DENSITY OF COMPOSITE PLACES IN FUNCTION FIELDS AND APPLICATIONS TO REAL HOLOMORPHY RINGS

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Abstract. Given an algebraic function field $F|K$ and a place $\wp$ on K, we prove that the places that are composite with extensions of $\wp$ to finite extensions of K lie dense in the space of all places of F, in a strong sense. We apply the result to the case of $K = R$ any real closed field and the fixed place on R being its natural (finest) real place. This leads to a new description of the real holomorphy ring of F which can be seen as an analogue to a certain refinement of Artin's solution of Hilbert's 17th problem. We also determine the relation between the topological space $M(F)$ of all $\mathbb{R}$-places of F (places with residue field contained in $\mathbb{R}$), its subspace of all $\mathbb{R}$-places of F that are composite with the natural $\mathbb{R}$-place of R, and the topological space of all R-rational places. Further results about these spaces as well as various classes of relative real holomorphy rings are proven. At the conclusion of the paper the theory of real spectra of rings will be applied to interpret basic concepts from that angle and to show that the space $M(F)$ has only finitely many topological components.

1. Introduction

1.1. The main results on places of algebraic function fields.

The Main Theorem of [26] showed the density (in a very strong sense) of certain types of places in the space of all places of a function field of characteristic 0 (by "function field" we will always mean an algebraic function field of transcendence degree at least 1). A modification of the Main Theorem was then applied to various classes of holomorphy rings, including the real and the p-adic. In a subsequent paper [22], the Main Theorem was generalized to arbitrary characteristic. The density of several important sets of places was shown, such as prime divisors, as well as the Abhyankar places which play a crucial role e.g. in [16, 24]. While the paper [26] only considered the space

$$S(F|K) \;=\; \{\xi \text{ place of } F \mid \xi|_K = \mathrm{id}_K\}$$

of all places of an algebraic function field $F|K$ that are trivial on K, the scope was widened in [22] to the spaces

$$S(F|K\,;\,\wp) \;=\; \{\xi \in S(F) \mid \xi|_K = \wp\}$$

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of places of F that extend a fixed place $\wp$ of K. We note that every $S(F|K\,;\,\wp)$ is a subset of the **Zariski space** $S(F)$ of all places of F, and that $S(F|K) = S(F|K\,;\,\mathrm{id}_K)$.

However, one interesting subset of these spaces was entirely missed: the set consisting of those places that factor over $S(F|K)$ (see below for the precise definition). In this paper we will adapt the proofs of the density theorems from [26, 22] so as to prove the density of this subset and show how this is used to obtain ample information on families of real holomorphy rings and the topologies of various spaces of places into formally real fields.

In order to present our central theorems, we need some preparations. In contrast to the usage in [16, 22, 23, 24, 25] we will treat places as usual functions and apply them to elements from the left, that is, the image of a under a place ξ will be denoted by $\xi(a)$. However, we will keep one convention: the residue field of F under ξ will be denoted by $F\xi$. Further, the valuation, valuation ring and valuation ideal associated with the place ξ will be denoted by v_ξ, $\mathcal{O}_\xi$ and $\mathcal{M}_\xi$, respectively. The value group of v_ξ on F will be denoted by $v_\xi F$.

Now we have all definitions in place to state our first central theorem.

Theorem 1.1. *Take an arbitrary field K with a place $\wp$, a function field F over K, a place $\xi \in S(F|K\,;\,\wp)$, and nonzero elements $a_1,\ldots,a_m \in F$. Choose $r \in \mathbb{N}$ such that $1 \le r \le s = \mathrm{trdeg}\,F|K$ and an arbitrary ordering on $\mathbb{Z}^r$; denote by Γ the so obtained ordered abelian group. If $\mathrm{trdeg}\,F|K > 1$ and $\wp$ is trivial while ξ is not, then we assume in addition that Γ is the lexicographic product $\Gamma' \times \mathbb{Z}$, where $\Gamma' = \mathbb{Z}^{r-1}$ endowed with an arbitrary ordering.*

Then there is a place $\lambda \in S(F|K)$ and an extension $\wp'$ of $\wp$ from K to $F\lambda$ such that, with $\xi' := \wp' \circ \lambda \in S(F|K\,;\,\wp)$,

(a) $F\lambda$ is a finite extension of K,

(b) $v_\lambda F \subseteq \Gamma$ with $(\Gamma : v_\lambda F)$ finite,

(c) if $a_i \in \mathcal{O}_\xi$, then $\lambda(a_i) \in \mathcal{O}_{\wp'}$ and $a_i \in \mathcal{O}_{\xi'}$.

The following assertions can also be realized if, in case $\wp$ is trivial, we assume that ξ is trivial:

(d) if $a_i \in \mathcal{M}_\xi$, then $\lambda(a_i) \in \mathcal{M}_{\wp'}$ and $a_i \in \mathcal{M}_{\xi'}$,

(e) if $\xi(a_i) \in K\wp$, then $\xi'(a_i) = \xi(a_i)$,

(f) $\lambda(a_i) \neq 0,\infty$ for $1 \le i \le m$.

If $\wp$ is trivial while ξ is not, then in addition to assertions (a), (b) and (c), we can realize also (d) and (e), or alternatively, (f).

We present two additions to the above theorem which work under stronger assumptions.

Proposition 1.2. *Assume that the setting is as in Theorem 1.1, and in addition, that $F\xi = K\wp$. In case $\wp$ is trivial, also assume that $\mathrm{trdeg}\,F|K > 1$. Then in addition to the results of Theorem 1.1 we can also obtain that $F\xi'|K\wp$ is a finite purely inseparable extension.*

Recall that a field K is **existentially closed in an extension field** F if every existential sentence in the language of rings with parameters from K which holds in F will also hold in K. For further explanations, see [26, Section 1]. Similarly, a valued field K (or an ordered field K with a valuation) is existentially closed in an extension field F with an ordering extending that of K (or ordering and valuation extending those of K, respectively) if every existential sentence in the language of rings with a relation symbol for a valuation (or with relation symbols for an ordering and a valuation, respectively) and with parameters from K which holds in F will also hold in K.

Proposition 1.3. *Assume that the setting is as in Theorem 1.1. If $(K,\wp)$ is existentially closed in (F,ξ), then in addition to the results of Theorem 1.1 we can obtain that $F\lambda = K$, $v_\lambda F = \Gamma$, $F\xi' = K\wp$ and $\wp' = \wp$.*

If $<$ is an ordering on the field F, then we will say that ξ (or its associated valuation v_ξ) is **compatible with** $<$ if $\mathcal{O}_\xi$ is convex relative to this given ordering. That a place ξ on F is compatible with some of the orderings on F is equivalent to the statement that $F\xi$ is a formally real field. This is one essential part of the Baer-Krull Theorem (see [3, Theorem 10.1.10]).

Here is the convention we will follow: places into formally real fields are called **real places**; valuation rings with a formally real residue field, i.e., the valuation rings of real places, are called **real valuation rings**; finally, places ξ with $F\xi \subseteq \mathbb{R}$ are denoted as **$\mathbb{R}$-places**.

Now we have all definitions in place to state our second central theorem, which adapts Theorem 1.1 to the real case.

Theorem 1.4. *Assume that $K = R$ is a real closed field and F is an ordered function field over K. Assume further that $\wp$ is a place on R compatible with its ordering and ξ is a place on F compatible with the ordering $<$ of F. Take elements $a_1, \ldots, a_m \in F$ and let $r \in \mathbb{N}$ and Γ be as in Theorem 1.1. Then there is a place $\lambda \in S(F|R)$ such that, with $\xi' := \wp \circ \lambda \in S(F|R\,;\,\wp)$,*

(a) $F\lambda = R$ and $F\xi' = R\wp$,

(b) $v_\lambda F = \Gamma$,

(c) if $a_i \in \mathcal{O}_\xi$, then $\lambda(a_i) \in \mathcal{O}_\wp$ and $a_i \in \mathcal{O}_{\xi'}$,

(c') if $a_i > 0$ and $\xi(a_i) \neq 0, \infty$, then $\xi'(a_i) > 0$.

The latter implies that if $\infty \neq \xi(a_i) > 0$, then $\xi'(a_i) > 0$.

The following assertions can also be realized if, in case $\wp$ is trivial, we assume that also ξ is trivial:

(d) if $a_i \in \mathcal{M}_\xi$, then $\lambda(a_i) \in \mathcal{M}_\wp$ and $a_i \in \mathcal{M}_{\xi'}$,

(e) if $\xi(a_i) \in R\wp$, then $\xi'(a_i) = \xi(a_i)$,

(f) $\lambda(a_i) \neq 0, \infty$ for $1 \leq i \leq m$,

(g) if $a_i > 0$, then $\lambda(a_i) > 0$.

If $\wp$ is trivial while ξ is not, then in addition to assertions (a), (b), (c) and (c'), we can realize also (d) and (e), or alternatively, (f) and (g).

Remark 1.5. In the case where $\wp$ is trivial while ξ is not, assertion *(d)* is incompatible with *(f)* and *(g)* because in this case, $\mathcal{O}_{\xi'} = \mathcal{O}_\lambda$. Hence if $0 \neq a_i \in \mathcal{M}_\xi$ and λ satisfies assertion *(d)*, then $a_i \in \mathcal{M}_\lambda$, hence $\lambda(a_i) = 0$ so that assertion *(f)* is not satisfied by λ; if in addition $a_i > 0$, then also assertion *(g)* is not satisfied by λ. $\diamondsuit$

Proposition 1.6. *Assume that the setting is as in one of the above theorems or propositions. Then there are infinitely many nonequivalent places λ and ξ' which satisfy all assertions of the respective theorem or proposition, except in the case where $\wp$ is trivial while ξ is not,* $\operatorname{trdeg} F|K = 1$ *and we wish that assertions (d) and (e) are satisfied.*

Remark 1.7. If $\operatorname{trdeg} F|K = 1$ and $\wp$ is trivial while ξ is not, then ξ itself satisfies assertions *(a)*, *(b)*, *(c)*, *(d)*, *(e)* of the theorems, and also assertion *(c')* in the setting of Theorem 1.4. This will be shown in the proofs of the two theorems. But it may be the only such place. If for instance $\wp$ is the identity, $F = K(x)$, ξ is the x-adic place and $a_1 = x$, then $\xi(x) = 0 \in K = K\wp$ so that by assertion *(e)*, $\lambda(x) = \xi'(x) = \xi(x) = 0$, whence $\lambda = \xi$. ♢

The above theorems and propositions will be proven in Section 2. The proof of Theorem 1.4 uses the fact that a real closed field is existentially closed in every formally real extension field; see parts 3) and 4) of Theorem 2.2. This allows us to apply Proposition 1.3 as well as Theorem 2.3, a main ingredient to our proofs which is a generalization of Theorem 23 of [22].

1.2. **Applications to spaces of places of algebraic function fields.**

Given two places ξ' and π', we will say that ξ' **factors over** π' (or in other words, is **composite with** π') if there is a place λ such that $\xi' = \pi' \circ \lambda$. Theorem 1.1 shows the strong density of the subset of $S(F|K\,;\,\wp)$ of all places ξ' that factor over $\wp'$ for a suitable finite extension $(K', \wp')$ of $(K, \wp)$. Let us describe one consequence of the strong density. Every set $S(F|K\,;\,\wp)$ carries the **Zariski topology**, for which the basic open sets are the sets of the form

$$(1) \qquad \{\xi \in S(F|K\,;\,\wp) \mid a_1\,, \ldots,\, a_k \in \mathcal{O}_\xi\}\ ,$$

where $k \in \mathbb{N} \cup \{0\}$ and $a_1, \ldots, a_k \in F$. With this topology, $S(F|K\,;\,\wp)$ is a spectral space (see [22, Appendix] for a proof, and [14] for details on spectral spaces); in particular, it is quasi-compact. Its associated **patch topology** (or **constructible topology**) is the finer topology whose basic open sets are the sets of the form

$$(2) \qquad \{\xi \in S(F|K\,;\,\wp) \mid a_1\,, \ldots,\, a_k \in \mathcal{O}_\xi\,;\, a_{k+1}\,, \ldots,\, a_{k+\ell} \in \mathcal{M}_\xi\}\ ,$$

where $k, \ell \in \mathbb{N} \cup \{0\}$ and $a_1, \ldots, a_{k+\ell} \in F$. With the patch topology, $S(F|K\,;\,\wp)$ is a totally disconnected compact Hausdorff space.

Note that every set $S(F|K\,;\,\wp)$ contains nontrivial places since $\operatorname{trdeg} F|K > 0$ by our general assumption. Therefore, from Theorem 1.1 (in particular assertions *(c)* and *(d)*) in connection with Proposition 1.6, we obtain:

Corollary 1.8. *Take a function field $F|K$ and a place $\wp$ on K. Then every nonempty open set in the Zariski topology of $S(F|K\,;\,\wp)$ contains infinitely many places that factor over $\wp'$ for a suitable finite extension $(K', \wp')$ of $(K, \wp)$. The same holds in the Zariski patch topology, unless* $\operatorname{trdeg} F|K = 1$ *and $\wp$ is trivial.*

Now we turn to our applications of Theorem 1.4. Let us give the definitions necessary to deal with formally real function fields F over real closed fields R. By

$M(F)$

we will denote the set of all **$\mathbb{R}$-places** of F, that is, places ξ of F with residue field $F\xi \subseteq \mathbb{R}$. These are exactly (up to equivalence of places) the places associated with the natural valuations of the orderings on F, where the natural valuations of an ordered field $(F,<)$ is the finest valuation compatible with that ordering. In particular, every real closed field R has a unique $\mathbb{R}$-place ξ_R, which we will call its **natural $\mathbb{R}$-place**.

