easily seen a pair (α, δ) belonging to $(\widetilde{K} \backslash K) \times \widetilde{G}$ is a (K, v)-minimal pair if and only if δ is strictly greater than each element of the set $M(\alpha, K)$ defined by

$$M(\alpha, K) = \{\tilde{v}(\alpha - \beta) \mid \beta \in \widetilde{K},\ [K(\beta) : K] < [K(\alpha) : K]\}.$$

In 2000, Khanduja, Popescu and Roggenkamp proved that $M(\alpha, K)$ has an upper bound in $\widetilde{G}$ if and only if $[K(\alpha) : K] = [\widehat{K}(\alpha) : \widehat{K}]$, where $(\widehat{K}, \hat{v})$ is completion of (K, v) (see [21, Theorem 3.1]). This gave rise to the invariant $\delta_K(\alpha)$ defined to be the supremum of the set $M(\alpha, K)$ (for the sake of definition of supremum, $\widetilde{G}$ may be viewed as a subset of its Dedekind order completion). The invariant $\delta_K(\alpha)$ is called the main invariant associated to α. In 2002, it was proved that the set $M(\alpha, K)$ has a maximum element for every α in $\widetilde{K} \setminus K$ if and only if each simple algebraic extension of (K, v) is defectless (see [1]). Since any finite extension of a complete discrete rank one valued field (K, v) is defectless, it follows that $M(\alpha, K)$ has a maximum element for all α in $\widetilde{K} \setminus K$ in case of such a valued field (K, v). This led Popescu and Zaharescu [25] to define the notion of distinguished pairs for these valued fields, which was later extended to arbitrary henselian valued fields as follows:

Definition. Let (K, v) be henselian valued field of arbitrary rank. A pair (θ, α) of elements of $\widetilde{K}$ with $[K(\theta) : K] > [K(\alpha) : K]$ is said to be a (K, v)-*distinguished pair* if α is an element of smallest degree over K for which $\tilde{v}(\theta - \alpha) = \delta_K(\theta)$.

Distinguished pairs give rise to distinguished chains in a natural manner. A chain $\theta = \theta_0, \theta_1, \ldots, \theta_r$ of elements of $\widetilde{K}$ will be called a complete distinguished chain for θ if (θ_i, θ_{i+1}) is a distinguished pair for $0 \leqslant i \leqslant r-1$ and $\theta_r \in K$. It is immediate from what has been said above that in case (K, v) is a complete discrete rank one valued field, then each θ in $\widetilde{K} \setminus K$ has a complete distinguished chain. In 2005, Aghigh and Khanduja [2] characterized those elements $\theta \in \widetilde{K} \setminus K$ for which there exists a complete distinguished chain when (K, v) is a henselian valued field of arbitrary rank. Indeed they proved the following theorem.

Theorem 4. *Let (K, v) and $(\widetilde{K}, \tilde{v})$ be as in the above theorem. An element $\theta \in \widetilde{K} \setminus K$ has a complete distinguished chain with respect to (K, v) if and only if $K(\theta)$ is a defectless extension of (K, v).*

It is also known that complete distinguished chains for an element θ in $\widetilde{K} \setminus K$ give rise to several invariants associated with θ which satisfy some fundamental relations as is clear from Theorems 5-7 stated below proved in [2]. These invariants happen to be the same for all K-conjugates of θ and hence are invariants of the minimal polynomial of θ over K.

