ROOTS OF IRREDUCIBLE POLYNOMIALS IN TAME HENSELIAN EXTENSION FIELDS

Ron Brown

Department of Mathematics
2565 McCarthy Mall
University of Hawaii
Honolulu, HI 96822, USA
(*email:* ron@math.hawaii.edu)

Abstract. A class of irreducible polynomials $\mathcal{P}$ over a valued field (F, v) is introduced, which is the set of all monic irreducible polynomials over F when (F, v) is maximally complete. A "best–possible" criterion is given for when the existence of an approximate root in a tamely ramified Henselian extension K of F to a polynomial f in $\mathcal{P}$ guarantees the existence of an exact root of f in K.

§1. Introduction and the main theorem

Throughout this paper (F, v) will be a valued field with nontrivial value group vF, residue class field $\overline{F}$ and valuation ring $\mathcal{O}$. For any $a \in \mathcal{O}$ and polynomial $h \in \mathcal{O}[x]$ we let $\overline{a}$ and $\overline{h}$ denote the canonical images of a and h in $\overline{F}$ and $\overline{F}[x]$, respectively.

We begin with an example. Suppose that q is an odd prime. Then $f(x) = x^2 - q$ is irreducible over the q-adic numbers $\mathbb{Q}_q$, and if K is an algebraic extension of $\mathbb{Q}_q$ with an element α with $v(f(\alpha)) > v(q)$ (where v is the q-adic valuation), then $2v(f'(\alpha)) = v(q)$ and hence by Hensel's Lemma (Engler and Prestel, 2005, Theorem 1.3.1) a root of f lies

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in K. More generally suppose that $g \in F[x]$ is a generalized Schönemann polynomial over F, i.e., that: $g = p^e + th$ where $e \geq 1$; $p \in \mathcal{O}[x]$ is monic with $\overline{p}$ irreducible; $h \in \mathcal{O}[x]$ has $\overline{p} \nmid \overline{h}$ and $\deg p^e > \deg h$; and $t \in \mathcal{O}$ has $v(t) \notin svF$ for all divisors $s > 1$ of e. Then g is also irreducible over F, and if (F, v) is Henselian and if a tamely ramified finite degree extension (K, u) of (F, v) has an element α with $u(g(\alpha)) > v(t)$, then K has a root of g (Brown, 2007, Lemma 4 and Corollary 7).

In this paper we generalize the above result to a large class $\mathcal{P}$ of irreducible polynomials over F. If v is discrete rank one, then $\mathcal{P}$ is the set of monic polynomials over F which are irreducible over the completion of (F, v), and if (F, v) is maximally complete (Schilling, 1950, Definition 9, p. 36), then $\mathcal{P}$ is the set of all monic irreducible polynomials over F. (See Remark 6(C) below.) The role of "$v(t)$" in this generalization will be played for each $h \in \mathcal{P}$ by an invariant γ_h of h lying in the divisible hull of vF which will be constructed along with the set $\mathcal{P}$, but which can be characterized intrinsically in several ways. (See Remark 6(B).) The construction and analysis of $\mathcal{P}$ and the invariants γ_h will be made in the next section. Anticipating that work, we close this introduction by stating our only theorem. It will be proved in Section 3.

Theorem 1. *Suppose that (F, v) is Henselian and $h \in \mathcal{P}$. Suppose that α is an element of a tamely ramified finite degree extension (K, u) of (F, v) with $u(h(\alpha)) > \gamma_h$. Then there is a root of h in K.*

The analysis of γ_h will show that for each $h \in \mathcal{P}$ it is best possible in the above theorem (see Remark 8 at the end of Section 3). The hypothesis that (K, u) is a tamely ramified extension of (F, v) says that the field extension $\overline{K}/\overline{F}$ is separable, the characteristic of $\overline{F}$

does not divide the ramification index $e_{u/v} = (uK : vF)$, and $[K : F]$ is the product of $e_{u/v}$ and the residual degree $f_{u/v} = [\overline{K} : \overline{F}]$.

There are examples in (Brown, 2007, Remark 2(B)) showing that Theorem 1 gives a stronger result in general than those obtained from some natural direct applications of Hensel's Lemma to the problem of the existence of roots of polynomials in $\mathcal{P}$.

§2. The class $\mathcal{P}$ of polynomials

We shall let $\mathcal{E}$ denote the set of all extensions of v to a valuation on $F[x]$ mapping into $\mathbb{Q}vF \bigcup\{\infty\}$ where we let $\mathbb{Q}vF$ denote a (fixed) divisible hull of vF. For extensions w of v to a valuation on $F[x]$ we allow $w^{-1}(\infty)$ to be a nonzero ideal of $F[x]$. The extensions w with $w^{-1}(\infty)$ trivial correspond precisely to extensions of v to the rational function field $F(x)$, and the extensions with $w^{-1}(\infty)$ nontrivial (in which case $w^{-1}(\infty)$ is a maximal ideal) correspond precisely to the extensions of v to the field $F[x]/w^{-1}(\infty)$. This paper is indeed about algebraic extensions of F; however, valuations on $F[x]$ give us a setting in which we can describe ways in which two extensions of v to possibly different simple algebraic extensions of F can be closely related.

A notational convention will be useful. Whenever w (respectively, w_i) is used to denote an extension of v to a valuation on $F[x]$, we will denote the associated (surjective) place by $\tau : F[x] \longrightarrow k \bigcup\{\infty\}$ (respectively, by $\tau_i : F[x] \longrightarrow k_i \bigcup\{\infty\}$).

Definition 2. Suppose that $n \geq 0$. A *strict system of polynomial extensions over* (F, v) *of length* $n+1$ is a finite sequence

$$g = \big((g_0, w_0, \gamma_0), \cdots, (g_{n+1}, w_{n+1}, \gamma_{n+1})\big)$$

of elements of $F[x] \times \mathcal{E} \times (\mathbb{Q}vF \bigcup \{-\infty\})$ such that for some $a \in F$,

(A) $g_0 = x - a$, $\gamma_0 = -\infty$, and $w_0(h) = v(h(a))$ for all $h \in F[x]$;

and when $0 \leq i \leq n$:

(B) $\deg g_{i+1} > \deg g_i$;

(C) $\gamma_{i+1} = w_i(g_{i+1})$;

(D) $w_{i+1}(g_{i+1}) = \infty$;

(E) $w_i(A_r)/(d_i - r) \geq w_i(A_0)/d_i > \gamma_i$ for all $r < d_i$ where

$$g_{i+1} = g_i^{d_i} + \sum_{r<d_i} A_r g_i^r$$

is the g_i–expansion of g_{i+1} (so $\deg A_r < \deg g_i$ for all $r < d_i$);

(F) if $e > 0$ is least with $ew_i(A_0) \in d_i w_i F[x]$, then the polynomial

$$Y^{d_i/e} + \sum_{r<d_i/e} \tau_i\big(s^{(d_i/e)-r} A_{er}\big) Y^r \tag{1}$$

is irreducible over k_i for all $s \in F[x]$ with $d_i w_i(s) = -ew_i(A_0)$.