Instead of the set $S(F|R)$ of all places of F that are trivial on R, we are rather interested in the set

$$M(F|R) \;=\; \{\lambda \in S(F|R) \mid F\lambda = R\}\;.$$

of **R-rational places**. The new object we study in this paper is the set

$$M_R(F) \;:=\; \{\xi_R \circ \lambda \mid \lambda \in M(F|R)\} \;\subseteq\; S(F|R\,;\,\xi_R)$$

of all $\mathbb{R}$-places of F that factor over ξ_R. Theorem 1.4 implies that for every $\mathbb{R}$-place ξ of F there is an $\mathbb{R}$-place ξ' of F that factors over ξ_R and is "very close to ξ".

Remark 1.9. Note that we usually do not identify equivalent real places. However, here any two equivalent places in $M_R(F)$ are equal since their residue fields are equal to the archimedean real closed field $R\xi_R \subseteq \mathbb{R}$ which does not allow any nontrivial order preserving embedding in $\mathbb{R}$. Also in $M(F|R)$, by its definition as a subset of $S(F|R)$, equivalent places are equal. Hence $M(F|R)$ is in general smaller than the set of all R-rational places of F, but every such place is equivalent to a place in $M(F|R)$. As we are interested in the compositions of R-rational places with the natural $\mathbb{R}$-place ξ_R of R, this constitutes no loss of information. Indeed, assume that λ_1 and λ_2 are equivalent R-rational places, and write $\lambda_2 = \sigma \circ \lambda_1$ for some isomorphism σ. As λ_2 is assumed to be R-rational, σ must be an automorphism of R. Since R is real closed, it is also order preserving. As $\mathcal{O}_R := \mathcal{O}_{\xi_R}$ is the convex hull of $\mathbb{Q}$ in R and $\mathbb{Q}$ is left elementwise fixed by σ, it follows that $\sigma\mathcal{O}_R = \mathcal{O}_R$. This implies that ξ_R and $\xi_R \circ \sigma$ are equivalent, and with the same argument as before, we find that they are equal. Thus, $\xi_R \circ \lambda_1$ and $\xi_R \circ \lambda_2 = \xi_R \circ \sigma \circ \lambda_1$ are equal. $\diamondsuit$

Theorem 1.4 is essential for the description of the relation between the sets $M(F|R)$, $M_R(F)$ and $M(F)$. It will be applied to formally real function fields F over arbitrary real closed fields R where we address the following issues.

The set $M(F)$, its subset $M_R(F)$ and the set $M(F|R)$ carry natural topologies. The topology of $M(F)$ as described by Dubois in [10] is compact and Hausdorff; we will denote it by $\operatorname{Top} M(F)$. It is a quotient topology of the space of orderings with the Harrison topology. Its basic open sets are

$$U(f_1,...,f_m) \;:=\; \{\xi \in M(F) \mid \xi(f_i) > 0 \text{ for } 1 \leq i \leq m\}$$

where $f_1,...,f_m$ lie in the **real holomorphy ring** $H(F)$ of the field F.

The concept of the real holomorphy ring $H(L)$ of any formally real field L is due to Dubois, cf. loc. cit.. It is defined to be the intersection of all real valuation rings of L; it is equal to the intersection of the valuation rings of all $\mathbb{R}$-places of L, and L is its field of fractions.

The holomorphy ring $H(L)$ turns out to be a Prüfer ring as the localizations at its prime ideals are exactly the real valuation rings of L. In particular, a valuation ring of L is a real valuation ring if and only if it contains $H(L)$. The basic theory of the real holomorphy ring is developed in [1, 30]; consult [12] for the general theory of Prüfer rings. Note that if R is a real closed field, then $H(R) = \mathcal{O}_R$.

In this paper, we will also be dealing with various relative real holomorphy rings in L which are defined as the intersections of some family of real valuation rings. These relative real holomorphy rings are overrings of the (absolute) real holomorphy ring $H(L)$. Hence, by the general theory of Prüfer rings they are Prüfer rings as well.

When we speak of the topological space $M(F)$, we will always refer to $\operatorname{Top} M(F)$, and the subset $M_R(F) \subseteq M(F)$ will always carry the subspace topology. So far, the topological space $M_R(F)$ has not found any attention in the literature. Yet, for our present study it is highly relevant. In particular, Proposition 3.1 will show that $M_R(F)$ is dense in $M(F)$, which is a very important fact.

In an analogous way, a topology $\operatorname{Top} M(F|R)$ on $M(F|R)$ will be introduced. It is then shown in Theorem 3.4 that the mapping

$$\iota_{F|R} : M(F|R) \to M(F), \lambda \mapsto \xi_R \circ \lambda \tag{3}$$

is a topological embedding with image $M_R(F)$. In the same theorem, it is shown that all three topological spaces have no isolated points.

Similar to the space $M_R(F)$, the space $M(F|R)$ has found little, if any, attention in real algebraic geometry. It was passed by in favour of stronger topological spaces, see e.g. [29]. In the concluding section of this paper we re-address these three topological spaces by invoking the theory of real spectra of rings, a cornerstone of modern real algebraic and semi-algebraic geometry, see [3]. As a surprising application we derive that the space $M(F)$, where F is a formally real function field over any real closed field, admits only finitely many connected components.

So far, various authors have already studied the following **relative real holomorphy rings**

$$H(F|R) := \{a \in F \mid \xi(a) \neq \infty \text{ for all real places } \xi \in S(F|R)\}$$

for function fields F over real closed fields R, and its extensions

$$H(F|R)D$$

(the smallest subring of F containing $H(F|R)$ and D), where D is a finitely generated R-algebra inside F, see [3, 5, 6, 15, 17, 20, 21, 26, 29, 30, 31]. Model theory or algebraic geometry or a combination of both theories have been used. Common to all of these approaches is that they use the fact that F admits many smooth models (projective or real complete affine ones), which in turn allows to study the behaviour of the elements in F as functions on the set $M(F|R)$.

In the case of a non-archimedean real closed base field R, this relationship seems to get lost once one turns to the absolute real holomorphy ring $H(F)$ in place of $H(F|R)$. However, using the set $M_R(F)$ of all $\mathbb{R}$-places of F that factor over the natural $\mathbb{R}$-place of R, we are able to prove representations for $H(F)$ and related rings that still retain the geometric flavour; see Section 3.2.

Theorem 1.4 allows much wider application to all composite places which factor over places in $M(F|R)$. It is this strength that allows to broadly extend previous results on relative real holomorphy rings. In fact, we can include the class of rings $H(F)D$ where D is a general finitely generated ring extension over any real valuation ring B of the base field R.

2. Proof of Theorems 1.1 and 1.4 and the related propositions

We will need the following fact, which has been shown in [2, Theorem 1.1]:

Proposition 2.1. *Let $L|K$ be an extension of finite transcendence degree, and v_ξ a nontrivial valuation on L with associated place ξ. If $v_\xi L/v_\xi K$ is not a torsion group or $L\xi|K\xi$ is transcendental, then (L, v_ξ) admits an immediate extension of infinite transcendence degree.*

The proofs of our central theorems and propositions are adaptations of the proof of the Main Theorem in [26], but instead of the Ax-Kochen-Ershov Theorem used there we will have to use other transfer principles. Namely, we will need analogues for algebraically closed fields, algebraically closed fields with valuation, ordered real closed fields, and ordered real closed fields with compatible valuation. We also include a result on divisible ordered abelian groups that is analogous to the one on algebraically closed (ordered) fields.

Theorem 2.2. *1) In the language of rings, an algebraically closed field K is existentially closed in every extension field F.*

2) In the language of rings with a relation symbol for a valuation, an algebraically closed nontrivially valued field K is existentially closed in every valued extension field F.

3) In the language of rings with a relation symbol for an ordering, a real closed field R is existentially closed in every ordered extension field F.

4) In the language of rings with relation symbols for an ordering and a valuation, a real closed field R with nontrivial compatible valuation is existentially closed in every ordered extension field F equipped with a compatible valuation which extends the valuation of R.

5) In the language of groups with a relation symbol for an ordering, a nontrivial divisible ordered abelian group Γ is existentially closed in every ordered abelian group extension Δ.

Proof. 1): Take an algebraic closure F^{ac} of F. By the model completeness of the theory of algebraically closed valued fields (see [28]), F^{ac} is an elementary extension of K in the language of rings. Every existential sentence in this language with parameters from K that holds in F also holds in F^{ac}, and by what we just have stated, it then also holds in K. This proves that K is existentially closed in F in this language.

2): Take an algebraic closure F^{ac} of F together with some extension of the valuation. By Abraham Robinson's theorem on the model completeness of the theory of algebraically closed nontrivially valued fields, see [28, Theorem 3.4.21], F^{ac} is an elementary extension of K in the language of rings with a relation symbol for a valuation. The remainder of the argument is as in the proof of part 1).

3): Take a real closure F^{rc} of F together with the corresponding extension of the ordering. Then the ordering on F^{rc} extends the unique ordering of the real closed field R. By [9, Theorem 4.5.1], F^{rc} is an elementary extension of R in the language of rings with a relation symbol for an ordering. Now our assertion follows as in the proof of part 1), with F^{rc} and R in place of F^{ac} and K, respectively.

4): Take a real closure F^{rc} of F together with the corresponding extensions of the ordering and the compatible valuation of F. Again, the ordering on F^{rc} extends the unique ordering of the real closed field R. As the compatible valuation on F^{rc} extends the one of F, which in turn extends the one of R, it also extends the one of R. By [9, Corollary 4.5.4] and the fact that the ordering is definable in a real closed field in the language of rings, F^{rc} is an elementary extension of R in the language of rings with relation symbols for an ordering and a valuation. Now our assertion follows as in the proof of part 3).

5): Take any divisible hull $\tilde{\Delta}$ of Δ. By Abraham Robinson's theorem on the model completeness of the theory of nontrivial divisible ordered abelian groups, see [28, Theorem 3.1.13], $\tilde{\Delta}$ is an elementary extension of Γ in the language of groups with a relation symbol for an ordering. The remainder of the argument is as in the proof of part 1). □

Further, we will need a generalization of Theorem 23 of [22].

Theorem 2.3. *Let $F|K$ be an algebraic function field and choose Γ as in Theorem 1.1. Take any nonzero elements $a_1, \ldots, a_m \in F$. Then there are infinitely many (nonequivalent) places $\lambda \in S(F|K)$ such that $F\lambda|K$ is finite, $v_\lambda F \subseteq \Gamma$ with $(\Gamma : v_\lambda F)$ finite, and $\lambda(a_i) \neq 0, \infty$ for $1 \leq i \leq m$.*

If in addition K is existentially closed in F, then these places can be chosen to be K-rational with $v_\lambda F = \Gamma$.

Proof. We adapt the proof of the lemma on p. 190 of [26]. In some algebraic closure F^{ac} of F we find an algebraic closure K_0 of K and let $F' := K_0.F$ be the field compositum of K_0 and F inside of F^{ac}. By part 1) of Theorem 2.2, K_0 is existentially closed in F'.

Since $K_0|K$ is algebraic, $\operatorname{trdeg} F'|K_0 = \operatorname{trdeg} F|K = s$. The extension $F'|K_0$ is separable and finitely generated, so we can pick in F' a separating transcendence basis $t_1, \ldots, t_s$ together with an element y separable algebraic over $K_0(t_1, \ldots, t_s)$ such that $F' = K_0(t_1, \ldots, t_s, y)$. Take $f \in K_0[t_1, \ldots, t_s, Y]$ to be an irreducible polynomial of y over $K_0[t_1, \ldots, t_s]$. We write $\underline{t} = (t_1, \ldots, t_s)$ and

$$a_i = \frac{g_i(\underline{t}, y)}{h_i(\underline{t})} \quad \text{for} \quad 1 \leq i \leq m \, , \tag{4}$$

where g_i and h_i are polynomials over K_0, with $h_i(\underline{t}) \neq 0$. Since the elements $t_1, \ldots, t_s, y$ satisfy

$$f(\underline{t}, y) = 0 \, , \; \frac{\partial f}{\partial Y}(\underline{t}, y) \neq 0 \quad \text{and} \quad h_i(\underline{t}) \neq 0 \; \text{ for } 1 \leq i \leq m \tag{5}$$

in F', we infer from K_0 being existentially closed in F' that there are $t'_1, \ldots, t'_s, y'$ in K_0 such that

$$f(\underline{t}', y') = 0 \, , \; \frac{\partial f}{\partial Y}(\underline{t}', y') \neq 0 \quad \text{and} \quad h_i(\underline{t}') \neq 0 \; \text{ for } 1 \leq i \leq m \, .$$

Now let K_1 be the subfield of K_0 which is generated over K by the following elements:

- $t'_1, \ldots, t'_s, y'$,
- the coefficients of f, g_i and h_i for $1 \leq i \leq m$.

We note that K_1 is a finite extension of K. We will now construct an extension K_4 of K_1 with K_1-rational place λ_4, which will contain an isomorphic copy of $K_1.F$. The construction will be done in such a way that the place λ induced on F through the resulting embedding of F in K_4 and the place λ_4 will satisfy the assertions of our theorem.