Theorem 5. *Let (K, v) and $(\widetilde{K}, \tilde{v})$ be as in the foregoing theorem. Let (θ, α) and (θ, β) be two (K, v)- distinguished pairs and $f(x)$, $g(x)$ be the minimal polynomials of α, β over K, respectively. Then $G(K(\alpha)) = G(K(\beta))$, $\overline{K(\alpha)} = \overline{K(\beta)}$ and $\tilde{v}(f(\theta)) = \tilde{v}(g(\theta))$.*

Theorem 6. *Let (K, v) and $(\widetilde{K}, \tilde{v})$ be as in the above theorem. If $\theta = \theta_0, \theta_1, \ldots, \theta_r$ and $\theta = \eta_0, \eta_1, \ldots, \eta_s$ are two complete distinguished chains for $\theta \in \widetilde{K} \setminus K$, then $r = s$ and $[K(\theta_i) : K] = [K(\eta_i) : K]$ for $1 \leq i \leq s$.*

Theorem 7. *With (K, v) and $(\widetilde{K}, \tilde{v})$ as above, let $\theta = \theta_0, \theta_1, \ldots, \theta_s$ and $\theta = \eta_0, \eta_1, \ldots, \eta_s$ are two complete distinguished chains for an element $\theta \in \widetilde{K} \setminus K$. If $f_i(x)$ and $g_i(x)$ denote respectively the minimal polynomials of θ_i and η_i over K, then the following hold for $1 \leq i \leq s$:*

(i) $G(K(\theta_i)) = G(K(\eta_i))$;

(ii) $\overline{K(\theta_i)} = \overline{K(\eta_i)}$;

(iii) $\tilde{v}(\theta_{i-1} - \theta_i) = \tilde{v}(\eta_{i-1} - \eta_i)$;

(iv) $\tilde{v}(f_i(\theta_{i-1})) = \tilde{v}(g_i(\eta_{i-1}))$.

It is clear form Theorems 6 and 7 that if $\theta = \theta_0, \theta_1, \ldots, \theta_s$ is a complete distinguished chain for an element θ of $\widetilde{K} \setminus K$, then the number s, called the length of the chain for θ, the chain of groups $G(K(\theta_0)) \supseteq G(K(\theta_1)) \supseteq \cdots \supseteq G(K(\theta_s)) = G(K)$ and the tower of fields $\overline{K(\theta_0)} \supseteq \overline{K(\theta_1)} \supseteq \cdots \supseteq \overline{K(\theta_s)} = \overline{K}$, together with the finite sequence $\tilde{v}(\theta - \theta_1) > \tilde{v}(\theta_1 - \theta_2) > \cdots > \tilde{v}(\theta_{s-1} - \theta_s)$ are invariants of θ. If $f_i(x)$ stands for the minimal polynomial of θ_i over K and λ_i for $\tilde{v}(f_i(\theta_{i-1}))$, then it follows from Theorem 7 (iv) that $\lambda_1, \lambda_2, \ldots, \lambda_s$ are also independent of the chain for θ. It may be pointed out that Ota [24] gave a method to determine these invariants when K is a finite extension of the field of p-adic numbers and $[K(\theta) : K]$ is not divisible by p. This method was extended in 2005 by Khanduja and Singh [20] to valued fields (K, v) of arbitrary rank. Indeed the following theorem was proved in this regard [20, Theorem 1.2].

Theorem 8. *Let (K, v) be a henselian valued field and $(\widetilde{K}, \tilde{v})$ be as before. Suppose that $K(\theta)$ is a finite tame extension of (K, v) of degree more than one and that $c_1 > c_2 > \cdots > c_r$ are all the distinct members of the set $\{\tilde{v}(\theta - \theta') \mid \theta' \neq \theta$ runs over K-conjugates of $\theta\}$. Then*

(a) any complete distinguished chain for θ with respect to (K, v) has length r;

(b) given a complete distinguished chain $\theta = \theta_0, \theta_1, \ldots, \theta_r$ for θ, the following hold for $1 \leq i \leq r$:

(i) $\delta_K(\theta_{i-1}) = \tilde{v}(\theta_{i-1} - \theta_i) = c_i$,

(ii) $K(\theta_i) \subseteq K(\theta_{i-1})$,

(iii) $[K(\theta) : K(\theta_i)] = t_1 + t_2 + \cdots + t_i + 1$, where t_i is the number of elements in the set $\{\theta' \mid \theta'$runs over K-conjugates of θ with $\tilde{v}(\theta - \theta') = c_i\}$.

