THE SPACE OF $\mathbb{R}-$PLACES ON A RATIONAL FUNCTION FIELD

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Abstract. Suppose that F is a field such that the value groups of the $\mathbb{R}-$places on F, i.e., places from F into the real numbers $\mathbb{R}$, are all trivial or countable. The path-connected components of the space $\mathcal{M}(F(x_1, x_2, \cdots, x_n))$ of $\mathbb{R}$-places on $F(x_1, x_2, \cdots, x_n)$ are shown then to correspond bijectively to those of $\mathcal{M}(F)$. For example, the space $\mathcal{M}(\mathbb{R}(x_1, x_2, \cdots, x_n))$ of $\mathbb{R}-$places on the rational function field $\mathbb{R}(x_1, x_2, \cdots, x_n)$ is path-connected, and similarly for $\mathbb{Q}(x_1, x_2, \cdots, x_n)$. A key tool is a homeomorphism in the case that F is a maximal field between the space of $\mathbb{R}-$places on $F(x)$ and a certain space of sequences related to the "signatures" of [1].

1. The Main Theorem

We begin by paraphrasing part of the introduction of the paper [5]: For any field F the space of $\mathbb{R}-$places on F, i.e., places from F to the field $\mathbb{R}$ of real numbers, will be denoted by $\mathcal{M}(F)$. This space is an important invariant of F for understanding the structure of the reduced Witt ring of quadratic forms over F [4]. It also plays a natural role in real algebraic geometry; given a formally real field F the points of $\mathcal{M}(F)$ correspond to the closed points of the real spectrum of the real holomorphy ring of F [17]. The topology on $\mathcal{M}(F)$ is that for which the *Harrison sets* $H(a) = \{\tau \in \mathcal{M}(F) : 0 < \tau(a) < \infty\}$ (for $a \in F$) form a subbasis [11, Section 1]. For discussion of the topology the reader could consult [4, 6, 13]. It is easy to show that the space of $\mathbb{R}$-places on the rational function field $\mathbb{R}(x)$ is a simple closed curve, and the space of $\mathbb{R}$-places on an algebraic function field in one variable over a field with finitely many orderings, all Archimedean, is known to be a (possibly empty) disjoint union of a finite number of simple closed curves [3, Theorem 2.1]. No analogous result is known for algebraic function fields or even rational function fields in more than one variable over

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$\mathbb{R}$. It is known that $\mathcal{M}(\mathbb{R}(x_1, x_2, \cdots, x_n))$ is compact, Hausdorff and connected [6, page 5], [8, Theorem 2.12], but not whether it contains a disk (and much less whether it contains a torus, a topic of the paper [12]).

In the paper [5] $\mathbb{R}-$places were called "real places", but the terminology used here has become more standard.

Throughout this paper F will denote a field. For any $n > 0$ restriction of mappings gives a function res from the space of $\mathbb{R}-$places of the rational function field $F(x_1, x_2, \cdots, x_n)$ to $\mathcal{M}(F)$.

1.1. **Main Theorem.** *Suppose that for each $\mathbb{R}-$place of F, the value group of the corresponding valuation is trivial or countable. Then res^{-1} gives a bijection from the set of path-connected components of $\mathcal{M}(F)$ to the set of path-connected components of $\mathcal{M}(F(x_1, x_2, \cdots, x_n))$.*

1.2. **Corollary.** *For any $n \geq 0$ $\mathcal{M}(\mathbb{R}(x_1, \cdots, x_n))$ is path-connected.*

The $n = 2$ case of the above Corollary was proven in [5]; a number of arguments of that paper are generalized in the proof of Theorem 1.1, which will occupy the next five sections.

1.3. *Remark.* Corollary 1.2 was our first application of the above theorem. We thank Professor Katarzyna Kuhlmann for a stimulating conversation which led to our dropping our original hypothesis that the field F contained $\mathbb{R}$. This gives us more simple applications, such as that in the next corollary.