The symbol $-\infty$ above is used with the convention that $-\infty < a$ for all $a \in \mathbb{Q}vF \cup \{\infty\}$. One should observe that in (F) of the above definition we have $e \mid d_i$; indeed

$$e = \left(w_i F[x] + \mathbb{Z}\frac{1}{d_i} w_i(A_0) : w_i F[x] \right) . \tag{2}$$

Also one can check that the irreducibility of the polynomial (1) above over k_i is independent of the choice of s and, using the inequalities of (E), that the coefficients of this polynomial are indeed finite.

We let $\mathcal{P} = \mathcal{P}(F)$ denote the set of all polynomials over F which appear as the first coordinate of some term (h, w, γ) of some strict system of polynomial extensions over (F, v).

That any such polynomial h uniquely determines both the corresponding coordinates w and γ independently of any strict system of polynomial extensions in which h appears will follow from Proposition 5 and Remark 6(B) below. Definition 2, while admittedly complicated, puts the notion of a sequence of $n+1$ *extensions of polynomials* from (Brown, 1972, Definition 5.8, p. 467, and §7) (with the degrees of the polynomials strictly increasing and the first polynomial linear) into a setting which allows an efficient inductive analysis in Proposition 5 of the polynomials in $\mathcal{P}$ and their associated valuations and invariants. While this paper uses ideas and some arguments from (Brown, 1972), we will not use the results of that paper except in Remark 6(C), which itself is not applied elsewhere in this paper.

Examples 3. Suppose that $g = p^e + th \in F[x]$ is a generalized Schönemann polynomial as in Section 1.

(A) If $\deg p = 1$ and $e > 1$, then $((p, u_0, -\infty), (g, u_1, v(t)))$ is a strict system of polynomial extensions of length one, where $u_0\big(\sum b_i p^i\big) = v(b_0)$ and $u_1(cg + \sum_{i<e} b_i p^i) = \min_i(v(b_i) + \frac{i}{e}v(t))$ for all $b_i \in F$, $c \in F[x]$. A concrete example over the field $\mathbb{Q}_3$ of 3-adic numbers of such a polynomial g would be $(x-1)^2 - 3$.

(B) If $\deg p > 1$, then $((x, w_0, -\infty), (p, w_1, 0))$ is a strict system of polynomial extensions of length one, where $w_0(c) = v(c(0))$ and $w_1(pc + \sum_{i<\deg p} b_i x^i) = \min_{i<\deg p} v(b_i)$ for all $c \in F[x]$, $b_i \in F$.

(C) If $\deg p > 1$ and $e > 1$, then $((x, w_0, -\infty), (p, w_1, 0), (g, w_2, v(t)))$ is a strict system of polynomial extensions of length 2, where $w_2(cg + \sum_{i<e} b_i p^i) = \min_i(w_1(b_i) + \frac{i}{e}v(t))$ for all $c \in F[x]$ and $b_i \in F[x]$ with $\deg b_i < \deg p$. An example of such a polynomial g over $\mathbb{Q}_3$

would be the classical Schönemann polynomial $(x^2-2)^2-3$.

The above examples generalize routinely to a class of polynomials considered by Khanduja and Saha (1997, Theorem 1.1) even without the separability requirement of their theorem.

Let $N \geq 0$. Our last two examples will be strict systems of polynomial extensions of length $N+1$. Both will have the form

$$g = \big((g_0, w_0, -\infty), (g_1, w_1, \gamma_1), \cdots, (g_{N+1}, w_{N+1}, \gamma_{N+1})\big)$$

where for each n we denote by w_n an extension of v to a valuation on $F[x]$ with $w_n(g_n) = \infty$.

(D) Let p be a rational prime and v be the p-adic valuation on $\mathbb{Q}$ with $v(p) = 1$. Let $g_0 = x$, $g_1 = x^2 - p$, and for each $n \geq 1$ let $g_{n+1} = g_n^2 - p^{2^n} g_{n-1}$ and let $\gamma_n = \frac{4^n-1}{3}\frac{1}{2^{n-1}}$. Then (i) the sequence g above is a strict system of polynomial extensions over $(\mathbb{Q}, v)$ of length $N+1$, and (ii) $w_{N+1}(g_N) = \gamma_{N+1}/2$. The $N = 0$ case of the assertion (i) is included in Example (A) above, and the general case of both assertions is easily proven by induction on N using Proposition 5 below. (To prove (i) apply the Proposition to w_N, and having proved (i), then prove (ii) by applying the Proposition to w_{N+1}. The key observation is that the induction hypothesis implies that $w_N(g_{N+1} - g_N^2) = \gamma_{N+1}$.) The Proposition implies that the w_n are uniquely determined.

(E) Let v be the t-adic valuation on the rational function field $\mathbb{Q}(t)$ with $v(t) = 1$. Let $g_0 = x$, $g_1 = x^2-2$, and for all $n \geq 1$ let $g_{n+1} = g_n^2 - t^{2^{n+1}-2^{n-1}-1} g_{n-1}$ and let $\gamma_n = 2^n - 2$. Then (i) g is a strict system of polynomial extensions over $(\mathbb{Q}(t), v)$ of length $N+1$, and (ii) g_N/t^{2^N-1} is a unit with respect to the valuation w_{n+1} and its residue class is a 2^{N+1}st root of 2. The $N = 0$ case of this assertion is a special case of Example (B) above, and the general case follows by induction on N using exactly the outline given parenthetically in the previous example (D).