Further the fields $K(\theta_i)$ are uniquely determined by θ.

Liftings and distinguished pairs are related to each other by the following theorem which was partially proved in [6, Proposition 2.3] and now refined in the following form in [18, Proposition 2.6].

Theorem 9. *Let (K, v) be a henselian valued field of arbitrary rank and (α, δ) be a (K, v)-minimal pair. Let $f(x)$ be the minimal polynomial of α over K of degree m and $\lambda, e, h(x)$ be as in Theorem 1. Let $g(x) \in K[x]$ be a lifting of a monic polynomial $T(Y)$ not divisible by Y of degree t belonging to $\overline{K(\alpha)}[Y]$ with respect to (α, δ). Then the following hold:*

(i) $\tilde{v}(\theta - \alpha) \leq \delta$ *for each root* θ *of* $g(x)$.

(ii) *Given any root* θ *of* $g(x)$, *there exists a* K*-conjugate* θ' *of* θ *such that* $\tilde{v}(\theta' - \alpha) = \delta$ *and* $\tilde{v}(f(\theta')) = \tilde{v}(f(\theta)) = \lambda$.

(iii) *If* θ_i *is a root of* $g(x)$ *with* $\tilde{v}(\theta_i - \alpha) = \delta$, *then the* $\tilde{v}$*- residue of* $\left(\frac{f(\theta_i)^e}{h(\alpha)}\right)$ *is a root of* $T(Y)$.

Using liftings of polynomials, a refinement of the classical Hensel's Lemma stated below as Theorem 10 has been proved for complete rank-1 valued fields (see [17, Theorem 1.1]); it can be easily seen that its statement is same as that of Hensel's Lemma when the minimal pair (α, δ) is $(0, 0)$ and $h = 1$. However it is an open problem whether Theorem 10 holds for henselian valued fields of arbitrary rank or not.

Theorem 10. *Let* (K, v) *be a complete rank-1 valued field with value group* G_v *and* $(\widetilde{K}, \tilde{v})$, (α, δ), $w_{\alpha,\delta}$, $f(x)$, m, λ *and* e *be as in Theorem 1. Assume that* $e\lambda$ *belongs to* G_v *with* $e\lambda = v(h)$ *for some* h *in* K. *Let* Z *denote the* $w_{\alpha,\delta}$*-residue of* $\frac{f(x)^e}{h}$ *and* $F(x)$ *belonging to* $K[x]$ *be such that* $w_{\alpha,\delta}(F(x)) = 0$. *If the* $w_{\alpha,\delta}$*-residue of* $F(x)$ *is the product of two coprime polynomials* $T(Z), U(Z)$ *belonging to* $\overline{K(\alpha)}[Z]$ *with* $T(Z)$ *monic of degree* $t \geq 1$, *then there exist* $G(x), H(x) \in K[x]$ *such that* $F(x) = G(x)H(x), \deg G(x) = etm$ *and the* $w_{\alpha,\delta}$*-residue of* $G(x), H(x)$ *are* $T(Z), U(Z)$ *respectively.*

Liftings have also been used to establish the irreduciblity of Eisenstein-Dumas polynomials defined below. Recall that a polynomial $g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_o$ be a polynomial with coefficients in $\mathbb{Z}$ is said to be an Eisenstein-Dumas polynomial with respect to a prime p if the exact power p^{r_i} dividing a_i (where $r_i = \infty$ if $a_i = 0$), satisfy $r_n = 0$, $(r_i/n - i) \geqslant (r_0/n)$ for $0 \leqslant i \leqslant n-1$ and $\gcd(r_0, n) = 1$. Similarly a polynomial $g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_o$ with coefficients in a valued field (K, v) is said to be an Eisenstein-Dumas polynomial with respect to v if $v(a_n) = 0, \dfrac{v(a_i)}{n-i} \geqslant \dfrac{v(a_0)}{n}$ for $1 \leq i \leq n-1$ and there does not exist any number $d > 1$ dividing n such that $v(a_0) \in dG$. The notion of an Eisenstein-Dumas polynomial has been extended to Generalized Schönemann polynomials defined in the following way, first studied in this generality by R. Brown [9].