1.4. **Corollary.** *Let K denote an algebraic extension of the field of rational numbers $\mathbb{Q}$. The number of path-connected components of $\mathcal{M}(K(x_1, x_2, \cdots, x_n))$ is the number of homomorphisms of K into $\mathbb{R}$.*

Proof. The space $\mathcal{M}(K)$ can be identified with the space of orderings of K [4], which is disconnected. Thus the path-connected components of $\mathcal{M}(K)$ are singletons, which of course correspond bijectively with the homomorphisms of K into $\mathbb{R}$. $\square$

For example, $\mathcal{M}(\mathbb{Q}[\sqrt{2}](x_1, \cdots, x_n))$ has exactly two path-connected components and $\mathcal{M}(\mathbb{Q}(x_1, \cdots, x_n))$ is path-connected.

2. The reduction to maximal fields

Suppose that $\sigma \in \mathcal{M}(F)$. We call (F, σ) *ultracomplete* if it is a maximal field and σ maps onto $\mathbb{R}$; we call an extension (E, ρ) of (F, σ) an *ultracompletion* of (F, σ) if it is ultracomplete and the value group of (E, ρ) is the value group of (F, σ). (Of course ρ here is assumed to be an $\mathbb{R}$-place.) This language comes from [2, Section 1], keeping in mind that any $\mathbb{R}-$place σ is associated with the "extended absolute value" obtained by composing σ with the absolute value on $\mathbb{R}$. From [2] we use only the facts that ultracompletions at $\mathbb{R}$-places admit only one $\mathbb{R}$-place and that they are unique up to place-preserving isomorphism. (Both facts are easy to prove directly using respectively the fact that ultracompletions are Henselian and the uniqueness of maximal immediate extensions [10, Theorem 5].) We let $\text{res} : \mathcal{M}(F(x)) \longrightarrow \mathcal{M}(F)$ denote the obvious map obtained by restriction of maps. For any $\sigma \in \mathcal{M}(F)$, let $(\tilde{F}_\sigma, \tilde{\sigma})$ denote an ultracompletion of (F, σ) and let $\Psi = \Psi_\sigma : \mathcal{M}(\tilde{F}_\sigma(x)) \longrightarrow \mathcal{M}(F(x))$ also be the map obtained by restriction of maps. Let $\text{comp} : \mathcal{M}(F) \longrightarrow \mathcal{M}(F(x))$ be given by composition of places in $\mathcal{M}(F)$ with the $x-$adic place on $F(x)$. All these maps are checked to be continuous and the composition $\text{res} \circ \text{comp}$ is the identity map on $\mathcal{M}(F)$. We will argue that Ψ maps onto $\text{res}^{-1}(\sigma)$; this argument will use the next lemma.

2.1. **Lemma.** *Suppose that $\sigma \in \mathcal{M}(F)$. Any ultracomplete field (K, δ) extending (F, σ) contains an ultracompletion of (F, σ).*

Proof. Let (L, ρ) be maximal among all unramified extensions of (F, σ) within (K, δ). We claim that (L, ρ) is an ultracompletion of (F, σ).

We begin by checking that $\rho(L) \supset \mathbb{R}$. Suppose otherwise. Let $t \in \mathbb{R} \setminus \rho(L)$. If t is transcendental over $\rho(L)$, choose $x \in K$ with $\delta(x) = t$. Then x is transcendental over L, and $L(x)$ is an unramified extension of (L, ρ) within (K, δ) [7, Corollary 2.2.2], a contradiction. On the other hand, if t is algebraic over $\rho(L)$, then we have a monic, necessarily irreducible, polynomial $f(y) \in L[y]$ with $\rho(f)$ the minimal polynomial of t over $\rho(L)$. As we do have $t \in \mathbb{R}$ (the residue class field of K), and K is henselian, by Hensel's lemma K contains a root of f, which by a degree argument would generate an unramified extension of (L, ρ) within (K, δ), again a contradiction. So we have $\rho(L) \supset \mathbb{R}$.