For the remainder of this section we assume that g is a strict system of polynomial extensions over (F, v) and use the notation for g in the above Definition 2. In the next Proposition we will also use the convention introduced just before Definition 2 as well as the following

Notation 4. (A) Write $J_0 = \{0\}$ and $g^0 = 1$. When $0 < i \leq n+1$ let $J_i = \prod_{j<i} \{0, 1, \ldots, d_j - 1\}$. For any $\sigma \in \mathbb{Z}^i$ we write $0 \leq \sigma$ if all the coordinates of σ are nonnegative, in which case we also write $\sigma = (\sigma(0), \cdots, \sigma(i-1))$ and $g^\sigma = \prod_{j<i} g_j^{\sigma(j)}$. In all cases, $\{g^\sigma : \sigma \in J_i\}$ is a basis for the F–space $F[x]_{\deg g_i}$, where for any $m \geq 1$ we write $F[x]_m = \{h \in F[x] : \deg h < m\}$. We will say that $\{g^\sigma : \sigma \in J_i\}$ is a *valuation basis* for $F[x]_{\deg g_i}$ with respect to an extension of v to a valuation w on $F[x]$ if for all choices of $a_\sigma \in F$ we have

$$w\left(\sum_{\sigma \in J_i} a_\sigma g^\sigma\right) = \min_{\sigma \in J_i} w(a_\sigma g^\sigma) \,.$$

(B) Let E denote either $F[x]$ or a field extension of F. For any extension u of v to a valuation on E let $e_{u/v} = (uE : vF)$ denote the ramification index and $f_{u/v} = [\overline{E} : \overline{F}]$ denote the residual degree of the extension.

Proposition 5. *Suppose that w is an extension of v to a valuation on $F[x]$ with $w(g_{n+1}) > \gamma_{n+1}$. Then for all integers i with $0 \leq i \leq n$ we have:*

(A) $w(g_i) = w_{i+1}(g_i) = \gamma_{i+1}/d_i$;

(B) $\{g^\sigma : \sigma \in J_{i+1}\}$ is a valuation basis for w on $F[x]_{\deg g_{i+1}}$;

(C) g_{i+1} *is irreducible over* F*;* w_{i+1} *is the unique extension of* v *to* $F[x]$ *mapping* g_{i+1} *to* ∞*; and* $\deg g_{i+1} = e_{w_{i+1}/v} f_{w_{i+1}/v}$*;*

(D) there is an $\overline{F}$*–homomorphism* $\Phi_{i+1} : k_{i+1} \longrightarrow k$ *with* $\Phi_{i+1}\tau_{i+1}(cg^\sigma) = \tau(cg^\sigma)$ *whenever* $c \in F$*,* $0 \leq \sigma \in \mathbb{Z}^{i+1}$*, and* $w(cg^\sigma) \geq 0$*.*

In (A) of the above Proposition we have identified (as one can do, uniquely) the subgroup of elements of $wF[x]$ which have a nonzero multiple in vF with a subgroup of our fixed divisible hull $\mathbb{Q}vF$ of vF. Part (C) above is equivalent to the assertion that if u is an extension of v to $F[\xi]$ where ξ is a root of g_{i+1}, then $\deg g_{i+1} = [F[\xi] : F] = e_{u/v} f_{u/v}$.

Remarks 6. We note some consequences of the Proposition above.

(A) Suppose that $0 \leq i \leq n$. Since $w_{i+1}(g_{i+1}) = \infty$, the place τ_{i+1} maps $F[x]_{\deg g_{i+1}}$ onto $k_{i+1} \bigcup \{\infty\}$ and w_{i+1} maps $F[x]_{\deg g_{i+1}}$ onto $w_{i+1}F[x] \bigcup \{\infty\}$. Combining (A), (B) and (D) of the Proposition shows that w and w_{i+1} agree on $F[x]_{\deg g_{i+1}}$ and that $\Phi_{i+1}\tau_{i+1}$ and τ agree on $F[x]_{\deg g_{i+1}}$.

(B) Part (C) of the above Proposition says that g_{i+1} uniquely determines w_{i+1} for all $i \leq n$. We now argue that g_{i+1} also uniquely determines γ_{i+1}. Specifically, γ_{i+1} is the minimal element of $\mathbb{Q}vF$ with the property that if u and u^* are extensions of v to $F[x]$ with $u(g_{i+1}) > \gamma_{i+1} < u^*(g_{i+1})$, then they agree on $F[x]_{\deg g_{i+1}}$. That γ_{i+1} has this property follows from the remarks in the previous paragraph; γ_{i+1} is the minimal element of $\mathbb{Q}vF$ with this property, since $w_i(g_{i+1}) = \gamma_{i+1} < w_{i+1}(g_{i+1})$, but w_i and w_{i+1} do not agree on $g_i \in F[x]_{\deg g_{i+1}}$. Since g_{i+1} uniquely determines γ_{i+1}, we can unambiguously denote γ_{i+1} by $\gamma_{g_{i+1}}$ (independently of any strict system of polynomial extensions in which g_{i+1} appears but of course depending on the choice of (F, v)). This is the notation used in the statement of Theorem 1. Another characterization of γ_{i+1} can be adapted from (Brown, 1972, Proposition 5.6, p. 467): γ_{i+1} is the minimal element of $\mathbb{Q}vF$ such that if $h \in F[x]_{\deg g_{i+1}}$ and $w_{i+1}(h) > \gamma_{i+1}$, then $g_{i+1} + h$ is irreducible. A third intrinsic characterization of γ_{i+1} follows from the fact that it is best possible in Theorem 1 (see Remark 7 below).

(C) First suppose that (F, v) is a maximally complete field (Schilling, 1950, p. 36). By the Proposition all the elements of $\mathcal{P}$ are monic irreducible over F. Conversely any monic irreducible polynomial h over F can be shown by the methods of (Brown, 1972) to be in $\mathcal{P}$. Basically one takes the (possibly transfinite) generating sequence of the augmented signature (Brown, 1972, Definition 7.5, p. 477) of the unique extension w of v to $F[x]$ with $w(h) = \infty$, and then deletes all terms whose degree is the same as that of a later term.

Next suppose that (F, v) is discrete rank one, say with completion $(\tilde{F}, \tilde{v})$. Any strict system of polynomial extensions over (F, v) lifts to one over $(\tilde{F}, \tilde{v})$ by just extending the valuations on $F[x]$ to extensions of $\tilde{v}$ to $\tilde{F}[x]$. Thus each element of $\mathcal{P}$ is irreducible over $\tilde{F}$. In fact, in this case $\mathcal{P}$ is exactly the set of monic polynomials over F which are irreducible over $\tilde{F}$. The key fact here is that for any monic polynomial h in $F[x]$ which is irreducible over $\tilde{F}$ there is a unique extension w of $\tilde{v}$ to $\tilde{F}[x]$ with $w(h) = \infty$, and if we take the systems of representatives A and B of (Brown, 1972, §4) to be in F, then the generating sequence of the augmented signature of w will be a sequence of polynomials in $F[x]$ with last term h.