We write $\Gamma = \bigoplus_{1\le i\le r} \mathbb{Z}\alpha_i$ with $\alpha_i > 0$. We adjoin r many algebraically independent elements $x_1, \ldots, x_r$ to K_1 and denote the resulting field by K_2. By [4, Chapter VI,§10.3, Theorem 1] (see also [22, Lemma 25]), there is a place λ_2 of K_2 whose restriction to K_1 is the identity, such that $K_2\lambda_2 = K_1$ and $v_{\lambda_2} x_i = \alpha_i$ for $1 \le i \le r$, whence $v_{\lambda_2} K_2 = \bigoplus_{1\le i\le r} \mathbb{Z}\alpha_i = \Gamma$.

Since $r \ge 1$, Proposition 2.1 shows that (K_2, λ_2) admits an immediate extension of transcendence degree $s - r$. We pick a transcendence basis $x_{r+1}, \ldots, x_s$ of this extension and take (K_3, λ_3) to be the immediate subextension which it generates over (K_2, λ_2). It follows that $\lambda_3|_{K_1} = \lambda_2|_{K_1} = \mathrm{id}_{K_1}$. We may choose the elements x_i such that $v_{\lambda_3} x_i > 0$, $r+1 \le i \le s$. We have the same for $1 \le i \le r$ since all α_i are positive.

Now we take (K_4, λ_4) to be the henselization of (K_3, λ_3). Since it is an immediate extension of (K_3, λ_3), which in turn is an immediate extension of (K_2, λ_2), we have that $v_{\lambda_4} K_4 = v_{\lambda_2} K_2 = \Gamma$ and $K_4\lambda_4 = K_2\lambda_2 = K_1$, as well as $\lambda_4|_{K_1} = \mathrm{id}_{K_1}$.

We wish to show that F can be embedded in K_4 over K. In fact, we find an embedding ι of $K_1.F$ over K_1 in K_4 as follows. We set $t_i^* := t_i' + x_i \in K_4$, $1 \le i \le s$; since $v_{\lambda_4} x_i = v_{\lambda_2} x_i = \alpha_i > 0$ we have that $\lambda_4(x_i) = 0$ and obtain that $\lambda_4(t_i^*) = t_i'$. Using Hensel's Lemma, we lift the simple root y' of $f(\underline{t}', Y)$ to an element $y^* \in K_4$ which satisfies $f(\underline{t}^*, y^*) = 0$ and $\lambda_4(y^*) = y'$.

By construction, $t_1^*, \ldots, t_s^*$ are algebraically independent over K_1, so we obtain the desired embedding by setting $\iota(t_i) = t_i^*$ and $\iota(y) = y^*$. Then we take λ to be the restriction of $\lambda_4 \circ \iota$ to F. As $x_i = t_i^* - t_i' \in \iota(K_1.F)$, the value group of $\lambda_4 \circ \iota$ on $K_1.F$ is equal to Γ and consequently, $v_\lambda F \subseteq \Gamma$. As $K_1.F|F$ is finite, so is $\Gamma/v_\lambda F$.

The restriction of λ to K is the identity because the same holds for λ_4 and λ is a restriction of $\lambda_4 \circ \iota$. Hence, $\lambda \in S(F|K)$. Further, $F\lambda \subseteq K_4\lambda_4 = K_1$, hence $F\lambda|K$ is finite.

We have that $\lambda(t_j) = \lambda_4(\iota(t_j)) = \lambda_4(t_j^*) = t_j'$ for $1 \le j \le s$, and $\lambda(y) = \lambda_4(\iota(y)) = \lambda_4(y^*) = y'$, whence $\lambda(g_i(\underline{t}, y)) = g_i(\underline{t}', y')$ and $\lambda(h_i(\underline{t})) = h_i(\underline{t}') \ne 0$ for $1 \le i \le m$. Therefore, $\lambda(a_i) \ne \infty$ for all i. By including also a_i^{-1} in the list for each i, we obtain in addition that $\lambda(a_i) \ne 0$ for all i.

Now suppose that we have already constructed places $\lambda_1, \ldots, \lambda_k \in S(F|K)$ which are finite on $a_1, \ldots, a_m$ and satisfy all additional assertions. Since $\operatorname{trdeg} F|K \ge 1$ by our general assumption, but $F\lambda_j|K$ is algebraic, the places λ_j are nontrivial. Hence there are elements $a_{m+j} \in F$ such that $\lambda_j(a_{m+j}) = \infty$ for $1 \le j \le k$. As shown above, there exists a place λ which is finite on $a_1, \ldots, a_{m+k}$ and satisfies all additional assertions. It follows that $\lambda(a_{m+j}) \ne \infty = \lambda_j(a_{m+j})$ and hence λ is not equivalent to λ_j for $1 \le j \le k$. This shows that there are infinitely many nonequivalent places which satisfy all assertions of the first part of our theorem.

If K is existentially closed in F, then $F|K$ is separable (see [25, Lemma 5.3]). In this case, the proof proceeds as above with K in place of K_0 and F in place of F'. We then have that $K_1 = K$, which implies that $F\lambda = K$. We also have that $t_i' \in K$ for all i, which yields that $x_i \in K(\underline{t}^*) \subseteq \iota(F)$. As a consequence, $\Gamma \subseteq v_\lambda F$, so that $v_\lambda F = \Gamma$. □

Now we are ready for the proof of Theorem 1.1.

Proof of Theorem 1.1:
Assume the setting as in the statement of our theorem. We write $\Gamma = \bigoplus_{1\leq i\leq r} \mathbb{Z}\alpha_i$ with $\alpha_i > 0$. We break our proof into several parts.

Part I: We will first assume that $\wp$ is a nontrivial place.

As in the proof of Theorem 2.3 we choose the fields K_0 and F', and the elements $t_1,\ldots,t_s, y,\ g_i$ and h_i satisfying (4) and (5). We consider the place ξ extended from F to $F' = K_0.F$. Then we take ξ_0 to be the restriction of ξ to K_0 . Note that ξ_0 is an extension of $\wp$ and that $F'\xi$ is algebraic over $F\xi$. For every i such that $\xi(a_i) \in K\wp$ we can choose $a'_i \in K$ such that

$$\xi(a_i) \;=\; \wp(a'_i) \;=\; \xi(a'_i)\,.$$

As an extension of $\wp$, also ξ_0 is nontrivial. Therefore, we can apply part 2) of Theorem 2.2 to obtain that (K_0, ξ_0) is existentially closed in (F', ξ). Hence there exist elements

$$t'_1\,, \ldots,\, t'_s\,,\, y' \in K_0$$

such that for $1 \leq i \leq m$,

$$\begin{array}{ll}
(i) & f(\underline{t}', y') = 0 \;\text{ and }\; \dfrac{\partial f}{\partial Y}(\underline{t}', y') \neq 0\,,\\[2mm]
(ii) & g_i(\underline{t}', y') \neq 0,\; h_i(\underline{t}') \neq 0\,,\\[2mm]
(iii) & v_{\xi_0} g_i(\underline{t}', y') \geq v_{\xi_0} h_i(\underline{t}') \quad \text{if} \quad a_i \in \mathcal{O}_\xi\,,\\[2mm]
(iv) & v_{\xi_0} g_i(\underline{t}', y') > v_{\xi_0} h_i(\underline{t}') \quad \text{if} \quad a_i \in \mathcal{M}_\xi\,,\\[2mm]
(v) & v_{\xi_0}\left(\dfrac{g_i(\underline{t}', y')}{h_i(\underline{t}')} \;-\; a'_i\right) \;>\; 0 \quad \text{if} \;\; \xi(a_i) \in K\wp\,,
\end{array}$$

since these assertions are true in F' for $\underline{t}, y$ in place of $\underline{t}', y'$ and v_ξ in place of v_{ξ_0} .

Now let K_1 be as in the proof of Theorem 2.3, and let $\wp_1$ denote the restriction of ξ_0 to K_1 . As before, K_1 is a finite extension of K and $\wp_1$ is an extension of $\wp$. The extension K_4 of K_1 with the K_1-rational place λ_4 and the embedding ι of F' over K_1 in K_4 are constructed as in the proof of Theorem 2.3. As before, we obtain $\lambda \in S(F|K)$ with $v_\lambda F \subseteq \Gamma$, $(\Gamma : v_\lambda F)$ finite and $F\lambda|K$ finite. We take $\wp'$ to be the restriction of $\wp_1$ to $F\lambda$. Then assertions *(a)* and *(b)* of our theorem are satisfied.

We still have to check assertions *(c)*, *(d)*, *(e)* and *(f)* on the elements a_i . Since $\lambda_4(t^*_i) = t'_i$ and $\lambda_4(y^*) = y'$, we have that

$$\lambda(g(\underline{t}, y)) \;=\; \lambda_4(\iota(g(\underline{t}, y))) \;=\; \lambda_4(g(\underline{t}^*, y^*)) \;=\; g(t', y')$$

for every polynomial $g \in K_1[X_1,\ldots,X_s, Y]$. Consequently, using that $h_i(t') \neq 0$ by *(ii)*,

$$\lambda(a_i) \;=\; \lambda\left(\frac{g_i(\underline{t}^*, y^*)}{h_i(\underline{t}^*)}\right) \;=\; \frac{g_i(\underline{t}', y')}{h_i(\underline{t}')}\,.$$

Hence *(ii)*, *(iii)* and *(iv)* imply that *(c)*, *(d)* and *(f)* hold (note that $\lambda(a_i) \in \mathcal{O}_{\wp'}$ implies that $a_i \in \mathcal{O}_{\wp'\circ\lambda} = \mathcal{O}_{\xi'}$ and that $\lambda(a_i) \in \mathcal{M}_{\wp'}$ implies that $a_i \in \mathcal{M}_{\wp'\circ\lambda} = \mathcal{M}_{\xi'}$). If $\xi(a_i) \in K\wp$, then by *(v)*,

$$\begin{array}{rcl}
\xi'(a_i) &=& \wp'(\lambda(a_i)) \;=\; \wp'\left(\dfrac{g_i(\underline{t}', y')}{h_i(\underline{t}')}\right) \;=\; \xi_0\left(\dfrac{g_i(\underline{t}', y')}{h_i(\underline{t}')}\right)\\[2mm]
&=& \xi_0\,(a'_i) \;=\; \xi\,(a'_i) \;=\; \xi\,(a_i)\;,
\end{array}$$

which shows that also assertion *(e)* holds.

Part II: We will now assume that $\wp$ is trivial. In this case we can assume that $\wp = \mathrm{id}_K$ since otherwise we replace ξ by $\xi \circ \sigma$ where σ is any monomorphism on F which extends $\wp^{-1}$. We then also choose every extension $\wp'$ of $\wp$ to be the identity. Further, we have that $\mathcal{O}_\wp = K$ and $\mathcal{M}_\wp = \{0\}$.

Part II.1: First we discuss the case where the place ξ is trivial. Then ξ is a monomorphism and we may assume that $\xi|_F = \mathrm{id}_F$ since otherwise, we apply the following proof to $F\xi$ and $\xi(a_i)$ in place of F and a_i and then replace the places λ of $F\xi$ that we obtain by the places $\lambda \circ \xi$ of F.

Since ξ is trivial, we have that $\mathcal{O}_\xi = F$ and $\mathcal{M}_\xi = \{0\}$. Hence assertion *(d)* of our theorem is satisfied for every $\lambda \in S(F|K)$ because $a_i \in \mathcal{M}_\xi$ would imply that $a_i = 0$, contrary to our choice of the elements a_i. Also assertion *(e)* is always satisfied, as the condition $\xi(a_i) \in K\wp$ means that $a_i \in K$, whence $\xi'(a_i) = \wp(a_i) = a_i = \xi(a_i)$ as λ and $\wp$ are trivial on K.

For any choice of finitely many elements $a_1, \ldots, a_m \in F$, Theorem 2.3 shows the existence of infinitely many places λ which satisfy assertions *(a)* and *(b)* as well as $\lambda(a_i) \neq 0, \infty$ for $1 \leq i \leq m$. The latter implies that they also satisfy assertions *(c)* and *(f)*.

Part II.2: Now we deal with the case of ξ being nontrivial.

Part II.2a: We wish to satisfy assertions *(d)* and *(e)*, but not necessarily assertion *(f)*.

Assume first that $\operatorname{trdeg} F|K = 1$. We claim that $\lambda = \xi$ satisfies assertions *(a)*–*(e)*. Indeed, *(a)* and *(b)* are satisfied since ξ is a nontrivial place of the function field $F|K$ of transcendence degree 1 which is trivial on K. As indicated before, we choose $\wp'$ on $F\xi$ to be the identity, so we have that $\xi' = \xi$, $\mathcal{O}_{\wp'} = F\xi$ and $\mathcal{O}_{\xi'} = \mathcal{O}_\xi$. Hence if $a_i \in \mathcal{O}_\xi$, then $\lambda(a_i) = \xi(a_i) \in \mathcal{O}_{\wp'}$ and $a_i \in \mathcal{O}_{\xi'}$, that is, λ also satisfies assertion *(c)*. Likewise, we have that $\mathcal{M}_{\wp'} = \{0\}$ and $\mathcal{M}_{\xi'} = \mathcal{M}_\xi$. Hence if $a_i \in \mathcal{M}_\xi$, then $\lambda(a_i) = \xi(a_i) = 0 \in \mathcal{M}_{\wp'}$ and $a_i \in \mathcal{M}_{\xi'}$, that is, λ also satisfies assertion *(d)*. Further, $\xi'(a_i) = \xi(a_i)$, so also assertion *(e)* is satisfied.