It remains to show that (L, ρ) is a maximal field. First, since (K, δ) is henselian, it contains a henselization of (L, ρ) [7, Theorem 5.2.2]. As henselizations are immediate, by the choice of (L, ρ), we must have that this henselization is (L, ρ) itself, i.e., (L, ρ) is henselian. It follows that (L, ρ) has no proper immediate algebraic extension [16, Ostrowski's Theorem, p. 236]. Now by [10, Theorem 4], if (L, ρ) is not maximal, there exists in L a pseudoconvergent sequence $\langle a_i \rangle$ with no pseudolimit in L; however it has a pseudolimit u in K. Then $\langle a_i \rangle$ cannot be of algebraic type, or by [10, Theorem 3] there would be a proper immediate algebraic extension of (L, ρ). But if $\langle a_i \rangle$ is of transcendental type, then [10, Theorem 2] implies $L(u)$ is an immediate extension of (L, ρ) in (K, δ). So we have a contradiction, establishing maximality.

□

2.2. **Lemma.** *Ψ maps $\mathcal{M}(\tilde{F}_\sigma(x))$ onto $res^{-1}(\sigma)$.*

This lemma generalizes [5, Lemma 2.2] whose proof unfortunately contained an error. The error can be corrected by making a more careful construction of the field E in that proof.

Proof. Suppose that $\rho \in \text{res}^{-1}(\sigma)$. Let (K, δ) be an ultracompletion of $(F(x), \rho)$ (such field extensions are easily constructed). By the previous lemma we have a homomorphism $\Delta : (\tilde{F}_\sigma, \tilde{\sigma}) \longrightarrow (K, \delta)$ fixing F. Let $\tau : F(x) \longrightarrow \tilde{F}_\sigma(x)$ be the inclusion map. Extend Δ to a map Δ' from $\tilde{F}_\sigma(x)$ to the rational function field $K(z)$ by mapping x to z, and let α be the natural place $K(z) \longrightarrow K \cup \{\infty\}$ mapping z to x. Then $\delta\alpha\Delta' \in \mathcal{M}(\tilde{F}_\sigma(x))$; we will show it maps under Ψ to ρ, which will complete the proof of the lemma. Thus it suffices to show that

$$\delta\alpha\Delta'\tau = \rho.$$

For any rational function $\sum a_i x^i / \sum b_i x^i \in F(x)$ we have (since Δ' is the identity on F and $\alpha\Delta'(x) = x$)

$$\delta\alpha\Delta'\tau(\sum a_i x^i / \sum b_i x^i) = \delta(\sum a_i x^i / \sum b_i x^i) = \rho(\sum a_i x^i / \sum b_i x^i),$$

completing the proof of the lemma.

□

2.3. **Lemma.** *Let C be a path-connected component of $\mathcal{M}(F)$. If for all $\sigma \in C$ the space $\mathcal{M}(\tilde{F}_\sigma(x))$ is path-connected, then $C^* := res^{-1}(C)$ is a path-connected component of $\mathcal{M}(F(x))$.*

Proof. Suppose $\alpha, \beta \in C^*$. Then by hypothesis there exits a path from $\text{res}(\alpha)$ to $\text{res}(\beta)$, i.e., a continuous map $f : [0,1] \longrightarrow C$ taking 0 to $\text{res}(\alpha)$ and 1 to $\text{res}(\beta)$. Then $\text{comp} \circ f$ is a path from $\text{comp} \circ \text{res}(\alpha)$ to $\text{comp} \circ \text{res}(\beta)$. Note that $\text{comp} \circ \text{res}(\alpha) \in \text{res}^{-1}(\text{res}(\alpha))$ and $\alpha \in \text{res}^{-1}(\text{res}(\alpha))$, so by the previous lemma, both α and $\text{comp} \circ \text{res}(\alpha)$ are in $\Psi_{\text{res}(\alpha)}(\mathcal{M}(\tilde{F}_{\text{res}(\alpha)}(x)))$, which is path-connected. Therefore there exists a path from α to $\text{comp} \circ \text{res}(\alpha)$ and similarly a path from $\text{comp} \circ \text{res}(\beta)$ to β. Combining these paths gives a path from α to β. Thus C^* is path-connected. Hence it lies in a path-connected component C^{**} of $\mathcal{M}(F(x))$. Just suppose there exists $\delta \in C^{**} \setminus C^*$. Then $\text{res}(\delta)$ lies in a path-connected component $C' \neq C$ of $\mathcal{M}(F)$. We can pick some $\sigma \in C$. By hypothesis there is a path, say g, from δ to $\text{comp}(\sigma)$. Then the composition $\text{res} \circ g$ is a path from $\text{res}(\delta)$ to