The remainder of this section is devoted to a proof of the Proposition. The proof of part (A) especially borrows heavily from the proof of the Fundamental Lemma in Section

8 of (Brown, 1972). We included these arguments here since it seemed unreasonable to ask the reader to extract them from (Brown, 1972), where the exposition involves complicated machinery, some of which assumes that (F, v) is maximally complete. The proof will show that when $0 \leq i \leq n$, then the value group of w_{i+1} is $w_i F[x] + \mathbb{Z}(\gamma_{i+1}/d_i)$ and the residue class field of w_{i+1} is isomorphic to an extension of the residue class field of w_i by a root of the polynomial (1) of Definition 2.

Given our strict system of polynomial extensions g, we set for each $i \leq n$:

(3) $$q_i = \gamma_{i+1}/d_i\,, \quad e_i = (w_i F[x] + \mathbb{Z} q_i : w_i F[x])\,, \quad \text{and} \quad f_i = d_i/e_i\,.$$

Note that $q_0 > \gamma_0 = -\infty$ (by Definition 2 applied with $i = 0$) and that if $0 \leq i \leq n$, then with A_0 as in Definition 2 we have

(4) $$d_i q_i = \gamma_{i+1} = w_i(g_{i+1}) = w_i(A_0) > d_i \gamma_i$$

(using the definition of q_i and parts (C), (D) and (E) of Definition 2, respectively). Hence, if $0 \leq i \leq n$, then

(5) $$q_i > \gamma_i \quad \text{and if} \quad i < n, \text{ then } \quad q_{i+1} > d_i q_i\,.$$

Finally, by Equation (2) for any $i \leq n$ the value of e in Definition 2(F) is exactly the e_i above.

By induction we may assume that the Proposition is true for all strict systems of polynomial extensions over (F, v) of length less than $n + 1$.

Our first task is to prove that $w(g_i) = q_i$ whenever $0 \leq i \leq n$. Just suppose that this is not true. Then there exists a smallest t with $0 \leq t \leq n$ and $w(g_t) \neq q_t$. Hence

$w(g_i) = q_i$ for all $i < t$, and by our induction hypothesis on n and the inequality (5) we have $w(g_i) \le \gamma_i < q_i$ whenever $t < i \le n$.

Claim 1. If $w(g_t) < q_t$, then $w(g_{i+1}) = d_i w(g_i)$ whenever $t \le i \le n$.

Proof of Claim 1. Suppose that $t \le i \le n$. We may suppose by induction on i that $w(g_{m+1}) = d_m w(g_m)$ whenever $t \le m < i$. We use the notation of Definition 2(E). For each $r < d_i$ we can write

$$A_r = \sum_{\sigma \in J_i} c_{r\sigma} g^\sigma$$

for some $c_{r\sigma} \in F$. If $i > 0$ we can apply our induction hypothesis on n to the valuation w_i to show that for each $r < d_i$

$$w_i(A_r) = \min_{\sigma \in J_i} \left(v(c_{r\sigma}) + \sum_{m<i} \sigma(m) q_m \right) . \tag{6}$$

Equation (6) is trivially true if $i = 0$. Thus by Equation (4) above and Definition 2(E) for all r and σ we have

$$\begin{aligned} w(c_{r\sigma} g^\sigma) &\ge w_i(A_r) + \sum_{m<i} \sigma(m)(w(g_m) - q_m) \\ &\ge (d_i - r) q_i + \sum_{m<i} \sigma(m)(w(g_m) - q_m) . \end{aligned}$$

Now $w(g_m) = q_m$ for all $m < t$, and $w(g_m) < q_m$ whenever $t \le m \le i$. Hence for all $r < d_i$

and $\sigma \in J_i$ setting $\sigma(i) = r$ yields

$$\begin{aligned}
w\big(c_{r\sigma} g^\sigma g_i^r\big) &= w(c_{r\sigma} g^\sigma) + r w(g_i) \\
&\geq d_i q_i + \sum_{m \leq i} \sigma(m)(w(g_m) - q_m) \\
&\geq d_i q_i + \sum_{t \leq m \leq i} (d_m - 1)(w(g_m) - q_m) \\
&= d_i w(g_i) + (q_t - w(g_t)) + \sum_{t \leq m < i} (q_{m+1} - d_m q_m) + \sum_{t \leq m < i} (d_m w(g_m) - w(g_{m+1})) \\
&> d_i w(g_i)
\end{aligned}$$

(since $q_t > w(g_t)$, and since by our induction hypothesis on i and formula (5), the last summation above is zero and the penultimate one is nonnegative). Therefore

$$w(g_{i+1}) = w\left(g_i^{d_i} + \sum_{r < d_i} \sum_{\sigma \in J_i} c_{r\sigma} g^\sigma g_i^r \right) = w(g_i^{d_i}) = d_i w(g_i)\,,$$

proving the claim.

If $w(g_t) < q_t$, then the above claim tells us that $w(g_{n+1}) = d_n w(g_n) < d_n q_n = \gamma_{n+1}$, a contradiction. Hence we may conclude that if $w(g_t) \neq q_t$, then $w(g_t) > q_t$. Thus by our induction on n, $\{g^\sigma : \sigma \in J_t\}$ is a valuation basis for w on $F[x]_{\deg g_t}$.

Claim 2. $w(g_{t+1}) = d_t q_t$ and $t < n$.

Proof of Claim 2. Note that w_t and w agree on $F[x]_{\deg g_t}$ by induction on n. We can write $g_{t+1} = g_t^{d_t} + \sum_{r < d_t} A_r g_t^r$ where $A_r \in F[x]_{\deg g_t}$ for each r. If $r \neq 0$, then

$$w(A_r g_t^r) = w_t(A_r) + r w(g_t) > \frac{d_t - r}{d_t} w_t(A_0) + r q_t = d_t q_t$$

(cf. display (4)). But $w(g_t^{d_t}) = d_t w(g_t) > d_t q_t$ and $w(A_0) = w_t(A_0) = d_t q_t$, so $w(g_{t+1}) = d_t q_t$, as claimed. Thus $t \neq n$ since otherwise $w(g_{n+1}) = d_n q_n = \gamma_{n+1} < w(g_{n+1})$. This completes the proof of Claim 2.

Claim 3. If $t < i \le n$, then $w(g_{i+1}) = d_i w(g_i)$.