Definition. Let v be a valuation of arbitrary rank of a field K with value group G and valuation ring R_v having maximal ideal M_v. Let $f(x)$ belonging to $R_v[x]$ be a monic polynomial of degree m such that $\bar{f}(x)$ is irreducible over R_v/M_v. Assume that $g(x) \in R_v[x]$ is a monic polynomial whose $f(x)$-expansion $f(x)^s + \sum_{i=0}^{s-1} g_i(x)f(x)^i$ satisfies $\frac{v^x(g_i(x))}{s-i} \geqslant \frac{v^x(g_0(x))}{s} > 0$ for $0 \leqslant i \leqslant s-1$ and $v^x(g_0(x)) \notin dG$ for any number $d > 1$ dividing s. Such a polynomial $g(x)$ will be referred to as a Generalized Schönemann polynomial with respect to v and $f(x)$.

Using distinguished pairs, we have proved that a translate $g(x+a)$ of a given polynomial $g(x)$ belonging to $K[x]$ having a root θ is an Eisenstein-Dumas polynomial with respect to an arbitrary henselian valuation v if and only if $K(\theta)/K$ is a totally ramified extension and (θ, a) is a distinguished pair. In particular, it is deduced that if some translate of a polynomial $g(x) = x^s + a_{s-1}x^{s-1} + \cdots + a_0$ is an Eisenstein-Dumas polynomial with respect to v with s not divisible by the characteristic of the residue field of v, then the polynomial $g\big(x - \frac{a_{s-1}}{s}\big)$ is an Eisenstein-Dumas polynomial with respect to v (cf. [7, Theorem 1.1, 1.2]). In fact using distinguished chains the following more general problem related to Generalized-Schönemann polynomials was solved in 2010.

> *Let $g(x)$ belonging to $R_v[x]$ be a monic polynomial over a henselian valued field (K, v) of arbitrary rank with $\bar{g}(x) = \phi(x)^s$, where $\phi(x)$ is an irreducible polynomial over R_v/M_v and θ is a root of $g(x)$. What are necessary and sufficient conditions so that $g(x)$ is a Generalized Schönemann polynomial with respect to v and some polynomial $f(x) \in R_v[x]$ with $\bar{f}(x) = \phi(x)$?*

In this regard, the next two theorems have been proved in 2010 (cf. [7]).

Theorem 11. *Let v be a henselian valuation of arbitrary rank of a field K with value group G and $f(x)$ belonging to $R_v[x]$ be a monic polynomial of degree $m > 1$ with $\bar{f}(x)$ irreducible over the residue field of v. Let $g(x) \in K[x]$ be a Generalized Schönemann polynomial with respect to v and $f(x)$ having $f(x)$-expansion $f(x)^s + \sum_{i=0}^{s-1} g_i(x)f(x)^i$ with $s > 1$. Let θ be a root of $g(x)$. Then for some suitable root θ_1 of $f(x)$, θ has a complete distinguished chain $\theta = \theta_0, \theta_1, \theta_2$ of length 2 with $G(K(\theta_1)) = G$, $\overline{K(\theta)} = \overline{K}(\bar{\theta})$ and $[G(K(\theta)) : G] = s$.*