$\mathrm{res} \circ \mathrm{comp}(\sigma) = \sigma$, contradicting that $\mathrm{res}(\delta)$ and σ lie in distinct path-connected components of $\mathcal{M}(F)$. Thus, C^* is actually a path-connected component of $\mathcal{M}(F(x))$. □

This section's title refers to the following proposition.

2.4. **Proposition.** *If for all $\sigma \in \mathcal{M}(F)$ the space $\mathcal{M}(\tilde{F}_\sigma(x))$ is path-connected, then res^{-1} gives a bijection from the set of path-connected components of $\mathcal{M}(F)$ to the set of path-connected components of $\mathcal{M}(F(x))$.*

Proof. By the previous lemma res^{-1} gives a map from the set of path-connected components of $\mathcal{M}(F)$ to the set of path-connected components of $\mathcal{M}(F(x))$. This map is surjective since if $\sigma \in \mathcal{M}(F(x))$, then the path-connected component of σ is the inverse image under res of the path-connected component of $\mathrm{res}(\sigma)$. On the other hand if α and α' lie in distinct path-connected components, say C and C', of $\mathcal{M}(F)$ but $\mathrm{res}^{-1}(C) = \mathrm{res}^{-1}(C')$, then there would be a path f from $\mathrm{comp}(\alpha)$ to $\mathrm{comp}(\alpha')$ so that $\mathrm{res} \circ f$ would be a path from α to α', a contradiction. Hence, our map is also injective. □

Our Main Theorem 1.1 follows by induction from the previous proposition combined with the next one.

2.5. **Proposition.** *If (F, σ) is ultracomplete and the value group of σ is trivial or countable, then $\mathcal{M}(F(x))$ is path-connected.*

The assertion of the above proposition is standard if the value group is trivial. The proof of this proposition when the value group is not trivial will occupy the next four sections of this paper. The next section introduces some necessary notation; in the remaining sections it will be convenient to assume that (F, σ) is ultracomplete with non-trivial value group.

3. Valuations and cuts

Let v be a valuation on F associated with some place $\sigma \in \mathcal{M}(F)$. Let $\mathbb{Q}vF$ denote a divisible hull of the value group vF of v. By a *cut* of $\mathbb{Q}vF$ we mean a subset C of $\mathbb{Q}vF$ such that if $a \in C$ and $b \in \mathbb{Q}vF$ with $b \leq a$, then $b \in C$. (Our "cuts" might elsewhere be considered the "lower set" of a cut.) We let Γ denote the union of $\mathbb{Q}vF$ and the set of cuts of $\mathbb{Q}vF$. The order $\leq$ on $\mathbb{Q}vF$ is extended to Γ in the natural way; for $\alpha, \beta \in \Gamma \setminus \mathbb{Q}vF$ and $\rho \in \mathbb{Q}vF$ we write $\alpha \geq \beta$ if and only if $\alpha \supseteq \beta$ and $\alpha \geq \rho$ if and only if $\rho \in \alpha$ (and write $\rho \geq \alpha$ otherwise). We will sometimes let ∞ denote the maximum of Γ, namely the cut $\mathbb{Q}vF$, and let $-\infty$ denote the minimum of Γ, namely the empty cut $\emptyset$.