Proof of Claim 3. We are of course assuming that $w(g_t) \ne g_t$ and hence that $n > t$ and $w(g_t) > q_t$. We my assume inductively that $w(g_{m+1}) = d_m w(g_m)$ whenever $t < m < i$. As in the proof of Claim 1 we write

$$g_{i+1} = g_i^{d_i} + \sum_{r<d_i} \sum_{\sigma \in J_i} c_{r\sigma} g^\sigma g_i^r \,.$$

Then for all $0 \le r < d_i$ and $\sigma \in J_i$ we have (again setting $\sigma(i) = r$)

$$\begin{aligned} w(c_{r\sigma} g^\sigma g_i^r) &\ge d_i q_i + \sum_{m \le i} \sigma(m)(w(g_m) - q_m) \\ &\ge d_i q_i + \sum_{t<m\le i} (d_m - 1)(w(g_m) - q_m) \\ &= \sum_{t<m<i} (q_{m+1} - d_m q_m + d_m w(g_m) - w(g_{m+1})) \\ &+ q_{t+1} - w(g_{t+1}) + d_i w(g_i) \\ &> d_i w(g_i) = w(g_i^{d_i}) \end{aligned}$$

(the last inequality uses formula (5) and our induction hypotheses on i and n). The claim follows immediately.

Since $n > t$, we can take $i = n$ in Claim 3 to obtain

$$w(g_{n+1}) = d_n w(g_n) \le d_n q_n = \gamma_{n+1} \,,$$

contradicting the hypothesis on w. Hence for all $i \le n$ we have $w(g_i) = q_i$. Applying this result with w_{i+1} in place of w shows that $w_{i+1}(g_i) = q_i = \gamma_{i+1}/d_i$. Thus our induction hypothesis on n implies (A) of the Proposition. In particular, $w(g_n) = q_n > \gamma_n$, so that parts (B), (C) and (D) of the Proposition are valid for all $i < n$ by our induction hypothesis on n. We now prove them for $i = n$.

Because $w_{n+1}(g_n) = w(g_n) = q_n > \gamma_n$, we may assume by our induction on n that we have $\overline{F}$–homomorphisms $\Phi_n : k_n \longrightarrow k$ and $\Phi'_n : k_n \longrightarrow k_{n+1}$ with $\Phi_n \tau_n(cg^\sigma) = \tau(cg^\sigma)$ and $\Phi'_n \tau_n(cg^\sigma) = \tau_{n+1}(cg^\sigma)$ whenever $c \in F$, $0 \le \sigma \in \mathbb{Z}^n$, and $w(cg^\sigma) \ge 0$. As usual we write

$$g_{n+1} = g_n^{d_n} + \sum_{r<d_n} A_r g_n^r \,.$$

There exist $b \in F$ and $\mu \in J_n$ such that $w(s^{f_n} A_0) = 0$ where $s = bg^\mu$ (if $n > 0$ then this follows from (B) with $i = n-1$). By Definition 2(F) the polynomial

$$G = G(Y) = Y^{f_n} + \sum_{r<f_n} \tau_n(s^{f_n - r} A_{re_n}) Y^r$$

is irreducible over k_n, and hence $\Phi_n(G)$ is irreducible over $\Phi_n(k_n)$.

We now show that $\tau(sg_n^{e_n})$ is a root of $\Phi_n(G)$. By hypothesis and formula (4)

$$w(g_{n+1}) > \gamma_{n+1} = w(A_0) = -w(s^{f_n}) \,,$$

and if $0 \le r < d_n$ and $e_n \nmid r$, then

$$w(A_r g_n^r) > \frac{d_n - r}{d_n} w(A_0) + r q_n = d_n q_n = -w(s^{f_n}) \,.$$

Hence

$$w\left((sg_n^{e_n})^{f_n} + \sum_{r<f_n} (A_{re_n} s^{f_n - r})(sg_n^{e_n})^r \right) = w\left(s^{f_n} \left(g_{n+1} - {\sum}' A_r g_n^r \right) \right) > 0$$

where ${\sum}'$ is the sum over all $r < d_n$ with $e_n \nmid r$. Thus $\tau(sg_n^{e_n})$ is indeed a root of

$$\tau\left(Y^{f_n} + \sum_{r<f_n} A_{re_n} s^{f_n - r} Y^r \right) = \Phi_n(G) \,.$$

(The equality above holds by the choice of Φ_n, since $\{g^\sigma : \sigma \in J_n\}$ is a valuation basis for w on $F[x]_{\deg g_n}$ and hence each $A_{re_n} s^{f_n - r}$ is a sum of integral elements of the form cg^σ where $c \in F$ and $0 \le \sigma \in \mathbb{Z}^n$.)

Replacing w by w_{n+1} and τ by τ_{n+1} in the above argument shows that the irreducible polynomial $\Phi'_n G$ over $\Phi'_n k_n$ has root $\tau_{n+1}(sg_n^{e_n})$ in k_{n+1}. This shows that $f_{w_{n+1}/v} \ge f_{w_n/v} f_n$. Moreover, it also shows that the unique $\overline{F}$–isomorphism $\psi_0 : \Phi'_n(k_n) \longrightarrow \Phi_n(k_n)$ with $\psi_0 \Phi'_n = \Phi_n$ lifts to an isomorphism $\psi : \Phi'_n(k_n)[\tau_{n+1}(sg_n^{e_n})] \longrightarrow \Phi_n(k_n)[\tau(sg_n^{e_n})]$ taking $\tau_{n+1}(sg_n^{e_n})$ to $\tau(sg_n^{e_n})$ (after all, $\psi_0(\Phi'_n(G)) = \Phi_n(G)$).

Since $w_n(g_n) = \infty$,

$$w_n F[x] = w_n(F[x]_{\deg g_n}) \setminus \{\infty\} = vF + \sum_{i<n} \mathbb{Z} q_i$$

(by induction on n, using (A) and (B) of the statement of the Proposition). By our proof of part (A) we have $w_{n+1} F[x] \supseteq w_n F[x]$. Hence $e_{w_{n+1}/v} \ge e_n e_{w_n/v}$ by the definition of e_n in the display (3).