Theorem 12. *Let (K, v) be as in the above theorem. Let $g(x)$ belonging to $R_v[x]$ be a monic polynomial such that $\bar{g}(x) = \phi(x)^s$, $s > 1$, where $\phi(x)$ is an irreducible polynomial over R_v/M_v of degree $m > 1$. Suppose that a root θ of $g(x)$ has a complete distinguished chain $\theta = \theta_0,\ \theta_1,\ \theta_2$ of length 2 with $G(K(\theta_1)) = G$, $\overline{K(\theta)} = \overline{K}(\bar{\theta})$ and $[G(K(\theta)) : G] = s$. Then $g(x)$ is a Generalized Schönemann polynomial with respect to v and $f(x)$, where $f(x)$ is the minimal polynomial of θ_1 over K.*

In 2011, we gave an explicit formula for the main invariant $\delta_K(\theta)$ associated to a root θ of a Generalized Schönemann polynomial $g(x) = f(x)^s + g_{s-1}(x)f(x)^{s-1} + \cdots + g_0(x)$. By virtue of Theorem 11, one can choose a suitable root α of $f(x)$ such that (θ, α) is a (K, v)-distinguished pair in case $\deg g(x) > \deg f(x)$. Using distinguished pairs, we have proved the following more general theorem for calculating $\delta_K(\theta)$ (cf. [8, Theorems 1.2, 1.3]).

Theorem 13. *Let $(K, v), (\widetilde{K}, \tilde{v})$ be as in the Theorem 1 and (θ, α) be a (K, v)-distinguished pair. Let $f(x), g(x)$ be the minimal polynomials of α, θ over K of degrees m, n respectively and $\sum\limits_{i=0}^{s} g_i(x)f(x)^i$ be the $f(x)$-expansion of $g(x)$ with $s = \dfrac{n}{m}$. Then $\delta_K(\theta) = \max\limits_{1 \leqslant i \leqslant m} \left\{ \dfrac{1}{i} \left[\dfrac{\tilde{v}\big(g_0(\alpha)\big)}{s} - \tilde{v}(c_i) \right] \right\}$ where $f(x) = \sum\limits_{i=1}^{m} c_i(x-\alpha)^i$, $c_i \in K(\alpha)$.*

Recently R. Brown and J. L. Merzel studied in great detail invariants including the main invariant of irreducible polynomials $g(x)$ over henselian valued field (K, v) such that for any root θ of $g(x)$, $K(\theta)/K$ is a defectless extension (such polynomials are referred to as defectless polynomials) using the approach of strict system of polynomial extensions (see [11], [12]). They also developed some significant connections between complete distinguished chains and strict systems of polynomial extensions. They proved that complete distinguished chains give rise to strict systems of polynomial extensions, and, in the tame case, the converse. In 2011, Khanduja and Khassa [15] proved the converse in full generality, establishing the equivalence of the two approaches and thereby giving new interpretations of the invariants studied by Brown and Merzel. Of particular interest is an invariant λ_g belonging to $\widetilde{G}$ introduced by Ron Brown in [10], associated to any defectless polynomial $g(x) \in K[x]$ which satisfies the property that whenever $K(\beta)$ is a tamely ramified extension of $(K, v), \beta \in \widetilde{K}$ and $\tilde{v}(g(\beta)) > \lambda_g$, then

$K(\beta)$ contains a root of $g(x)$; moreover the constant λ_g is the smallest with this property. The constant λ_g has been named as Brown's constant and a method to determine it explicitely has been given in [14] using complete distinguished chains. It has also been shown that the condition $\tilde{v}(g(\beta)) > \lambda_g$ is in general weaker than the analogous condition $\tilde{v}(g(\beta)) > 2\tilde{v}(g'(\beta))$ in Hensel's Lemma for guaranteeing the existence of a root of $g(x)$ in $K(\beta)$ (see [14, Corollaries 1.2, 1.5]).

References

[1] K. Aghigh and S. K. Khanduja, On the main invariant of elements algebraic over a henselian valued field, *Proc. Edinburgh Math. Soc.,* **45** (2002) 219–227.

[2] K. Aghigh and S. K. Khanduja, On chains associated with elements algebraic over a henselian valued field, *Algebra Colloq.*, **12** (2005) 607–616.