Given a cut $\gamma \in \Gamma$ one can verify that we have an ordered group $\mathbb{Q}vF \oplus \mathbb{Z}$ where $(a, n) \geq 0$ if and only if either $n = 0$ and $a \geq 0$ or $n > 0$ and $-a/n \in \gamma$ or, finally, $n < 0$ and $-a/n \notin \gamma$. (Alternately, Corollary 2.4.2 of [18] asserts the existence of an extension of $\mathbb{Q}vF$ generated by an element δ to an ordered abelian group with δ greater than every element of the set γ and less than every element of $\mathbb{Q}vF$ not in γ. Such a group must have the order described above.) Given such a group we will usually write $(a, n) = a + n\gamma$, identifying a with $(a, 0)$ and γ with $(0, 1)$.

For any $\beta \in \mathbb{Q}vF$ we let β^+ be the smallest cut containing β, and β^- denote the cut consisting of all elements less than β. Note that for $\alpha \in \mathbb{Q}vF$ and γ a cut, we have $\alpha \leq \gamma$ if and only if $\alpha^+ \leq \gamma$, and $\gamma \leq \alpha$ if and only if $\gamma \leq \alpha^-$. Thus, for example, $\alpha^- \leq \alpha \leq \alpha^+$.

3.1. **Lemma.** *(A) Suppose that Δ is an ordered abelian group containing vF such that Δ/vF is torsion. Then there exists a unique order isomorphism θ fixing vF from Δ to a subgroup of $\mathbb{Q}vF$.*

(B) Suppose G is an ordered abelian group of the form $G = \Delta + d\mathbb{Z}$ where Δ is a subgroup of $\mathbb{Q}vF$ containing vF and d has no nonzero multiple in vF. Then there exists a unique cut δ of $\mathbb{Q}vF$ and unique order isomorphism from G to the subgroup $\Delta+\delta\mathbb{Z}$ of the group $\mathbb{Q}vF+\delta\mathbb{Z}$ which fixes Δ and maps d to δ.

Because of part (A) of the above lemma, we will often identify groups Δ as in (A) with subgroups of $\mathbb{Q}vF$.

In the proof below and elsewhere in the paper, notation of the form $A := B$ is used to indicate that the symbol A is being defined by the expression B.

Proof. (A) Note that if $\delta \in \Delta$ has $m\delta \in vF$ for some positive integer m, then θ must map δ to $\frac{1}{m}(m\delta)$, considered as an element of $\mathbb{Q}vF$.

(B) Let δ be the cut of $\mathbb{Q}vF$ generated by $\delta_0 := \{\frac{1}{s}\alpha : 0 < s \in \mathbb{Z}, \alpha \in vF, \alpha < sd\}$. We of course have a group isomorphism $\theta : \Delta + d\mathbb{Z} \longrightarrow \Delta + \delta\mathbb{Z}$ fixing Δ and mapping d to δ. Suppose that $\alpha + sd > 0$ where $\alpha \in \Delta$ and $s \in \mathbb{Z}$. We claim that $\alpha + s\delta > 0$ (so that θ preserves the order). This is trivial if $s = 0$; suppose $s > 0$. Then $sd > -\alpha$, so $\frac{1}{s}(-\alpha) < \delta$ and so $\alpha+s\delta > 0$ as claimed. Now suppose that $s < 0$. Then for all $\beta/t \in \delta_0$ (with the obvious notation), we have $-\alpha/s > d > \frac{1}{t}\beta$, so that $-\alpha/s > \delta$, whence $\alpha + s\delta > 0$.

It remains to show that δ is unique. Suppose we have another cut δ' and order isomorphism $\theta' : \Delta + d\mathbb{Z} \longrightarrow \Delta + \delta'\mathbb{Z}$ fixing Δ and mapping d to δ'. First suppose that $\delta > \delta'$. Then there exists $0 < s \in \mathbb{Z}$ and

$\alpha \in vF$ with $\delta > \frac{1}{s}\alpha > \delta'$ so $s\delta > \alpha > s\delta'$, and hence

$$0 < \theta'\theta^{-1}(s\delta - \alpha) = s\delta' - \alpha < 0,$$

a contradiction. The case when $\delta < \delta'$ is handled similarly.