Because $w_{n+1}(g_{n+1}) = \infty$, there is a monic irreducible factor h of g_{n+1} with $w_{n+1}(h) = \infty$. Then w_{n+1} induces an extension u of v to the quotient field $F[x]/(h)$ and we have

$$\begin{aligned}
\deg g_{n+1} &\ge \deg h = [F[x]/(h) : F] \\
&\ge e_{u/v} f_{u/v} = e_{w_{n+1}/v} f_{w_{n+1}/v} \\
&\ge e_n f_n e_{w_n/v} f_{w_n/v} \\
&= e_n f_n \deg g_n = \deg g_{n+1}
\end{aligned}$$

(using our induction hypothesis on n). This shows that $g_{n+1} = h$ is irreducible and that

$$[F[x]/(g_{n+1}) : F] = \deg g_{n+1} = e_{w_{n+1}/v} f_{w_{n+1}/v} = e_{u/v} f_{u/v} \, .$$

Thus u is the unique extension of v to $F[x]/(g_{n+1})$, cf. (Engler and Prestel, 2005, Theorem 3.3.4). It follows that w_{n+1} is the unique extension of v to $F[x]$ with $w_{n+1}(g_{n+1}) = \infty$. This completes the proof of (C) when $i = n$; it also shows that $k_{n+1} = \Phi'_n(k_n)[\tau_{n+1}(sg_n^{e_n})]$ and hence that our homomorphism ψ has domain k_{n+1}. Thus in order to prove (D) when $i = n$ it suffices to show that $\psi\tau_{n+1}(cg^\sigma) = \tau(cg^\sigma)$ whenever $c \in F$, $0 \leq \sigma \in \mathbb{Z}^{n+1}$, and $w(cg^\sigma) = v(c) + \sum_{j \leq n} \sigma(j)q_j \geq 0$. This is obvious if $w(cg^\sigma) > 0$, so we suppose that $w(cg^\sigma) = 0$. This condition implies that we can write $cg^\sigma = cg^\rho g_n^{e_n j}$ where $0 \leq \rho \in \mathbb{Z}^n$ and $0 \leq j \in \mathbb{Z}$. We can also write $w(s^j) = -w(dg^\delta)$ for some $d \in F$ and $\delta \in J_n$. Then

$$\begin{aligned}
&\psi(\tau_{n+1}(s^j dg^\delta))\psi(\tau_{n+1}(cg^\sigma)) \\
&= \psi\tau_{n+1}(cdg^{\delta+\rho})\psi\tau_{n+1}((sg_n^{e_n})^j) \\
&= \psi_0\Phi'_n\tau_n(cdg^{\delta+\rho})\tau((sg_n^{e_n})^j) \\
&= \Phi_n\tau_n(cdg^{\delta+\rho})\tau((sg_n^{e_n})^j) \\
&= \tau(cdg^{\delta+\rho}(sg_n^{e_n})^j) \\
&= \psi(\tau_{n+1}(s^j dg^\delta))\tau(cg^\sigma) \neq 0\,,
\end{aligned}$$

so indeed $\psi\tau_{n+1}(cg^\sigma) = \tau(cg^\sigma)$, proving (D) for $i = n$.

It remains to prove (B) when $i = n$. Consider any nonzero polynomial

$$B = \sum_{j < d_n} B_j g_n^j \in F[x]_{\deg g_{n+1}}$$

where each $B_j \in F[x]_{\deg g_n}$. Using induction on n, it suffices to prove that

$$w(B) = \min_{j < d_n} w(B_j g_n^j)\,.$$

We may without loss of generality suppose that all the nonzero terms of B have the same value and that $B_0 \neq 0$ (divide out the largest power of g_n dividing all the terms). This implies that $B_j = 0$ whenever $e_n \nmid j$. With s as above we may pick $b \in F[x]_{deg\, g_n}$ with $w(bB_0 s^{f_n}) = 0$. Then all nonzero terms of

$$s^{f_n} bB = \sum_{j<f_n} B_{je_n} s^{f_n - j} b(sg_n^{e_n})^j$$

have the same value, namely, $w(B_0 s^{f_n} b) = 0$. Hence $H(Y) := \sum_{j<f_n} \tau(B_{je_n} s^{f_n-j} b) Y^j$ is a nonzero polynomial over $\Phi_n(k_n)$ of degree less than f_n, the degree of the minimal polynomial $\Phi_n(G)$ of $\tau(sg_n^{e_n})$ over $\Phi_n(k_n)$. Hence $0 \neq H(\tau(sg_n^{e_n}))$, and therefore

$$0 = w\Big(\sum_{j<f_n} B_{je_n} s^{f_n-j} b(sg_n^{e_n})^j\Big) = w(s^{f_n} bB)\,.$$

Thus $w(B) = -w(s^{f_n} b) = w(B_0)$, which was to be proved. This completes the proof of (B) in the case $i = n$ and hence the proof of the Proposition.

Remark 7. Let w be as in Proposition 5. We record here some corollaries of the proof of Proposition 5 above. First, $e_{w_{n+1}/v} = e_0 \cdots e_n$ and $w_{n+1} F[x] = vF + \sum_{0 \le i \le n} \mathbb{Z} q_i$. Next, if we treat the $\overline{F}$-homomorphisms $\Phi_i : k_i \to k$ as identifications, then whenever $0 \le i \le n$ we can regard k_i as a subfield of k_{i+1} and the extension k_{i+1}/k_i is generated over k_i by a root of the polynomial (1) of Definition 2.

§3. Proof of Theorem 1

Let β denote a root of h in some algebraic extension of K. Since (F, v) is Henselian, v has unique extensions to $F(\alpha)$ and to $F(\beta)$, both of which we also denote by v. We let w_α and w_β denote the extensions of v to $F[x]$ taking each $f \in F[x]$ to $v(f(\alpha))$ and $v(f(\beta))$, respectively. The idea of the proof is that the Proposition of Section 2 can be applied to w_α and w_β to obtain an injective homomorphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{F(\beta)}^\bullet & \longrightarrow & F(\beta)^\bullet/1+\mathcal{M}_\beta & \longrightarrow & vF(\beta) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \overline{F(\alpha)}^\bullet & \longrightarrow & F(\alpha)^\bullet/1+\mathcal{M}_\alpha & \longrightarrow & vF(\alpha) & \longrightarrow & 0 \end{array}$$

where $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ are the maximal ideals of the valuation rings of $F(\alpha)$ and $F(\beta)$ and where for any field E we let $E^\bullet$ denote the multiplicative group of nonzero elements of E. This is sufficient to guarantee that $F(\beta)$ is F-isomorphic to a subfield of $F(\alpha)$, and hence that h has a root in $F(\alpha)$. A convenient vehicle for expressing the argument formally is the functor Δ of Brown and Harrison (1970), which we now review.