Assume now that $\operatorname{trdeg} F|K > 1$. Since ξ is nontrivial, there is some $x \in F$ such that $\xi(x) = 0$. We denote the x-adic place of $K(x)$ by ξ_x. We apply the already proven part of our theorem to the function field $F|K(x)$, with $\wp$ replaced by ξ_x, to obtain a place $\lambda' \in S(F|K(x))$ such that, with ξ_x extended to $F\lambda'$ and $\lambda := \xi_x \circ \lambda' \in S(F|K)$,

(a$'$) $F\lambda'$ is a finite extension of $K(x)$,

(b$'$) $v_{\lambda'}F \subseteq \Gamma'$ with $(\Gamma' : v_{\lambda'}F)$ finite,

(c$'$) if $a_i \in \mathcal{O}_\xi$, then $\lambda'(a_i) \in \mathcal{O}_{\xi_x}$ and, consequently, $a_i \in \mathcal{O}_\lambda$,

(d$'$) if $a_i \in \mathcal{M}_\xi$, then $\lambda'(a_i) \in \mathcal{M}_{\xi_x}$ and, consequently, $a_i \in \mathcal{M}_\lambda$,

(e$'$) if $\xi(a_i) \in K(x)\xi_x = K$, then $\lambda(a_i) = \xi(a_i)$ for $1 \leq i \leq m$,

(f$'$) $\lambda'(a_i) \neq 0, \infty$ for $1 \leq i \leq m$.

Now (a$'$) implies that $F\lambda = (F\lambda')\xi_x$ is a finite extension of $K(x)\xi_x = K$, so assertion *(a)* of our theorem is satisfied. Since $\operatorname{trdeg} F\lambda'|K = \operatorname{trdeg} K(x)|K = 1$, we obtain that $v_{\xi_x}(F\lambda') = \mathbb{Z}$, so that $v_{\xi_x}F$ is the lexicographic product of $v_{\lambda'}F$ with $\mathbb{Z}$, which by (b$'$) is a subgroup of Γ, with $(\Gamma : v_\lambda F)$ finite.

To see that assertions *(c)* and *(d)* of our theorem are satisfied, we recall that we take the extension $\wp'$ of the trivial place $\wp$ of K to $F\lambda$ to be the identity. Consequently, $\mathcal{O}_\lambda = \mathcal{O}_{\wp'\circ\lambda}$ and $\mathcal{M}_\lambda = \mathcal{M}_{\wp'\circ\lambda}$. To see that assertion *(e)* of our theorem is satisfied, we use statement (e′) above and observe that $\xi' = \wp' \circ \lambda = \lambda$.

Part II.2b: We wish to satisfy assertion *(f)*, but not necessarily assertions *(d)* and *(e)*. In the present setting where $\wp$ is trivial, assertion *(c)* follows directly from assertion *(f)*. Hence as a matter of fact, the given place ξ does not play any role. Therefore we obtain infinitely many places λ with the required properties by just applying Theorem 2.3. □

For the next proof, we will use a version of the Ax-Kochen-Ershov Theorem for **tame fields** as presented in [25]. These are henselian valued fields (K, v) whose absolute ramification field is algebraically closed. Here, the **absolute ramification field** of (K, v) with respect to an extension of the valuation v to the separable algebraic closure K^{sep} of K is the ramification field of the extension $(K^{\mathrm{sep}}|K, v)$.

Further, we need the following notation. If E is any field, we will denote by E^{1/p^∞} its perfect hull (which is equal to E if $\operatorname{char} E = 0$).

Proof of Proposition 1.2:
We will first assume that $\wp$ is nontrivial. We modify the proof of Theorem 1.1 for this case as follows. We take (L, ξ) to be a maximal algebraic extension of (F, ξ) with the property of having $(F\xi)^{1/p^\infty} = (K\wp)^{1/p^\infty}$ as its residue field. Then (L, ξ) will have a divisible value group. For the construction of such an extension, see Section 2.3 of [23]. Further, (L, ξ) is algebraically maximal (i.e., does not admit nontrivial immediate algebraic extensions) and therefore, it is a tame field by [25, Theorem 3.2].

This time we take K_0 to be the relative algebraic closure of K in L, and ξ_0 the restriction of ξ to K_0 ; as before, ξ_0 is an extension of $\wp$. Since $L\xi = (K\wp)^{1/p^\infty}$ is algebraic over $K\wp$, [25, Lemma 3.7] shows that (K_0, ξ_0) is a tame field with $K_0\xi_0 = L\xi = (K\wp)^{1/p^\infty}$ and $v_{\xi_0}K_0$ equal to the divisible hull of $v_\wp K$.

Since a divisible ordered abelian group is existentially closed in every ordered abelian group extension by part 5) of Theorem 2.2, and since ξ_0 is nontrivial, we can apply [25, Theorem 1.4] to obtain that (K_0, ξ_0) is existentially closed in (L, ξ). By [25, Lemma 3.1], the tame field K_0 is perfect, hence again $K_0.F|K_0$ is separably generated.

From here, the construction proceeds as before. Since $K_1|K$ is finite and $K_1\wp_1$ is contained in the purely inseparable extension $L\xi$ of $K\wp$, we conclude that $K_1\wp_1$ is a finite purely inseparable extension of $K\wp$. Since $\iota(F) \subseteq K_4$, we have that $F\xi' \subseteq (K_4\lambda_4)\wp_1 = K_1\wp_1$. Therefore, $F\xi'|K\wp$ is a finite purely inseparable extension.

Now we assume that $\wp$ is trivial and that $\operatorname{trdeg} F|K > 1$. By our general assumption on function fields $F|K$, we have that $F \neq K$. Hence if both $\wp$ and ξ are trivial, the condition $F\xi = K\wp$ is never satisfied, and therefore, the assertion of our proposition is trivially true. Therefore, we now assume that ξ is nontrivial.

From our assumption that $F\xi = K\wp$ it follows that ξ is nontrivial. With the element x chosen as in the corresponding part of the proof of Theorem 1.1, we have that $K(x)\xi_x = K\wp = F\xi$. Then the condition of our proposition is satisfied for $(K(x), \xi_x)$ in place of $(K, \wp)$. Since ξ_x is nontrivial, from the already proven part of our proposition we infer that in addition to the results of Theorem 1.1 we can choose λ' such that $F(\xi_x \circ \lambda') = (F\lambda')\xi_x$ is a finite purely inseparable extension of $K(x)\xi_x$. As $\xi' = \wp' \circ \xi_x \circ \lambda' = \xi_x \circ \lambda'$ and $K(x)\xi_x = K = K\wp$, this yields that $F\xi'|K\wp$ is a finite purely inseparable extension. □

Proof of Proposition 1.3:
Assume that $(K,\wp)$ is existentially closed in (F,ξ). In the case of nontrivial $\wp$, we modify the proof of Theorem 1.1 by setting $K_0 = K$. Then we will also have that $K_1 = K$. The further construction proceeds as in the proof of Theorem 1.1, yielding a place λ such that $F\lambda = K$.

In the case of trivial $\wp$, the hypothesis yields that also ξ is trivial, and the corresponding part of the proof of Theorem 1.1 can be combined with the second assertion of Theorem 2.3.

In both cases the places λ constructed satisfy $F\lambda = K$, which gives us that $\wp' = \wp$ and $F\xi' = K\wp$. Further, one shows as in the proof of Theorem 2.3 that $v_\lambda F = \Gamma$. □

Proof of Theorem 1.4:
Under the assumptions of Theorem 1.4 we know from part 4) of Theorem 2.2 that in the language of rings with relation symbols for an ordering and a valuation, K is existentially closed in F. Hence from Proposition 1.3, we obtain places λ of F such that in addition to the assertions of Theorem 1.1, we have that $F\lambda = R$, $v_\lambda F = \Gamma$, $F\xi' = R\wp$ and $\wp' = \wp$.

To prove assertion *(c')*, we assume that $a_i > 0$ and $\xi(a_i) \neq 0,\infty$ for some i. This means that $a_i, a_i^{-1} \in \mathcal{O}_\xi$. Hence, in view of assertion *(c)* we can choose λ such that $\lambda(a_i)$, $\lambda(a_i^{-1}) \in \mathcal{O}_\wp$, so $\xi'(a_i) = \wp(\lambda(a_i)) \neq 0,\infty$. Since $\lambda(a_i) > 0$ and $\wp$ is compatible with the ordering on K, this implies that $\xi'(a_i) = \wp(\lambda(a_i)) > 0$. This proves assertion *(c')*.

Since ξ is compatible with the ordering on F, $\infty \neq \xi(a_i) > 0$ implies that $a_i > 0$. Thus assertion *(c')* implies that $\xi'(a_i) > 0$.

It remains to deal with assertion *(g)*. We have to show that in all cases where we can get λ to satisfy assertion *(f)*, we can also get it to satisfy assertion *(g)*. To this end, we replace F by a larger ordered function field in which every positive a_i is a square. When λ satisfies assertion *(f)*, we have that $\lambda(a_i) \neq 0,\infty$. If a_i is a square, it then follows that also $\lambda(a_i)$ is a nonzero square, hence positive. □

Proof of Proposition 1.6:
It suffices to show the assertion for the places λ. This is seen as follows. The valuation ring of λ is an overring of $\wp' \circ \lambda$ and the overrings of a valuation ring in a field are linearly ordered by inclusion. Hence if λ_1 and λ_2 are such that $\wp' \circ \lambda_1$ and $\wp' \circ \lambda_2$ have the same valuation ring, then the valuation rings of λ_1 and λ_2 must be comparable by inclusion. But since $F\lambda_1$ and $F\lambda_2$ are algebraic over K, this implies that the two valuation rings are equal.

In all cases where the constructed places λ satisfy assertion *(f)* of our theorem, the argument given in the proof of Theorem 2.3 shows the existence of infinitely many nonequivalent places λ.

It remains to prove our assertion in the case where $\wp$ is trivial while ξ is not and $\operatorname{trdeg} F|K > 1$, and we want the places to satisfy assertions *(d)* and *(e)*. In this case, in the corresponding part of the proof of Theorem 1.1, we constructed places λ' satisfying assertion (f$'$), hence by what we just said, there are infinitely many nonequivalent such places λ'. By our above argument (with λ replaced by λ' and $\wp$ replaced by ξ_x), also the resulting places $\lambda = \xi_x \circ \lambda'$ are nonequivalent. □

3. Applications to topologies and holomorphy rings

In this section, F always denotes a formally real function field over a real closed base field R.

3.1. **Sets of real places and their topologies.**

From Theorem 1.4 we will deduce:

Proposition 3.1. *1) The set $M_R(F)$ is dense in $M(F)$ with respect to* Top $M(F)$.
2) If in addition R is non-archimedean, then every nonempty intersection of an open set in the Zariski patch topology of $M(F)$ with an open set in Top $M(F)$ *contains infinitely many places from $M_R(F)$.*

Proof. Assume that there is an $\mathbb{R}$-place $\xi \in U(f_1, ..., f_m)$ and choose a compatible ordering $<_\xi$ on F. Then there are positive rational numbers q_1 and q_2 such that

$$q_1 <_\xi f_i <_\xi q_2 \ \text{ for } 1 \leq i \leq m\,.$$

Using Theorem 1.4, we obtain a place $\lambda \in M(F|R)$ such that

$$q_1 < \lambda(f_i) < q_2$$

in R. Composing λ with ξ_R we obtain

$$q_1 \leq \xi_R \circ \lambda(f_i) \leq q_2\,,$$

which shows that the $\mathbb{R}$-place $\xi_R \circ \lambda$ is in $U(f_1, ..., f_m)$. This proves the first assertion of Proposition 3.1.

In order to prove the second assertion, assume in addition that R is non-archimedean. Then ξ_R is a nontrivial place. Further, take elements $a_1, \ldots, a_{k+\ell+m} \in F$ and an $\mathbb{R}$-place ξ of F such that $a_1, \ldots, a_k \in \mathcal{O}_\xi$, $a_{k+1}, \ldots, a_{k+\ell} \in \mathcal{M}_\xi$ and $\xi(a_{k+\ell+1}) > 0, \ldots, \xi(a_{k+\ell+m}) > 0$. Note that $\xi|_R = \xi_R$. Hence by Theorem 1.4 in connection with Proposition 1.6, there are infinitely many R-rational places λ of F and places $\xi' = \xi_R \circ \lambda$ such that:

(1) $a_1, \ldots, a_k \in \mathcal{O}_{\xi'}$ and $a_{k+1}, \ldots, a_{k+\ell} \in \mathcal{M}_{\xi'}$;

(2) $\xi'(a_{k+\ell+1}) > 0, \ldots, \xi'(a_{k+\ell+m}) > 0$. □

We observe the following equivalences that hold for all $a \in F$:

$$\lambda(a) > 0 \quad \Leftrightarrow \quad \lambda\left(\frac{a}{1+a^2}\right) > 0\,, \tag{6}$$

$$\lambda(a) \neq 0, \infty \quad \Leftrightarrow \quad \lambda\left(\frac{a^2}{1+a^2}\right) \;>\; 0\,. \tag{7}$$

We introduce a topology Top $M(F|R)$ on $M(F|R)$ through the basic open sets

$$V(f_1, ..., f_m) \;:=\; \{\lambda \in M(F|R) \mid \lambda(f_i) > 0 \text{ for } 1 \leq i \leq m\}$$

where $f_1, ..., f_m \in H(F|R)$. Note that $\frac{f_i}{1+f_i^2} \in H(F) \subseteq H(F|R)$. Using the equivalence (6) we can thus replace the condition "$f_1, ..., f_m \in H(F|R)$" by "$f_1, ..., f_m \in H(F)$" without changing the collection of basic sets.