□

The following application of the above lemma is used in the next section to describe bijections from the spaces of Proposition 2.5 to certain sets of sequences.

3.2. **Lemma.** *Suppose that $\tau \in \mathcal{M}(F(x))$ is an extension of σ. Then there exists a unique valuation v_τ of $F(x)$ associated with τ and extending v such that either $v_\tau F(x) \subset \mathbb{Q}vF$ or else there exists a monic $f \in F[x]$ such that $v_\tau(f)$ is a cut of $\mathbb{Q}vF$; $v_\tau F(x) = \Delta + \mathbb{Z}v_\tau(f)$ for some subgroup Δ of $\mathbb{Q}vF$; and $v_\tau(g) \in \Delta$ for all polynomials $g \in F[x]$ of degree less than the degree of f.*

Proof. Let u be an extension of v associated with τ. By Lemma 3.1(A) we may uniquely identify the subgroup of $uF(x)$ of elements with some positive multiple in vF with a subgroup Δ of $\mathbb{Q}vF$. Note that the set of such elements depends only on τ and is independent of the choice of u. (After all, $u(f) \in \Delta$ if and only if $\tau(f^b r) \in \mathbb{R}^\bullet$ for some $0 < b \in \mathbb{Z}$ and $0 \neq r \in F$.) If $uF(x) = \Delta$, we are done. Otherwise there exists a monic polynomial $f \in F[x]$ of minimal degree such that $u(f)$ is not in Δ. Then for all polynomials g_i of degree less that that of f we have $u(\sum g_i f^i) = \min(u(g_i) + iu(f))$, so that $uF(x) = \Delta + \mathbb{Z}u(f)$. Hence by Lemma 3.1(B) there is a unique cut γ of $\mathbb{Q}vF$ and a unique order isomorphism $\theta : uF \longrightarrow \Delta + \mathbb{Z}\gamma$ taking $u(f)$ to γ and fixing Δ. Then $v_\tau := \theta u$ is an extension of v to a valuation associated with τ satisfying the conditions of the lemma. If w, with corresponding monic polynomial g, is another such extension, then since $\deg(f - g) < \deg g$ and $w(f) \notin \Delta$,

$$w(f) = w(f - g + g) = \min(w(g), w(f - g)) = w(g).$$

Then $w(f)$ is a cut of $\mathbb{Q}vF$ and hence

$$\begin{aligned} v_\tau(f) &= \{\frac{1}{s}v(d) : 0 \neq d \in F, 0 < s \in \mathbb{Z}, v_\tau(f) > \frac{1}{s}v(d)\} \\ &= \{\frac{1}{s}v(d) : 0 \neq d \in F, 0 < s \in \mathbb{Z}, \tau(f^s/d) = 0\} \\ &= \{\frac{1}{s}v(d) : 0 \neq d \in F, 0 < s \in \mathbb{Z}, w(f^s/d) > 0\} = w(f). \end{aligned}$$

Hence for g_i as above,

$$w(\sum g_i f^i) = \min(w(g_i) + iw(f)) = v_\tau(\sum g_i f^i),$$

so $w = v_\tau$.

□

The notation v_τ will be used frequently below.

4. The bijection $\mathcal{M}(F(x)) \longrightarrow \mathcal{S}$

As indicated at the end of Section 2, we will assume now that (F, σ), with associated (nontrivial) valuation v, is ultracomplete, i.e., a maximal field with σ mapping onto $\mathbb{R}$. We may therefore assume without loss of generality that in fact F is a power series field $\mathbb{R}(t^{vF}, c)$ where c is a suitable factor set [10, Theorem 6]. We generalize the arguments of [5], again using ideas that go back to [14, 1], to give a bijection from $\mathcal{M}(F(x))$ to a set $\mathcal{S}$ of sequences.