Let $S(F,\Gamma,r)$ denote the generalized Laurent series field with residue class field F, value group Γ, and symmetric factor set r, so $S(F,\Gamma,r)$ consists of formal sums $\sum\limits_{\gamma\in\Gamma} c_\gamma t^\gamma$ with well–ordered support $\{\gamma \in \Gamma : c_\gamma \neq 0\}$ (Schilling, 1950, pp. 23–24); if the factor set is trivial we write $S(F,\Gamma)$ for $S(F,\Gamma,1)$. For any valued field (E,w) we let $\Delta E = B/\mathfrak{b}$ where

$$B = B_E = \left\{\sum b_\gamma t^\gamma \in S(E,wE) : w(b_\gamma) \geq \gamma \quad \forall \gamma \in wE\right\}$$

and

$$\mathfrak{b} = \mathfrak{b}_E = \left\{\sum b_\gamma t^\gamma \in S(E,wE) : w(b_\gamma) > \gamma \quad \forall \gamma \in wE\right\},$$

and we let $\Delta w : B/\mathfrak{b} \longrightarrow wE \bigcup \{\infty\}$ be defined by the formula

$$\Delta w\left(\sum b_\gamma t^\gamma + \mathfrak{b}\right) = \inf\{\gamma \in wE : w(b_\gamma) = \gamma\}$$

(setting $\inf \phi = \infty$). Then $\Delta(E,w) := (\Delta E, \Delta w)$ is a valued field (Brown and Harrison, 1970), and Δ is a functor on the category of valued fields. (Here, morphisms $(E,w) \longrightarrow$

(E', w') are pairs (φ, φ^*) where $\varphi : E \longrightarrow E'$ is a homomorphism and $\varphi^* : wE \longrightarrow w'E'$ is an injective order homomorphism with $\varphi^* w = w'\varphi$; Δ acts on morphisms in the obvious way.)

Since $h \in \mathcal{P}$ we may assume that $h = g_{n+1}$ where g is a strict system of polynomial extensions of length $n+1$ as in Section 2, whose notation we use here and below. Thus $\gamma_h = \gamma_{n+1}$ (cf. Remark 6(B)). Also by Proposition 5(C), $w_\beta = w_{n+1}$. Finally, we set $w = w_\alpha$, so that w satisfies the hypotheses of Proposition 5.

It is convenient to set

$$\mathcal{C} = F[x]_{\deg h} \bigcup \{ag^\sigma : a \in F,\ 0 \leq \sigma \in \mathbb{Z}^{n+1}\}\,.$$

By parts (B) and (D) of the Proposition there exists an $\overline{F}$–homomorphism $\varphi : k_{n+1} \longrightarrow k$ such that $\varphi\tau_{n+1}$ and τ agree on $\{f \in \mathcal{C} : w(f) \geq 0\}$. Also by (A) and (B) of the Proposition for all $f \in \mathcal{C}$ we have $w(f) = w_{n+1}(f)$, and so

$$w_{n+1}F[x] = vF + \sum_{0 \leq i \leq n} \mathbb{Z}q_i \subseteq wF[x] \subseteq \mathbb{Q}vF\,.$$

Now let T be a system of representatives in F for vF, so that for each $\gamma \in vF$ there exists a unique $a_\gamma \in T$ with $v(a_\gamma) = \gamma$. By formula (3)

$$T_\beta := \{ag^\sigma : a \in T,\ 0 \leq \sigma \in \mathbb{Z}^{n+1},\ \sigma(i) < e_i\ \forall i \leq n\}$$

is a system of representatives in $F[x]$ for $w_{n+1}F[x]$. Since $w(f) = w_{n+1}(f)$ for all $f \in T_\beta$, there is a system of representatives $T_\alpha \supseteq T_\beta$ in $F[x]$ for $wF[x]$. For each $\gamma \in wF[x]$ let A_γ denote the unique element of T_α with $w(A_\gamma) = \gamma$. Thus $T_\beta = \{A_\gamma : \gamma \in w_{n+1}F[x]\}$ and $A_\gamma = a_\gamma$ for all $\gamma \in vF$.

These observations on $F[x]$ let us establish connections between $F(\alpha)$ and $F(\beta)$. After all, we may identify $vF(\alpha)$ with $wF[x]$; $vF(\beta)$ with $w_{n+1}F[x]$; $\overline{F(\alpha)}$ with k; and $\overline{F(\beta)}$ with k_{n+1}. Thus we will regard φ as an $\overline{F}$–homomorphism from $\overline{F(\beta)}$ to $\overline{F(\alpha)}$ with $\varphi\big(\overline{f(\beta)}\big) = \overline{f(\alpha)}$ for all $f \in \mathcal{C}$ with $w(f) \geq 0$. Similarly we have $vF(\beta) \subseteq vF(\alpha)$ and $v(f(\alpha)) = v(f(\beta))$ for all $f \in \mathcal{C}$. Finally, $T_\alpha^* := \{f(\alpha) : f \in T_\alpha\}$ and $T_\beta^* := \{f(\beta) : f \in T_\beta\}$ are systems of representatives in $F(\alpha)$ and $F(\beta)$ for $vF(\alpha)$ and $vF(\beta)$, respectively. Indeed for any $\gamma \in vF(\beta)$, the elements of T_β, T_α^* and T_β^* of value γ (under the valuations w_{n+1}, v, and v) are A_γ, $A_\gamma(\alpha)$ and $A_\gamma(\beta)$, respectively.

The systems of representatives T_α^* and T_β^* yield symmetric factor sets $r_\alpha : vF(\alpha) \times vF(\alpha) \longrightarrow \overline{F(\alpha)}^{\bullet}$ and $r_\beta : vF(\beta) \times vF(\beta) \longrightarrow \overline{F(\beta)}^{\bullet}$. For example, for all $\delta,\, \gamma \in vF(\alpha)$ we have

$$r_\alpha(\delta, \gamma) = \overline{A_\delta(\alpha) A_\gamma(\alpha) A_{\delta+\gamma}(\alpha)^{-1}}\,.$$

As in (Brown and Harrison, 1970, Proposition, p. 372) we have an isomorphism

$$\Theta_\alpha : \Delta F(\alpha) \longrightarrow S\big(\overline{F(\alpha)}, vF(\alpha), r_\alpha\big)$$

mapping each formal sum $\sum b_\gamma t^\gamma + \mathfrak{b}_{F(\alpha)}$ to $\sum \overline{b_\gamma A_\gamma(\alpha)^{-1}}\, t^\gamma$, and an analogous isomorphism

$$\Theta_\beta : \Delta F(\beta) \longrightarrow S(\overline{F(\beta)}, v(\beta), r_\beta)\,.$$

Suppose that $\delta,\, \gamma \in vF(\beta)$ and set $\rho = \delta + \gamma$. Then since any product of the A_μ (for