[3] V. Alexandru and N. Popescu, Sur une classse de prolongements à K(x) dúne valuation sur une corps K, *Rev. Roumaine Math. Pures Appl.,* **33** (1988) 393-400.

[4] V. Alexandru, N. Popescu and A. Zaharescu, A theorem of characterization of residual transcendental extension of a valuation, *J. Math. Kyoto Univ.,* **28** (1988) 579–592.

[5] V. Alexandru, N. Popescu and A. Zaharescu, Minimal pairs of definition of a residual transcendental extension of a valuation, *J. Math. Kyoto Univ.,* **30** (1990) 207–225.

[6] S. Bhatia and S. K. Khanduja, On extensions generated by roots of lifting polynomials, *Mathematika.,* **49** (2002) 107–118.

[7] A. Bishnoi and S. K. Khanduja, On Eisenstein-Dumas and generalized Schönemann polynomials, *Comm. Algebra.,* **38** (2010) 3163–3173.

[8] A. Bishnoi and S. K. Khanduja, On generalized Schönemann polynomials, *Comm. Algebra.,* (to appear).

[9] R. Brown, Roots of generalized Schönemann polynomials in henselian extension fields, *Indian J. Pure Appl. Math.,* **39** (2008) 403–410.

[10] R. Brown, Roots of irreducible polynomials in tame henselian extension fields, *Comm. Algebra.,* **37** (2009) 2169–2183.

[11] R. Brown and J. L. Merzel, Invariants of defectless irreducible polynomials, *J. Algebra Appl.*, **9** (2010) 603–631.

[12] R. Brown and J. L. Merzel, The main invariant of a defectless polynomial, *J. Algebra Appl.*, (DOI No: 10.1142/S0219498812501228).

[13] A. J. Engler and A. Prestel, Valued Fields, Springer-Verlag, New York 2005.

[14] S. K. Khanduja, On Brown's constant associated with irreducible polynomials over henselian valued fields, *J. Pure Appl. Algebra,* **214** (2010) 2294–2300.

[15] S. K. Khanduja and R. Khassa, On invariants and strict systems of irreducible polynomials over Henselian valued fields, *Comm. Algebra,* **39** (2011) 584–593.

[16] F. V. Kuhlmann, A classification of Artin-Schreier defect extensions and a characterization of defectless fields, *Illinois J. Math.,* **54** (2010) 397–448.

[17] S. K. Khanduja and S. Kumar, On irreducible factors of polynomials over complete fields, *J. Algebra Appl.*, (DOI No: 10.1142/S0219498812501253).

[18] S. K. Khanduja and S. Kumar, On prolongations of valuations via Newton polygons and liftings of polynomials, *J. Pure Appl. Algebra*, **216** (2012) 2648–2656.

[19] S. K. Khanduja and J. Saha, On a generalization of Eisenstein's irreducibility criterion, *Mathematika,* **44** (1997) 37–41.

[20] S. K. Khanduja and A. P. Singh, On finite tame extensions of valued fields, *Comm. Algebra.,* **33** (2005) 1095–1105.

[21] S. K. Khanduja, N. Popescu and K. W. Roggenkamp, On minimal pairs and residually transcendental extensions of valuations, *Mathematika*, **49** (2002) 93–106.

[22] M. Nagata, A theorem on valuation rings and its applications, *Nagoya Math. J.*, **29** (1967) 85–91.

[23] J. Ohm, The ruled residue theorem for simple transcendental extensions of valued fields, *Proc. Amer. Math. Soc.*, **89** (1983) 16–18.

[24] K. Ota, On saturated distinguished chains over a local field, *J. Number Theory,* **79** (1999) 217–248.

[25] N. Popescu and A. Zaharescu, On the structure of the irreducible polynomials over local fields, *J. Number Theory,* **52** (1995) 98–118.

[26] P. Ribenboim, *The Theory of Classical Valuations.* Springer-Verlag, New York, 1999.