We first use the set Γ introduced in the previous section to generalize the "signatures" of [5]. We will need to consider sequences $\langle a_n \rangle_{i<n}$ where now we allow n to be an arbitrary ordinal number and where the i range over all ordinal numbers less than n. We will let 0 denote the first ordinal number and denote the successor of an ordinal number n by $n+1$.

4.1. *Definition and Notation.* A *presignature* is a sequence $S = \langle (q_i, \theta_i) \rangle_{i<n}$ of elements of $\Gamma \times \mathbb{R}$, usually abbreviated $\langle q_i, \theta_i \rangle_{i<n}$, such that for all $i < n$,

$$\theta_i = 0 \Leftrightarrow q_i \notin \mathbb{Q}vF \qquad \text{and} \qquad \theta_i = 0 \Rightarrow i+1 = n.$$

The *degree* of S is $(\Gamma_S : vF)$ where for $m \leq n$ we set $\Gamma_m = vF + \sum_{i<m} \mathbb{Z}q_i$ and set $\Gamma_S = \Gamma_n$; the *length* of S is n. We call S a *signature* if $(\Gamma_i : vF) < \infty$ when $i < n$ and if the sequence $\langle q_i/(\Gamma_i : vF) \rangle_{i<n}$ is strictly increasing. Finally, we let $\mathcal{S}$ denote the set of signatures of infinite degree. It should be noted that these definitions are related to but different from those of [1].

Set $e_i = e_i^S = (\Gamma_{i+1} : \Gamma_i)$ for all $i < n$, and for each $m \leq n$,

$$J_m = J_m^S = \{\sigma \in \bigoplus_{0 \leq i < m} \mathbb{Z} : 0 \leq \sigma(i) < e_i \qquad \forall i < m\}.$$

Then the number of elements of J_n is the degree of S. If $g = \langle g_i \rangle_{i<m}$ is a sequence of polynomials in $F[x]$, then for each $\sigma \in J_m$ we set $g^\sigma = \prod_{i<m} g_i^{\sigma(i)}$. If $k < m$ we will identify J_k with the subset of J_m consisting of elements σ with $\sigma(i) = 0$ for all $i \geq k$; this identification does not affect the meaning of g^σ.

We next describe a bijection from $\mathcal{M}(F(x))$ to $\mathcal{S}$. We begin by showing how, given a signature $S = \langle q_i, \theta_i \rangle_{i<n}$, to construct *its associated*

sequence $g = \langle g_i \rangle_{i<n'}$ of polynomials in $F[x]$ where $n' = n$ if S has infinite degree and $n' = n+1$ otherwise. The case when S has infinite degree (so $n' = n$) applies directly to the bijection Φ of Theorem 4.3(A) below, and the case when S has finite degree (where $n' = n+1$) is critical for the application of Theorem 4.3(C), which plays a fundamental role in many of our arguments. The construction of g will be inductive; we start by setting $g_0 = x$. Now suppose that for some ordinal $k < n'$ we have constructed g_i for all $i < k$. If k is not a limit ordinal, say $k = m+1$, then we set

$$g_k = g_m^{e_m} + \theta_m t^{\alpha_m} g^{\sigma_m} \tag{4.1}$$

where $\alpha_m \in vF$ and $\sigma_m \in J_m$ are uniquely determined by the condition that

$$e_m q_m = \alpha_m + \sum_{i<m} \sigma_m(i) q_i. \tag{4.2}$$

Next suppose that k is a limit ordinal. Since $(\Gamma_k : vF) < \infty$, there is a least $k_0 < k$ with $e_i = 1$ whenever $k_0 \le i < k$. We then set

$$g_k = g_{k_0} + \sum_{\sigma \in J_k} \left(\sum \theta_j t^{\alpha_j} \right) g^{\sigma} \tag{4.3}$$

where for each $\sigma \in J_k$, the inside sum above is over all j with $k_0 \le j < k$ and $\sigma_j = \sigma$. These inside sums are in our power series field since for all $\sigma \in J_k$ the sequence $\langle \alpha_j \rangle_{k_0 \le j < k}$ is strictly increasing. (Recall the definition of α_j and the fact that $\langle q_i/(\Gamma_i : vF) \rangle_{i<n}$ is strictly increasing.)