Proposition 3.2. *Consider elements $f_1, \ldots, f_k \in H(F)$ and nonzero elements $a_1, \ldots, a_\ell \in F$. If the basic open set $V(f_1, \ldots, f_k)$ of $\operatorname{Top} M(F|R)$ is nonempty, then there are infinitely many places in*

$$\{\lambda \in V(f_1, \ldots, f_k) \mid \lambda(a_j) \neq 0, \infty \textit{ for } 1 \leq j \leq \ell\} . \tag{8}$$

Proof. Take $\lambda \in V(f_1, \ldots, f_k)$. Then λ is an R-rational place with $\lambda(f_i) > 0$ for $1 \leq i \leq k$ and therefore F admits an ordering P which is compatible with λ and under which $f_1, \ldots, f_k$ are positive. We next pass to the function field $F' = F(a_{l+1}, \ldots, a_{l+k})$ where $f_i = a_{l+i}^2$ for $i = 1 \leq i \leq k$. The ordering P of F can be extended to F', so F' is also a formally real function field over R. The extension $F'|F$ is finite and we note that all $a_i, i = 1, \ldots, l + k$, are nonzero elements.

Since R is real closed and F' is formally real, we know that the field R is existentially closed in F'. By Theorem 2.3 there are infinitely many places in $M(F'|R)$ which do not take the value ∞ on $a_1, \ldots, a_{\ell+k}, a_1^{-1}, \ldots, a_{\ell+k}^{-1}$. Hence they do not take the values $0, \infty$ on $a_1, \ldots, a_{\ell+k}$. In particular, they send $f_1, \ldots, f_k$ to squares $\neq 0, \infty$ which consequently are positive elements of R. The restrictions of these places are places in $M(F|R)$ as their residue fields are subfields of finite degree below R, hence equal to R. So they yield the desired places in the set (8). Indeed, since $F'|F$ is finite, the restriction of infinitely many places of F' yields infinitely many places of F. □

Remark 3.3. Alternatively, one may prove the last proposition by passing to regular affine R-algebras A with quotient field F' which contain at least all elements $a_1, \ldots, a_{\ell+k}, a_1^{-1}, \ldots, a_{l+k}^{-1}$. One then applies the Artin-Lang Homomorphism Theorem (see [3, 4.1.2]) and uses the fact that every regular R-point is the center of a rational R-place. Yet, as shown in [3] the Artin-Lang Theorem is a kind of geometric version of model theoretic facts. ♢

A point x in a topological space is called **isolated** if the singleton $\{x\}$ is an open set.

Theorem 3.4. *Let ξ_R denote the natural $\mathbb{R}$-place of R.*

1) The mapping $\iota_{F|R} : M(F|R) \to M_R(F)$ defined in (3) is a bijection.

2) $\iota_{F|R}$ is a topological embedding of $M(F|R)$ into $M(F)$.

3) All nonempty open sets in $M(F|R)$, $M(F)$ and $M_R(F)$ are infinite.

4) In particular, none of the spaces $M(F|R)$, $M(F)$ and $M_R(F)$ admit any isolated points.

Proof. 1): This is a special instance of Proposition 3.6 in the next section.

2): We first prove that $\iota_{F|R} : M(F|R) \to M(F)$ is continuous. Take $\lambda \in M(F|R)$ and $f \in H(F)$ such that $\xi := \iota_{F|R}(\lambda) = \xi_R \circ \lambda \in U(f)$. Then there are positive rationals c, d such that $c < \xi(f) < d$. Then also $c < \lambda(f) < d$, so

$$\lambda \in V(f - c) \cap V(d - f) \ =: \ V$$

and V is an open neighbourhood of λ. We will show that $\iota_{F|R}(V) \subseteq U(f)$. If $\lambda' \in V$, then $c < \lambda'(f) < d$, whence $c \leq \xi_R \circ \lambda'(f) \leq d$. Thus $\xi_R \circ \lambda' \in U(f)$. Hence, $\iota_{F|R}$ is shown to be continuous.

Next, we prove that $\iota_{F|R} : M(F|R) \to M_R(F)$ is an open map. To this end, take an arbitrary subbasic open set $V(f) = \{\lambda \in M(F|R) \mid \lambda(f) > 0\}$ where we may take $f \in H(F)$. We have to show that $\iota_{F|R}(V(f))$ is open in the subspace topology on $M_R(F)$. Take any $\lambda \in V(f)$ and set $\xi = \xi_R \circ \lambda$. Then $a := \lambda(f) \in H(R), a > 0$. Set $g := \frac{af}{a^2+f^2}$. One sees that $g \in H(F)$. We obtain that $\lambda(g) = \frac{1}{2} = \xi(g)$ and therefore $\xi \in U(g) \cap M_R(F)$. We want to show that the whole neighbourhood $U(g) \cap M_R(F)$ of ξ is contained in $\iota_{F|R}(V(f))$.

If $\xi' \in U(g) \cap M_R(F)$, then $\xi' = \xi_R \circ \lambda'$ with $\lambda' \in M(F|R)$, and $\xi_R(\lambda'(g)) = \xi'(g) > 0$ implies that $\lambda'(g) > 0$, whence $\lambda'(f) > 0$. This yields that $\lambda' \in V(f)$, and the inclusion $U(g) \cap M_R(F) \subseteq \iota_{F|R}(V(f))$ is proven.

3): The assertion about $M(F|R)$ follows from Proposition 3.2. From this the assertion about $M_R(F)$ follows by part 2) of our theorem, which together with the density of $M_R(F)$ in $M(F)$ (see Proposition 3.1) implies the assertion for $M(F)$.

4): The assertions follow directly from part 3). □

Remark 3.5. Here is an even simpler proof of the fact that $M(F)$ has no isolated points (from which the same follows for $M_R(F)$ and $M(F|R)$ via the density of $M_R(F)$ in $M(F)$ and part 2) of the above theorem). We have that F is a finite extension of some rational function field $R(x_1, ..., x_n)$. Assume that ξ is an isolated point in F, i.e., $U := \{\xi\}$ is an open subset of $M(F)$. Take the inverse image V of U in the space of orderings of F. It is open since $M(F)$ is a quotient space of the space of orderings of F, and it has only finitely many elements by the Baer-Krull Theorem (note that the value group of the restriction of any place $\xi \in M(F)$ to the real closed field R has divisible value group and as $F|R$ has finite transcendence degree, it follows that $v_\xi(F)/2v_\xi(F)$ is finite). Consider the set of all orderings on $R(x_1, ..., x_n)$ induced by the orderings in V. By the openness of the restriction function for orderings (see [11, Theorem 4.4]), this set is open in the space of orderings of $R(x_1, ..., x_n)$. As it contains a finite number of elements, this is impossible, as [7, Theorem 10] shows that the space of orderings of $R(x_1, ..., x_n)$ does not have isolated points. ♢

3.2. **Holomorphy rings.**

In the introduction we alluded to the rings

$$H(F)B[x_1, \ldots, x_n]\,,$$

B any real valuation ring of R. We will show that these rings admit a description as an intersection of valuation rings of a family $\mathcal{F}$ of composite places, or in other words: they are the holomorphy ring of this family. This section begins with a general study of rings which are intersections of families of valuation rings of composite places. It turns out that this property is closely related to a certain type of Nullstellensatz, a fact which was first observed by H.-W. Schülting, see [30, Section 2] in the context considered there.

Two further issues will be discussed in this section. We look at the existence of minimal representations as an intersection of valuation rings of composite places. Secondly, a new description of the real holomorphy ring $H(F)$ will be presented which can be seen as an analogue to a certain refinement of Artin's solution of Hilbert's 17th problem.

Given any subring D of F, the relative real holomorphy ring $H(F|D)$ is defined as follows:

$$H(F|D) := \bigcap\{\mathcal{O} \mid \mathcal{O} \text{ a real valuation ring of } F \text{ and } D \subseteq \mathcal{O}\}\,.$$

As said above, relative real holomorphy rings are overrings of $H(F)$, hence they are Prüfer rings with F as their field of fractions. This applies to our special case. In addition, we will be using that Prüfer rings are the intersection of their valuation overrings and that the valuation overrings of $H(F)$ are exactly the valuation rings of the real places of F. We find that

$$\begin{aligned} H(F|D) &= H(F)D\,, \\ H\left(F|B[x_1,\ldots,x_n]\right) &= H(F|B)[x_1,\ldots,x_n] \;=\; H(F)B[x_1,\ldots,x_n]\,, \end{aligned}$$

for any family $x_1,\ldots,x_n$ of elements of F.

In fact, one checks that the rings to be compared admit the same set of valuation overrings.

Note: if a subring $A \subseteq F$ is the intersection of real valuation rings of F, then it must contain $H(F)$. Hence, for a general discussion we will impose the condition

$$H(F) \subseteq A$$

throughout, if not stated otherwise. Under this condition, the ring A is a Prüfer ring. Hence it is the intersection of all valuation overrings which are real valuation rings as they contain $H(F)$. However, we are not interested in this sort of presentation of A as an intersection of general real valuation rings. As mentioned before, we want to study rings A which admit an intersection presentation by valuation rings of composite places.

The real valuation rings of the base field R are just the overrings of $H(R) = \mathcal{O}_R$. They will be denoted by, say, B and C, and their canonical places by π_B and π_C. Recall that if B is a valuation ring of R with maximal ideal $\mathcal{M}_B$, then π_B sends every $a \in \mathcal{O}_B$ to $a + \mathcal{M}_B \in \mathcal{O}_B/\mathcal{M}_B$, and every $a \in R \setminus \mathcal{O}_B$ to ∞. If $\wp$ is a real place on R, then π_B is equivalent to $\wp$. In particular, if $B = \mathcal{O}_R$, then π_B is equivalent to ξ_R, and if $B = R$, then π_B is equivalent to id_R.

These real places are pairwise non-equivalent and they represent the equivalence classes of real places on R. The objects of central interest in this paper are the **composite places** $\wp \circ \lambda$, $\wp$ any real place on R and $\lambda \in M(F|R)$. They are equivalent to the places $\pi_B \circ \lambda$, B any real valuation ring of R, $\lambda \in M(F|R)$. This follows from the fact that the valuation ring of $\wp \circ \lambda$ equals $\lambda^{-1}(B)$ where $B = \mathcal{O}_\wp = \mathcal{O}_{\pi_B}$. In addition, $\mathcal{O}_{\pi_B \circ \lambda}$ has the maximal ideal $\lambda^{-1}(\mathcal{M}_B)$ and its residue field equals the residue field of B.

From the following result we then deduce the fact that the family $\pi_B \circ \lambda, B$ a real valuation ring of $R, \lambda \in M(F|R)$, is a complete system of representatives of the family of composite places as defined above.

Proposition 3.6. *Let B, C be real valuation rings of R and $\lambda, \mu \in M(F|R)$. If the places $\pi_B \circ \lambda$ and $\pi_C \circ \mu$ are equivalent, then*

$$B = C,\ \lambda = \mu.$$

Proof. Let V denote the valuation ring of the composite place $\pi_B \circ \lambda$. Then $V \cap R = B$. Hence, $B = C$ follows. Clearly, $V \subseteq \mathcal{O}_\lambda, V \subseteq \mathcal{O}_\mu$. Therefore, these two valuation rings are comparable, say $\mathcal{O}_\lambda \subseteq \mathcal{O}_\mu$. Pick any $a \in \mathcal{O}_\mu$. then $\mu(a - \mu(a)) = 0$. As the maximal ideal of the larger valuation ring $\mathcal{O}_\mu$ is contained in the maximal ideal of $\mathcal{O}_\lambda$, we find that $\lambda(a - \mu(a)) = 0$, whence $\lambda = \mu$. □

The concepts and results we are presenting in this paper only depend on the equivalence class of the composite places, as one may check. Therefore, it is sufficient to work with the distinguished family

$$\mathcal{C}(F) := \{\pi_B \circ \lambda \mid B \text{ real valuation ring of } R,\ \lambda \in M(F|R)\}$$

of composite places, which we will abbreviate as $\mathcal{C}$, and for a given ring subring A of F with the set

$$\mathcal{C}_A := \{\xi \in \mathcal{C} \mid \xi \text{ finite on } A\} = \{\pi_C \circ \lambda \mid \lambda \in M(F|R),\ \lambda(A) \subseteq C\}\,.$$

In particular,

$$\mathcal{C}_{H(F|B)} = \{\pi_C \circ \lambda \mid \lambda \in M(F|R),\ B \subseteq C\}\,,$$

since for each $\lambda \in M(F|R)$ we have that $\lambda(H(F)) = H(R) = \mathcal{O}_R$ and therefore $\lambda(H(F|B)) = \lambda(H(F)B) = B$. Since

$$\mathcal{C}_{A[x]} = \{\xi \in \mathcal{C}_A \mid \xi(x) \neq \infty\}$$

and $\lambda(H(F|B)[x]) = B[\lambda(x)]$ if λ is finite on $H(F|B)[x]$, we obtain that

$$\mathcal{C}_{H(F|B)[x]} = \{\pi_C \circ \lambda \mid \lambda \in M(F|R),\ B \subseteq C \text{ and } \lambda(x) \in C\}\,. \tag{9}$$

We will make use of the following

Lemma 3.7. *Assume that $H(F) \subseteq A,\ x_1, \ldots, x_n \in F$, and $\mathfrak{a}$ is a finitely generated ideal of A. Then:*

1) there is $x \in F$ with $A[x_1, \ldots, x_n] = A[x] = A[1 + x^2]$,

2) $\sqrt{\mathfrak{a}}$ is the radical of a principal ideal.