$\mu \in vF(\beta))$ is in $\mathcal{C}$, we have

$$\begin{aligned}\varphi(r_\beta(\delta,\gamma)) &= \varphi\left(\overline{A_\delta(\beta)A_\gamma(\beta)A_\rho(\beta)^{-1}} \quad \overline{A_{-\rho}(\beta)A_{-\rho}(\beta)^{-1}}\right)\\ &= \varphi(\overline{(A_\delta A_\gamma A_{-\rho})(\beta)})\left(\varphi(\overline{(A_\rho A_{-\rho}))(\beta)})\right)^{-1}\\ &= \overline{(A_\delta A_\gamma A_{-\rho})(\alpha)} \quad \overline{(A_\rho A_{-\rho})(\alpha)}^{-1}\\ &= \overline{A_\delta(\alpha)A_\gamma(\alpha)A_\rho(\alpha)^{-1}} = r_\alpha(\delta,\gamma)\,.\end{aligned}$$

It follows that φ induces a homomorphism of valued fields

$$\Phi : S\big(\overline{F(\beta)}, vF(\beta), r_\beta\big) \longrightarrow S\big(\overline{F(\alpha)}, vF(\alpha), r_\alpha\big)$$

taking each formal sum $\sum b_\gamma t^\gamma$ to $\sum \varphi(b_\gamma)t^\gamma$.

Combining Φ with the isomorphisms Θ_α and Θ_β above yields a ΔF–homomorphism

$$\Upsilon_0 : \Delta(F(\beta), v) \longrightarrow \Delta(F(\alpha), v)\,.$$

We now extract some arguments from (Brown and Harrison, 1970) (which do not depend on the hypothesis there that (F, v) is maximally complete) to show that the existence of Υ_0 implies the existence of an F–homomorphism $F(\beta) \longrightarrow F(\alpha)$. This of course implies that there exists a root of h in $F(\alpha)$, and hence it will complete the proof of the Theorem.

By hypothesis $F(\alpha)$ and $F(\beta)$ are tamely ramified over F (apply Remark 6(A) and Proposition 5(C) with i=n). Hence both are separable extensions of F. (If $F(\alpha)$ is not a separable extension of F, then there exists a proper, purely inseparable and tamely ramified extension $F(\alpha)/E$. The residue class field of $F(\alpha)$ is therefore both separable and inseparable over that of E, so the residual degree is one. Similarly the ramification index is one, and this contradicts that the field extension is proper and tamely ramified. The argument for $F(\beta)$ is the same.) Hence $F(\alpha)$ and $F(\beta)$ are contained in a Galois extension L of F. Let G denote the large ramification group of L/F, so G is the set of $\sigma \in \mathrm{Gal}(L/F)$ with $v(\sigma(a) - a) > v(a)$ for all $a \in L^{\bullet}$. Our hypothesis that (F, v) is Henselian implies that G is a normal subgroup of $\mathrm{Gal}(L/F)$, so L^G/F is a Galois field extension. That (F, v) is Henselian also implies that the extension L^G/F is tamely ramified (Zariski and Samuel, 1960, pp. 67–78). Further, L^G contains every tamely ramified subextension E/F of L/F, including both $F(\alpha)$ and $F(\beta)$. After all, the large ramification group H of an extension L/E with E/F tamely ramified is clearly contained in G, but the extension L^H of L^G cannot be proper because it is both tamely ramified (since L^H/E and E/F are both tamely ramified) and wildly ramified (since it is a subextension of L/L^G, cf. (Zariski and Samuel, 1960, pp. 67-78)). Thus we may as well suppose that L/F is tamely ramified and G is trivial. It follows that the natural homomorphism arising from the functor Δ, call it

$$D : \mathrm{Gal}(L/F) \longrightarrow \mathrm{Gal}(\Delta L/\Delta F)\,,$$

is injective (one checks directly that in general the kernel is the large ramification group G of L/F). Thus D is surjective, since both the domain and codomain of D have order $[L : F] = \big[\overline{L} : \overline{F}\big](vL : vF) = [\Delta L : \Delta F]$. By Galois theory the ΔF–homomorphism Υ_0 extends to an automorphism $\Upsilon \in \mathrm{Gal}(\Delta L/\Delta F)$ with $\mathrm{Gal}(\Delta L/\Delta F(\alpha)) \subseteq \Upsilon\,\mathrm{Gal}(\Delta L/\Delta F(\beta))\Upsilon^{-1}$. One checks that D maps $\mathrm{Gal}(L/F(\alpha))$ onto $\mathrm{Gal}(\Delta L/\Delta F(\alpha))$ and similarly for $F(\beta)$. Applying the isomorphism D^{-1} we therefore have

$$\mathrm{Gal}(L/F(\alpha)) \subseteq D^{-1}(\Upsilon)\,\mathrm{Gal}(L/F(\beta))(D^{-1}(\Upsilon))^{-1}\,.$$

Hence $F(\beta)$ is F-isomorphic to a subfield of $F(\alpha)$. Theorem 1 is proved.

Remark 8. We now argue that γ_h is best possible in Theorem 1 and in fact in a superficial generalization of Theorem 1, in which we replace the hypothesis that (F, v) is Henselian by the hypothesis that (K, u) is finite degree and tamely ramified over a Henselization of (F, v). This fact gives another characterization of γ_h independent of any strict system of polynomial extensions in which h happens to appear. Without loss of generality h is not linear, and hence we may assume that $h = g_{n+1}$ for some strict system of polynomial extensions g over (F, v) as in Definition 2. Then g can be modified to give a strict system of polynomial extensions over (F_H, v_H) where (F_H, v_H) is a Henselization of (F, v) by simply replacing each valuation w_i by the unique extension w_i^* of v_H to $F_H[x]$ with $w_i^*(g_i) = \infty$. (Uniqueness follows from the fact that an extension (E, u) of (F, v) by a root of g_i has degree $f_{u/v}e_{u/v}$ (Proposition 5(C)), and hence the same is true for an extension of (F_H, v_H).) Let ξ be a root of g_n in a field extension of F_H. Then $F_H[\xi]$ is a tamely ramified finite degree extension of F_H (say with valuation u) with $u(h(\xi)) = w_n(g_{n+1}) = \gamma_{n+1} = \gamma_h$. However, $F_H[\xi]$ cannot contain a root of h since its degree over F_H is $\deg g_n$, which is strictly less than the degree of the irreducible polynomial $h = g_{n+1}$ over F_H.

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