Proof. 1): We show that $A[x_1, \ldots, x_n] = A[1+\sum_1^n x_i^2]$. The inclusion "$\supseteq$" is clear, while the inclusion "$\subseteq$" follows from the fact that $\frac{x_i}{1+\sum_k x_k^2} \in H(F)$ for all i, and $H(F) \subseteq A$. We set $x = 1 + \sum_1^n x_i^2$ and observe that a similar argument shows that $A[x] = A[1 + x^2]$ because $\frac{x}{1+x^2} \in H(F)$.

2): Let $\mathfrak{a} = (f_1, \ldots, f_n)$. Then $\mathfrak{a}^2 = (\sum_1^n f_i^2)$ as $\frac{f_i f_j}{\sum_k f_k^2} \in H(F) \subseteq A$. Now our assertion follows since $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{a}^2}$. □

In view of part 1) of this lemma, whenever we will consider a finitely generated ring extension A' of A inside F, we may always assume it to be of the form $A' = A[1+x^2]$ for some $x \in F$.

We say that the ring A satisfies the **intersection property** if

$$A = \bigcap_{\xi \in \mathcal{C}_A} \mathcal{O}_\xi\,,$$

or in other words, if A is the **holomorphy ring of the family** $\mathcal{C}_A$.

Next, we consider an ideal $\mathfrak{a}$ of A, from which we obtain the zero set

$$V(\mathfrak{a}) := \{\xi \in \mathcal{C}_A \mid \xi = 0 \text{ on } \mathfrak{a}\}\,.$$

Likewise, from a subset $V \subseteq \mathcal{C}_A$ we obtain the vanishing ideal

$$I(V) := \{a \in A \mid \xi(a) = 0 \text{ for all } \xi \in V\}\,.$$

Clearly, $\sqrt{\mathfrak{a}} \subseteq I(V(\mathfrak{a}))$. Following the usual terminology, we say that the ideal $\mathfrak{a}$ satisfies the **Nullstellensatz** if $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

The following proposition extends Schülting's result [30, 2.6].

Proposition 3.8. *Assume that $H(F) \subseteq A$. Then the following statements are equivalent:*

1) A satisfies the Nullstellensatz for finitely generated ideals,

2) every finite ring extension $A[x_1, \dots, x_n]$, where $n \in \mathbb{N}$, $x_1, \dots, x_n \in F$, satisfies the Nullstellensatz for finitely generated ideals,

3) every finite ring extension $A[x_1, \dots, x_n]$, where $n \in \mathbb{N}$, $x_1, \dots, x_n \in F$ has the intersection property.

Proof. The implication 2) ⇒ 1) is trivial, as A is its own finite ring extension. Once the equivalence of 1) and 3) is proven for **all** overrings of $H(F)$, the implication 1) ⇒ 2) also follows: if 1) holds, then by 3), every finite ring extension of $A' := A[x_1, \dots, x_n]$ of A, being also a finite ring extension of A, has the intersection property, which implies that 1) holds for A'.

1) ⇒ 3): As stated after the previous lemma, we know that

$$A[x_1, \dots, x_n] \;=\; A[1 + x^2]$$

for some $x \in F$. Set $A' = A[1 + x^2]$ and consider any $f \in \bigcap_{\xi \in \mathcal{C}_{A'}} \mathcal{O}_\xi$.
From the definition we deduce that

$$\xi \in \mathcal{C}_{A'} \Leftrightarrow \xi \in \mathcal{C}_A,\ \xi(1 + x^2) \neq \infty.$$

Hence, for all $\xi \in \mathcal{C}_A$, the implication $\xi(1 + x^2) \neq \infty \Rightarrow \xi(1 + f^2) \neq \infty$, holds, and so its contraposition

$$\xi\left(\frac{1}{1+f^2}\right) = 0 \;\Rightarrow\; \xi\left(\frac{1}{1+x^2}\right) = 0\,. \tag{10}$$

We observe that $\frac{1}{1+x^2}, \frac{1}{1+f^2} \in H(F) \subseteq A$. Consider the principal A-ideal $\mathfrak{a} = \left(\frac{1}{1+f^2}\right)$. From (10) we obtain that $\frac{1}{1+x^2} \in I\,(V(\mathfrak{a}))$. We infer from 1) that $\left(\frac{1}{1+x^2}\right)^k = a\frac{1}{1+f^2}$ for some $k \in \mathbb{N}$, $a \in A$ and $1 + f^2 \in A' = A[1 + x^2]$. The Prüfer ring A' is integrally closed, so $f \in A'$ as was to be shown.

3) ⇒ 1): Take any finitely generated ideal $\mathfrak{a}$ of A. In the proof of statement 2) of the previous lemma it was shown that $\mathfrak{a}^2 = (f)$ for some $f \in A$. In view of $V(\mathfrak{a}) = V((f))$ and $\sqrt{\mathfrak{a}} = \sqrt{(f)}$ we may assume that $\mathfrak{a}$ is a principal ideal (f). Consider $g \in I(V(f))$, so $\xi(f) = 0 \Rightarrow \xi(g) = 0$ holds for all $\xi \in \mathcal{C}_A$. The composite places which are finite on the extension $A[\frac{1}{g}]$ are just the composite ones which are finite on A and satisfy $\xi(\frac{1}{g}) \neq \infty$. By the contrapositive of the above implication, these places also satisfy $\xi(\frac{1}{f}) \neq \infty$. By assumption, the extension $A[\frac{1}{g}]$ has the intersection property, so we obtain that $\frac{1}{f} \in A[\frac{1}{g}]$. From this, $g^k \in (f)$ follows for some $k \in \mathbb{N}$. □

Remark 3.9. 1) That a ring A admits the intersection property does not imply that the Nullstellensatz holds for finitely generated ideals of A. To obtain this implication one really needs the hypothesis for all finitely generated extensions as above. For an example, take $A = \mathcal{O}_\lambda$ for some $\lambda \in M(F|R)$. Then $\mathcal{C}_A = \{\lambda\}$ and the intersection property trivially holds. But if the rank of λ is greater than 1, then there is $f \in \mathcal{O}_\lambda$ with $\sqrt{(f)} \neq \mathcal{M}_\lambda$, while $I(V(f)) = \mathcal{M}_\lambda$.

Pick any $x \in F \setminus \mathcal{O}_\lambda$. Then there is no composite place which is finite on the extension $\mathcal{O}_\lambda[x]$. Hence this ring does not have the intersection property.

2) Assume that A has the intersection property. Then $\mathcal{O}_R$ is contained in A since it is contained in $\mathcal{O}_\xi$ for all $\xi \in \mathcal{C}_A$. It follows that for every $\lambda \in M(F|R)$ that is finite on A, $\lambda(A)$ is a ring containing $\mathcal{O}_R$, hence a real valuation ring of R. This leads to the representation

$$A = \bigcap \{\mathcal{O}_{\pi_{\lambda(A)}\circ\lambda} \mid \lambda \in M(F|R) \text{ finite on } A\} ,$$

since the valuation rings on the right hand side are the minimal ones among all valuation rings $\mathcal{O}_\xi$ with $\xi \in \mathcal{C}_A$. ♢

Theorem 3.10. *For every real valuation ring $B \subseteq R$, each finite ring extension of $H(F|B)$ within F satisfies the Nullstellensatz for finitely generated ideals and has the intersection property.*

Proof. By Proposition 3.8, it suffices to prove that any finite extension A of $H(F|B)$ has the intersection property. Due to Lemma 3.7 we may write $A = H(F|B)[x]$ for some nonzero $x \in F$. Suppose that there exists $f \in \bigcap_{\xi\in\mathcal{C}_A} \mathcal{O}_\xi$ with $f \notin A$. As said above, A is the intersection of all real valuation rings in which it is contained. Hence we find a real place ξ_0 with $A \subseteq \mathcal{O}_{\xi_0}$, $x \in \mathcal{O}_{\xi_0}$ and $f \notin \mathcal{O}_{\xi_0}$. Applying Theorem 1.4 with ξ_0 in place of ξ and $\wp$ the restriction of ξ_0 to R, we obtain a place $\lambda \in M(F|R)$ which satisfies assertions c) and d) of the theorem with $a_1 = x$, $a_2 = \frac{1}{f}$. Then $\lambda(x) \in \mathcal{O}_\wp$ and $\lambda(\frac{1}{f}) \in \mathcal{M}_\wp$, whence $\lambda(f) \notin \mathcal{O}_\wp$. We set $C := \mathcal{O}_\wp = \mathcal{O}_{\xi_0} \cap R$, so $\lambda(x) \in C$. Since $H(F|B) \subseteq A \subseteq \mathcal{O}_{\xi_0}$, we also have that $B \subseteq C$. Hence by (9), $\pi_C \circ \lambda \in \mathcal{C}_A$. But $\lambda(f) \notin \mathcal{O}_\wp$ implies that $f \notin \mathcal{O}_{\pi_C\circ\lambda}$, a contradiction to our choice of f. □

The three distinguished cases of $A = H(F|B)$, $A = H(F) = H(F|\mathcal{O}_R)$ and $A = H(F|R)$ deserve special attention:

$$H(F|B) = \bigcap \{\mathcal{O}_{\pi_B\circ\lambda} \mid \lambda \in M(F|R)\} , \tag{11}$$

$$\begin{aligned} H(F) &= \bigcap \{\mathcal{O}_{\xi_R\circ\lambda} \mid \lambda \in M(F|R)\} \\ &= \bigcap \{\mathcal{O}_\xi \mid \xi \in M_R(F)\} , \end{aligned} \tag{12}$$

$$H(F|R) = \bigcap \{\mathcal{O}_\lambda \mid \lambda \in M(F|R)\} . \tag{13}$$

The presentation of $H(F|B)$ immediately yields the equality $H(F|B) \cap R = B$; in other words: $H(F|B)$ is the smallest relative real holomorphy which extends the real valuation ring B. It should be mentioned that this fact can be deduced more easily. In fact, choose any ordering P of F and consider the convex closure V of $\mathbb{Q}$ in F with respect to P. Then V is a real valuation ring of F extending $\mathcal{O}_R$, and one deduces that the real valuation ring VB extends B. This implies the nontrivial inclusion $H(F|R) \cap R \subseteq B$.

As listed above, $H(F)$ **is the intersection of the family of valuation rings of the real places in** $M_R(F)$**.** This is a straightforward and appealing geometric generalization of the situation in case of $R = \mathbb{R}$.

In what follows we address the question whether there are minimal representations for the relative real holomorphy rings $H(F|B)$ of the type above. More precisely, we will study subfamilies $\mathcal{F} \subseteq M(F|R)$ such that $H(F|B) = \bigcap_{\lambda\in\mathcal{F}} \mathcal{O}_{\pi_B\circ\lambda}$ and look at the existence of minimal families $\mathcal{F}$. This is a topic dealt with by

Schülting in [5, 3.13] and [31, 1.3 ff.] for the case $B = R$. His results are incorporated. More generally, we allow B to range over all real valuation rings of the base field R.

Theorem 3.11. *Let B, C be two real valuation rings of R. Then we have:*

1) If $H(F|B) = H(F|C)$, then $B = C$;

2) if $B \neq R$, then the following statements are equivalent for each subset $\mathcal{F}$ of $M(F|R)$:

(a) $H(F|B) = \bigcap_{\lambda\in\mathcal{F}} \mathcal{O}_{\pi_B\circ\lambda}$,

(b) $\mathcal{F}$ is dense in $M(F|R)$;

3) if $B \neq R$, then there is no representation of the form (a) with a minimal $\mathcal{F}$;

4) $H(F|C)$ admits a representation of the form (a) with a minimal $\mathcal{F}$ if and only if $C = R$ and $\operatorname{trdeg} F|R = 1$. *In the case of a minimal representation we necessarily have that $\mathcal{F} = M(F|R)$.*

Proof. 1): We showed above that $H(F|B) \cap R = R$; the same holds for C.

2): Assume that $\mathcal{F}$ is not dense in $M(F|R)$. Hence by part 2) of Theorem 3.4, $\iota_{F|R}(\mathcal{F})$ is not dense in $M_R(F)$ and thus also not in $M(F)$. Let N be the closure of $\iota_{F|R}(\mathcal{F})$ in $M(F)$ and take $\eta \in M(F) \setminus N$. By the Separation Criterion given in [27, Proposition 9.13], there is $f \in H(F)$ such that $N \subseteq U(-f)$ and $\eta \in U(f)$. Since $M_R(F) = \iota_{F|R}(M(F|R))$ is dense in $M(F)$ by Proposition 3.1, there is $\lambda_0 \in M(F|R)$ such that $\xi_R \circ \lambda_0 \in U(f)$ and thus $a := \lambda_0(f)$ is an element of the set $E^+(R)$ of positive units of $\mathcal{O}_R$. For $\lambda \in \mathcal{F}$ we have that $-\lambda(f) \in E^+(R)$. Define $g := \frac{1}{a-f}$. We have that $\lambda_0(g) = \infty$ and therefore $g \notin \mathcal{O}_{\pi_B\circ\lambda_0}$, whence $g \notin H(F|B)$. But for $\lambda \in \mathcal{F}$ we have $\lambda(g) \in E^+(R) \subseteq \mathcal{O}_R \subseteq B$ and therefore $g \in \mathcal{O}_{\pi_B\circ\lambda}$. Hence (a) cannot be true. This proves that (a) implies (b).

Now we prove that (b) implies (a). We have that $H(F|B) \subseteq \bigcap_{\lambda\in\mathcal{F}} \mathcal{O}_{\pi_B\circ\lambda}$. Suppose that equality does not hold. Then there is some $f \in F$ and $\lambda_0 \in M(F|R)$ such that $\lambda_0(f) \notin B$ but $\lambda(f) \in B$ for all $\lambda \in \mathcal{F}$. Then either $\lambda_0(f) = \infty$ or $\lambda_0(f) \in R \setminus B$. In the first case we choose $a \in R \setminus B$ with $a > 0$ (note that $B \subsetneq R$ since $B \subsetneq C$). In the second case we choose a such that $2a = \lambda_0(f)$; switching f to $-f$ if necessary, we may again assume that $a > 0$. In both cases we see that $-a < b < a$ for all $b \in B$ as B is convex under the ordering $<$ of R. We consider the set

$$S := \{\lambda \in M(F|R) \mid \lambda(f) = \infty \text{ or } |\lambda(f)| > a\}$$

and note that it contains λ_0 . With $g := \frac{f}{1+f^2} \in H(F)$, the condition defining S holds if and only if $\lambda(g) = 0$ or $|\lambda(g)| < \frac{1}{1+a^2} =: b$, which means that $-b < \lambda(g) < b$. This shows that S is an open subset of $M(F|R)$. By the density of $\mathcal{F}$ in $M(F|R)$ there is some $\lambda \in \mathcal{F} \cap S$. Consequently, $\lambda(f) = \infty$ or $|\lambda(f)| > a$ so that $\lambda(f) \notin B$, a contradiction to our assumption on $\lambda(f)$. Thus equality, and hence (a), must hold.

3:) Assume that $\mathcal{F}$ is a dense subset of $M(F|R)$ and that $\eta \in \mathcal{F}$. We wish to show that $\mathcal{F} \setminus \{\eta\}$ is still dense in $M(F|R)$. Suppose not. Then there is an nonempty open subset V of $M(F|R)$ such that $\mathcal{F} \cap (V \setminus \{\eta\}) = (\mathcal{F} \setminus \{\eta\}) \cap V = \emptyset$. But by part 3) of Theorem 3.4, V is infinite and hence $V \setminus \{\eta\}$ is a nonempty open set, so we obtain a contradiction to the density of $\mathcal{F}$.

4:) Assume first that $H(F|C)$ admits a minimal representation. Due to part 3) of our theorem we obtain that $C = R$. We are therefore dealing with the case that

$H(F|R)$ admits a minimal representation. Then necessarily trdeg $(F|R) = 1$; this is proven by Schülting in [31, Section 1] for $R = \mathbb{R}$. Transferring the arguments, one can deduce this from [5, 3.13] also for the case of any real closed base field.

Now consider $H(F|R)$ in the case of trdeg $(F|R) = 1$ and choose an arbitrary $\eta \in M(F|R)$. We are going to prove that even the family $M(F|R) \setminus \{\eta\}$ does not provide the intersection (a) for $H(F|R)$. This will imply claim 4).

We start by constructing a smooth affine model X on which all places in $M(F|R)$ admit a center. Assume $F = R(x_1, \ldots, x_n)$, then set $x_0 = 1$ and $y_i = x_i / \sum_{j=0}^{n} x_j^2$ for $i = 0, \ldots, n$. The elements y_i belong to $H(F|R)$ and generate the function field F over R. Setting $S = R[y_0, \ldots, y_n]$ we have found an affine R-algebra inside $H(F|R)$ with quotient field F.

Take the integral closure T of S in F. We still have that $T \subseteq H(F|R)$ as the Prüfer ring $H(F|R)$ is integrally closed. Due to the Noether Normalization Theorem we obtain that T is an affine R-algebra, say $T = R[t_1, \ldots, t_m]$. We observe that T is a noetherian, integrally closed ring of dimension 1; in other words, we have constructed a smooth affine model X with the Dedekind domain $T \subseteq H(F|R)$ as its coordinate R-algebra.

Given any $\lambda \in M(F|R)$, it is finite on $H(F|R)$, so it induces a R-epimorphism $\phi : T \to R$ the kernel of which is the maximal ideal $\mathfrak{m}$ of a point $a \in X(R)$, the center of λ. As T is a Dedekind domain, the local ring at the point a, i.e., $T_{\mathfrak{m}}$, is a valuation ring; it is also contained in the valuation ring of λ. Using again that T is a Dedekind ring, we see that both valuation rings must be equal. Finally, we find that λ is determined by the induced epimorphism ϕ.

Now consider the place $\eta \in M(F|R)$ chosen above. The elements $s_i = t_i - \eta(t_i)$, $i = 1, \ldots, m$, also generate the algebra T. Thus if $\lambda \neq \eta$, then $\lambda(s_i) \neq \eta(s_i) = 0$ for some i. It follows that for $s = \sum_{1 \leq i \leq m} s_i^2$, the element $1/s$ lies in the valuation ring of λ but not in the valuation ring of η. This shows that $\bigcap_{\lambda \in M(F|R) \setminus \{\eta\}} \mathcal{O}_\lambda$ is strictly larger than $H(F|R)$. □

For a function field F over a non-archimedean real closed field R we will give yet another, geometric representation of $H(F)$ which makes use of of the family of all smooth projective models of F. In doing so we base our arguments on Hironaka's celebrated theory of the resolution of singularities in characteristic zero. One may consult Kollar's excellent presentation of this theory, in particular Chapter 3 of his book [18].

Consider any smooth projective model X of F. For $x \in X$ we denote by $\mathcal{O}_x$ the local ring in x and by $X(R)$ the set of rational points of X. For $f \in F$, by X_f we denote the set of those rational points for which $f \in \mathcal{O}_x$, i.e., f is defined in x. Note that X_f is open in the Zariski topology on $X(R)$ and there exists an open nonempty affine subvariety Y with $f \in \mathcal{O}(Y)$, implying that $Y(R) \subseteq X_f$. As X is smooth, every point in $X(R)$ is the center of some $\lambda \in M(F|R)$.

Using these facts, Artin's solution to Hilbert's 17th Problem can be rephrased as follows:

$$f \in \sum F^2 \Leftrightarrow f(x) \geq 0 \text{ for every } x \in X_f\,.$$

The implication "⇒" can be proven as follows: the point $x \in X_f$ is the center of a place $\lambda \in M(F|R)$ and $f \in \mathcal{O}_x$, hence $f = \sum_1^k f_i^2$ is contained in the valuation ring V of λ. Now, this valuation ring is a real valuation ring, which implies that each $f_i \in V$. Therefore, $f(x) = \lambda(f) = \sum_i \lambda(f_i)^2 = \sum_i f_i(x)^2 \geq 0$. Concerning the other implication, we note that f is defined on an affine model Y of F and non-negative on $Y(R)$. Artin's theorem then states that $f \in \sum F^2$.

The above characterization suggests to look for a geometric characterization of $H(F)$ by appealing to the sets X_f whenever $f \in H(F)$.

Take a function $f \in H(F)$ and take $x \in X(R)$ such that $f \in \mathcal{O}_x$. Since x is the center of some $\lambda \in M(F|R)$, we obtain that $f(x) = \lambda(f)$. Since $\xi_R \circ \lambda$ is an $\mathbb{R}$-place and $f \in H(F)$, we have that $\xi_R \circ \lambda(f) \neq \infty$. Therefore, $\lambda(f) \in \mathcal{O}_R = H(R)$. We have shown:

$$f \in H(F) \Rightarrow f(x) \in H(R) \text{ for every } x \in X_f\,.$$

The converse is in general not true. If R is an archimedean real closed field, then $H(R) = R$ and the right hand side of the implication is always true, while the left hand side is not.

To understand what is going on, we have to turn again to the relative real holomorphy ring $H(F|R)$ and its geometrical description given by Schülting in [29]:

$$H(F|R) \;=\; \{f \in F \mid f \text{ is bounded on } X_f \text{ by elements of } R\}\,. \tag{14}$$

For a smooth real projective variety X, define

$$H_X \;:=\; \{f \in F \mid f(x) \in H(R) \text{ for every } x \in X_f\}\,.$$

Then $H(F) \subseteq H_X$. But functions in H_X are not necessarily bounded on X_f in the case of an archimedean ordered base field R. But in the non-archimedean case every function in H_X is bounded by the elements with negative values under v_R. Therefore, for R non-archimedean we have:

$$H(F) \;\subseteq\; H_X \;\subseteq\; H(F|R)\,, \tag{15}$$

where the latter inclusion follows from (14).

Proposition 3.12. *Take a function field F over a non-archimedean real closed field R. Then $H(F)$ is the intersection of the sets H_X where X runs through all smooth projective models of F.*

Proof. As we observed before, $H(F) \subseteq H_X$ for any smooth model of F. Therefore, $H(F) \subseteq \bigcap\{H_X \mid X \text{ smooth projective model of } F\}$.

Assume that f is in the intersection of the sets H_X. Since R is non-archimedean, (15) shows that f is in $H(F|R)$. By a theorem of Schülting (see [29, page 437]) there is a smooth projective model X_0 such that f is regular in every point of $X_0(R)$. Take any $\mathbb{R}$-place ξ such that $\xi = \xi_R \circ \lambda$, $\lambda \in M(F|R)$. Since $f \in H(F|R)$, we have $f \in \mathcal{O}_\lambda$. The place λ has a center $c(\lambda)$ on the projective model, so $c(\lambda) \in X_0(R)$. Then $\lambda(f) = f(c(\lambda)) \in H(F)$ by our assumption. This means that $\xi(f) \neq \infty$, so $f \in \mathcal{O}_\xi$ for every $\xi \in M_R(F) = \{\xi_R \circ \lambda \mid \lambda \in M(F|R)\}$. Thus $f \in \bigcap\{\mathcal{O}_{\xi_R \circ \lambda} \mid \lambda \in M(F|R)\}$, which by (12) is equal to $H(F)$. □

Note that this theorem is not true for an archimedean real closed field R since in this case $H_X = F$ for every smooth projective model X of F.

4. The Real Spectrum of $H(F|R)$ and $H(F)$

As before, F denotes a formally real function field over a real closed base field R.
The topologies on $M(F|R)$ and $M(F)$ find natural interpretations via the theory of the real spectrum $\mathrm{Spec}_r(A)$ of a commutative ring A. Regarding general concepts and results we refer to [3, Chapter 7] and [17, Kapitel III]; however, note that the authors of the latter reference are using the notation *Sper A* for the real spectrum of A. The real spectrum $\mathrm{Spec}_r(A)$ is a quasi-compact space; we reserve the term "compact", in contrast to the use in [3], for quasi-compact Hausdorff spaces. It is its compact subspace of closed points $\mathrm{MaxSpec}_r(A)$ that we are mainly interested in.

In our situation, we will prove:

Proposition 4.1. *There is a commutative diagram*

$$\begin{array}{ccc} M(F|R) & \xrightarrow{i} & \mathrm{MaxSpec}_r(H(F|R)) \\ \Big\downarrow \iota_{F|R} & /// & \Big\downarrow \tau \\ M(F) & \xrightarrow{j} & \mathrm{MaxSpec}_r(H(F)) \end{array}$$

where

(1) *the maps $i, \iota_{F|R}$ are topological embeddings with dense images,*
(2) *the map j is a homeomorphism,*
(3) *the map τ is continuous and surjective.*

Using this proposition and results from real algebraic geometry over arbitrary real closed fields (see [3, 8]), we can prove:

Proposition 4.2. *$M(F)$ has only finitely many connected components.*

It was already known that the space $M(F)$ of a rational function field $F = R(X_1, \dots, X_n)$ is connected, see [13, Theorem 2.12].

Let A denote any commutative ring. By definition, the real spectrum $\mathrm{Spec}_r(A)$, as a set, is the collection of all **prime cones** $\alpha \subsetneq A$ satisfying the conditions

$$\alpha + \alpha \subseteq \alpha,\ \alpha \cdot \alpha \subseteq \alpha,\ \alpha \cup \alpha = A,\ \alpha \cap -\alpha \text{ is a prime ideal of } A\,.$$

Let a prime cone α be given. We set $\mathrm{supp}\,(\alpha) = \alpha \cap -\alpha$. This prime ideal is called the **support** of α. By the **residue field of** α we will mean the quotient field of $A/\mathrm{supp}\,(\alpha)$. Given any $a \in A$ and $\alpha \in \mathrm{Spec}_r(A)$, we write $a(\alpha) := a + \mathrm{supp}\,(\alpha)$. An ordering $\bar{\alpha}$ with order relation $\leq_\alpha$ (or in short, $\leq$) is induced by requiring, for all $a \in A$,

$$0 \leq a(\alpha) \ \Leftrightarrow\ a \in \alpha$$

hence $a(\alpha) > 0 \Leftrightarrow a \in \alpha \wedge -a \notin \alpha$.

The topology on $\mathrm{Spec}_r(A)$ is defined by the following family of basic open sets:

$$\widetilde{\mathcal{U}}(a_1, \dots, a_n) = \{\alpha \mid a_1(\alpha) > 0, \dots, a_n(\alpha) > 0\}$$

for all $n \in \mathbb{N}, a_1, \dots, a_n \in A$. As ring homomorphisms $\phi : A \to B$ are well behaved with respect to the assignment $A \mapsto \mathrm{Spec}_r(A)$, we are dealing with a contravariant functor Spec_r from the category of rings to the category of quasi-compact spaces.

Here we will only be using the simplest case, where A is a subring of the ring B. It is readily seen that we obtain a continuous map, the restriction

$$\mathrm{res} \;=\; \mathrm{res}_{A,B} : \mathrm{Spec}_r(B) \to \mathrm{Spec}_r(A), \;\; \alpha \mapsto \alpha \cap A \,.$$

If $\alpha, \beta \in \mathrm{Spec}_r(A)$ satisfy $\alpha \subseteq \beta$, then β is called a **specialization** of α and α a **generalization** of β. The specializations of a given prime cone α form a totally ordered set with respect to inclusion, and there is a unique maximal specialization of α, denoted by $\rho(\alpha)$. The maximal prime cones are exactly the closed points in $\mathrm{Spec}_r(A)$. For example, a prime cone whose support is a maximal ideal is a maximal prime cone. We set

$$\mathrm{MaxSpec}_r(A) \;=\; \{\alpha \in \mathrm{Spec}_r(A) \mid \alpha \text{ maximal}\} \,.$$

It turns out that the subspace $\mathrm{MaxSpec}_r(A)$ is compact and that the specialization map

$$\rho \;=\; \rho_A : \mathrm{Spec}_r(A) \to \mathrm{MaxSpec}_r(A), \;\; \alpha \mapsto \rho(\alpha)$$

is continuous and a closed retraction, see [3, 7.1.25] and [17, p.128, Satz 5]. By composing the assignment $A \mapsto \mathrm{Spec}_r(A)$ with the specialization map, we obtain a functor $A \mapsto \mathrm{MaxSpec}_r(A)$ into the category of compact spaces. In the case where A is a subring of B we obtain the continuous map

$$\tau = \tau_{A,B} := \rho_A \circ \mathrm{res}_{A,B} : \mathrm{MaxSpec}_r(B) \to \mathrm{MaxSpec}_r(A) \,.$$

In what follows we will use the following, easily proven observation: if β is a specialization of α and $\mathrm{supp}\,(\alpha) = \mathrm{supp}\,(\beta)$, then $\alpha = \beta$. As already stated above, if $\mathrm{supp}\,(\alpha)$ is a maximal ideal, then α is a maximal prime cone.

We now turn to the proof of Proposition 4.1.

Proof. The map $\iota_{F|R}$ has already been introduced and shown in Theorem 3.4 to be a topological embedding of $M(F|R)$ into $M(F)$; by Proposition 3.1, its image $M_R(F)$ is dense in $M(F)$. The map τ on the right hand side equals $\tau_{A,B}$ for $A = H(F), B = H(F|R)$. So it is continuous. Surjectivity follows once the statements on the maps i, j and the commutativity of the diagram have been shown; this is seen as follows. As we are dealing with compact spaces the image of τ is closed, and furthermore, it contains the image of the dense subspace $M_R(F)$ under the homeomorphism j. All this implies that τ is surjective.

To define the map $i : M(F|R) \to \mathrm{MaxSpec}_r(H(F|R))$ and study its properties we need the following facts. A place $\lambda \in M(F|R)$ induces a R-epimorphism $\phi : H(F|R) \to R$ whose kernel $\mathfrak{p}_\lambda$ is a maximal ideal of $H(F|R)$. As this ring is a Prüfer ring we see that the valuation ring V of λ is just the localization $H(F|R)_{\mathfrak{p}_\lambda}, V = \{a/b \mid a, b \in H(F|R), b \notin \mathfrak{p}_\lambda\}$, hence $\lambda(a/b) = \phi(a)/\phi(b)$. Altogether we obtain that the places in $M(F|R)$ are determined by their restriction to $H(F|R)$. Using the unique ordering on R we now define the natural map

$$\begin{aligned} i : M(F|R) \;&\to\; \mathrm{MaxSpec}_r(H(F|R)), \\ \lambda \;&\mapsto\; \alpha_\lambda := \{a \in H(F|R) \mid \lambda(a) \geq 0\} \,. \end{aligned}$$

We observe that $\mathrm{supp}\,(\alpha_\lambda) = \mathfrak{p}_\lambda$, so indeed, α_λ is a maximal prime cone, as its support is a maximal ideal. As each $\lambda \in M(F|R)$ is the identity on R we find that for each $a \in H(F|R)$ we have $a - \lambda(a) \in \mathfrak{p}_\lambda$. From this the injectivity of i follows: indeed, if $\alpha_\lambda = \alpha_\mu$, then $\mathfrak{p}_\lambda = \mathfrak{p}_\mu$, so $\mu(a - \lambda(a)) = 0$ and therefore $\mu(a) = \lambda(a)$ for every $a \in H(F|R)$, whence $\mu = \lambda$. In addition we obtain that $a(\alpha_\lambda) = \lambda(a)$ for any $a \in H(F|R)$, from which we deduce:

$$i^{-1}\left(\widetilde{\mathcal{U}}(a_1, \dots, a_n) \cap \operatorname{MaxSpec}_r(H(F|R))\right) = V(a_1, \dots, a_n)\,.$$

This means that the map i is a topological embedding. To prove that the image is dense, consider a nonempty basic open subset

$$\overline{\mathcal{U}} = \widetilde{\mathcal{U}}(a_1, \dots, a_n) \cap \operatorname{MaxSpec}_r(H(F|R))$$

and pick one of its elements α. Set $\mathfrak{p} = \operatorname{supp}(\alpha)$. The residue field of the valuation ring $H(F|R)_\mathfrak{p}$ equals the residue field of α. Therefore we can pull back the ordering $\bar{\alpha}$ to construct an ordering $>$ on F which satisfies $a_i > 0$ for $1 \le i \le n$. Now the arguments presented in the proof of Proposition 3.2 yield the existence of $\lambda \in M(F|R)$ with $\lambda(a_i) > 0$ for $1 \le i \le n$. We see that $\alpha_\lambda \in \overline{\mathcal{U}}$.

In the case of the map j we follow a similar route. A place $\xi \in M(F)$ induces a homomorphism $H(F) \to \mathbb{R}$. We define

$$j : M(F) \to \operatorname{Spec}_r(H(F)), \xi \mapsto \alpha_\xi := \{a \in H(F) \mid \xi(a) \ge 0\}\,.$$

This time however, the kernel $\mathfrak{p}_\xi = \operatorname{supp}(\alpha_\xi)$ need not be a maximal ideal. Nevertheless, $\alpha_\xi \in \operatorname{MaxSpec}_r(H(F))$. To see this, first note that the residue field of ξ equals the residue field of α_ξ, which embeds into $\mathbb{R}$. Hence the induced ordering $\overline{\alpha_\xi}$ is nothing but the pullback of the natural ordering on $\mathbb{R}$. Thus it is an archimedean ordering of the residue field.

Now assume that $\alpha_\xi \subsetneq \beta$ for some $\beta \in \operatorname{Spec}_r(H(F))$; we wish to deduce a contradiction. Then, due to the above observation, we obtain that $\mathfrak{p} := \operatorname{supp}(\alpha_\xi) \subsetneq \mathfrak{q} := \operatorname{supp}(\beta)$. Then we can choose $a \in \mathfrak{q} \setminus \mathfrak{p}$, and we can assume that $a \in \alpha_\xi$ since otherwise, we can replace a by $-a$. For each rational number $r > 0$ we have that $r + a \in \alpha_\xi$ but also $r - a \in \alpha_\xi$: if not, then we would obtain that $r - a \in -\alpha_\xi \subseteq -\beta$ and $r - a \in \beta$ as $a \in \pm\beta$. This would imply that $r - a \in \mathfrak{q}$, which leads to the contradiction $r \in \mathfrak{q}$. Passing to the residue field we see that the non-zero element $\bar{a}$ is infinitesimally small relative to the archimedean ordering $\overline{\alpha_\xi}$: a contradiction to our assumption. Thus the image of j is contained in $\operatorname{MaxSpec}_r(H(F))$.

To prove the injectivity of j assume that $\alpha_\xi = \alpha_\zeta$. Then both places have the same valuation ring and the same residue field on which they induce embeddings $\bar{\xi}, \bar{\zeta}$ into $\mathbb{R}$, subject to the condition $\bar{\xi}(\bar{a}) > 0 \Leftrightarrow \bar{\zeta}(\bar{a}) > 0$ for every $a \in H(F)$. As $\mathbb{Q}$ is dense in $\mathbb{R}$ we find that $\bar{\xi} = \bar{\zeta}$, whence $\xi = \zeta$.

From the equivalence $a(\alpha_\xi) > 0 \Leftrightarrow \xi(a) > 0$ we find that j is a topological embedding of $M(F)$ into $\operatorname{MaxSpec}_r(H(F))$.

Now we show that j is surjective. Consider any $\alpha \in \operatorname{Spec}_r(H(F))$. We want to show that $\alpha \subseteq \alpha_\xi$ for some $\xi \in M(F)$. This, of course, will settle our claim. Set $\mathfrak{p} = \operatorname{supp}(\alpha)$. Then $H(F)_\mathfrak{p}$ is the valuation ring of a place $\zeta : F \to k(\mathfrak{p}) \cup \infty$, where $k(\mathfrak{p})$ is the quotient field of $H(F)/\mathfrak{p}$. It is known that $H(F)/\mathfrak{p} = H(k(\mathfrak{p}))$, see [30, 1.4]. The ordering $\bar{\alpha}$ induces a real place $\lambda_{\bar{\alpha}}$ with a valuation ring which contains $H(k(\mathfrak{p})) = H(F)/\mathfrak{p}$. Using the residue map $\pi : H(F) \to H(k(\mathfrak{p}))$, we find $\xi \in M(F)$, determined by the condition $\xi|_{H(F)} = \lambda_{\bar{\alpha}} \circ \pi$. One readily checks that $\alpha \subseteq \alpha_\xi$.

It remains to address the commutativity of the diagram. Starting with $\lambda \in M(F|R)$ we have to show that

$$\rho\,(\alpha_\lambda \cap H(F)) = \alpha_\xi \text{ with } \xi = \xi_R \circ \lambda\,.$$

As α_ξ is a maximal prime cone it is sufficient to prove that $\alpha_\lambda \cap H(F) \subseteq \alpha_\xi$. Pick any $a \in H(F)$ with $\lambda(a) \geq 0$. Then $\lambda(a) \in H(R)$ and consequently, $\xi_R(\lambda(a)) \geq 0$, i.e., $a \in \alpha_\xi$. □

Next, the proof of Proposition 4.2 will be sketched.

Proof. We know that τ is continuous and surjective. Therefore, once we know that $\mathrm{MaxSpec}_r(H(F|R))$ has only finitely many connected components, we can derive the same for $\mathrm{MaxSpec}_r(H(F))$. We list the arguments needed to show that $\mathrm{MaxSpec}_r(H(F|R))$ decomposes into finitely many connected components. First of all, for any given ring A the specialization map $\rho : \mathrm{Spec}_r(A) \to \mathrm{MaxSpec}_r(A)$ induces a bijection between the set of connected components of $\mathrm{Spec}_r(A)$ and that of $\mathrm{MaxSpec}_r(H(A))$, see for instance [17, p.129, Satz 6]. Consequently, we are facing the problem to show that $\mathrm{Spec}_r(H(F|R))$ admits only finitely many connected components. This follows from Schülting's result [29, p. 436, Theorem] as it is known that algebraic sets over real closed fields decompose into finitely many semi-algebraically connected components. By the way, they are exactly the semi-algebraic path connected components, see [3, Sections 2.4.,2.5] and [8, Theorem 4.1]. □

Note that the surjectivity of τ can be obtained in a more direct way by appealing to the Baer-Krull Theorem. But we preferred to convey the present argument for the sake of a coherent presentation.

Remark 4.3. Without providing any further details, we want to conclude by another observation. The number of connected components s_F of $\mathrm{Spec}_r(H(F|R))$, which is a geometric invariant of F, is an upper bound for the number of connected components t_F of $M(F)$. This is a consequence of the last proof. However, it may happen that $s_F > t_F$, as we will show now.

Take a non-archimedean real closed field R, and denote by R^+ the set of its positive elements and by I^+ the set of its positive infinitesimals. Take $a \in I^+$. Let F be the function field of the real complete affine curve C given by

$$y^2 = (x^2 - a^2)(1 - x^2)\ .$$

The relative real holomorphy $H(F|R)$ equals the coordinate ring $A := R[C]$ and is a Dedekind ring. The curve C has two semialgebraic connected components separated by the function x.

The real spectrum $\mathrm{Spec}_r(A)$ consists of the prime cones

$$P(\alpha, \beta)\ :=\ \{f \in A \mid f(\alpha, \beta) \geq 0\}$$

attached to the points $(\alpha, \beta) \in C$ and the prime cones $P \cap A$, where P runs through the orderings of F. The first ones are maximal prime cones. A prime cone of the second type is maximal if and only if (F, P) is archimedean over A, and this holds if and only if (F, P) is archimedean over R (see [17, Corollary 5, p. 134]).

Take the ordering

$$P\ :=\ \{f \mid \exists d \in I^+ \exists e \in R^+ \setminus I^+ : f(c) > 0 \text{ for all } c \in (d, e)\}$$

of $R(x)$. The ordering P has exactly one extension P' to F in which y is positive. Take the automorphism σ of F such that $\sigma(x) = -x$ and $\sigma(y) = y$. Then $Q' = \sigma(P')$ is an ordering of F such that $\lambda_{P'} = \lambda_{Q'}$. Since P is archimedean over R, the same is true for P' and Q'. The function x is positive in P' and negative in Q', therefore

$P' \cap A$ and $Q' \cap A$ belong to different components of $\mathrm{Spec}_r(H(F|R))$. But the map τ from Proposition 4.1 sends the maximal prime cones $P' \cap A$ and $Q' \cap A$ to prime cones related with the real place $\lambda_{P'} = \lambda_{Q'}$, which shows that the number of components drops.

The example above was also studied in the paper [19], where the relation between cuts on the real curve and the orderings of its function field was described. In general, the study of t_F and its comparison to s_F seem to be an interesting task. $\diamondsuit$

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