4.2. *Notation.* Suppose $S = \langle q_i, \theta_i \rangle_{i<n}$ is a signature. We write $S \rhd T$ if T is a signature which is an initial segment (not necessarily proper and possibly empty) of S. If $m \le n$ we will also set $S_m := \langle q_i, \theta_i \rangle_{i<m}$; thus, for example, $S \rhd S_m$.

If S has finite degree, then we let $\operatorname{cut}(S)$ denote the cut of $\mathbb{Q}vF$ determined by $\{(\Gamma_n : \Gamma_i) q_i : i < n\}$, i.e., the smallest cut containing this set. Note that by the above definiton of a signature, for all $i < n$ we have $q_i \ge \operatorname{cut}(S_i)$.

4.3. **Theorem.** *There exists a unique bijection $\Phi : \mathcal{M}(F(x)) \longrightarrow \mathcal{S}$ such that for all $S \in \mathcal{S}$ and $\tau \in \mathcal{M}(F(x))$ we have*

(A) $\Phi(\tau) = S$ if and only if $v_\tau(g_i) = q_i$ for all $i < n$;

(B) If $\Phi(\tau) = S$, then $\{g^\sigma : \sigma \in J_n\}$ is a valuation basis over v for the restriction of v_τ to $F[x]$, and $v_\tau F(x) = \Gamma_S$;

(C) for all $\rho \in \mathcal{M}(F(x))$ and $k < n$, we have $\Phi(\rho) \rhd S_k$ if and only if $v_\rho(g_k) \ge q$ for all $q \in cut(S_k)$.

(D) if $\Phi(\tau) = S$ and $k < n$, then g_k is irreducible over F and on all polynomials over F of degree less than that of g_k, v_τ agrees with

the composition of the natural map $F[x] \longrightarrow F[x]/(g_k)$ with the unique extension of v to $F[x]/(g_k)$.

The notation introduced in this section with respect to a signature S has been used in the statement of the above theorem. In particular, we have let $g = \langle g_i \rangle_{i<n}$ denote the associated sequence of polynomials of S. (In the proof below, g will be defined in a different way, but it will turn out to be the associated sequence of polynomials of a signature S.) Note that in (B) above we are asserting that for all $d_\sigma \in F$ the value of v_τ on $\sum d_\sigma g^\sigma$ is the minimum of the values $v_\tau(d_\sigma g^\sigma)$. The uniqueness assertion in (D) follows from the fact that (F, v) is Henselian and $F[x]/(g_k)$ is an algebraic field extension of F.

4.4. *Remark.* The condition "$v_\rho(g_k) \geq q$ for all $q \in \mathrm{cut}(S_k)$" of part (C) above is equivalent to saying that $v_\rho(g_k) \geq \mathrm{cut}(S_k)$ because $v_\rho(g_k)$ is either in $\mathbb{Q}vF$ or it is a cut of $\mathbb{Q}vF$. After all, if it is not in $\mathbb{Q}vF$ then using part (B) above we deduce that g_k is a monic polynomial of minimal degree with value under v_ρ not in $\mathbb{Q}vF$, and so by the proof of Lemma 3.2 its value under the map v_ρ is indeed a cut of $\mathbb{Q}vF$.

Proof. The map Φ will be obtained from the bijection of [1, Corollary 4.3] by restriction of the domain and codomain of that bijection and by modifying it so as to take advantage of the significant simplifications possible in the context of this theorem. (References as this one to [1, Section 4] should be assumed to include their generalizations indicated in [1, Sections 7 and 8].)

The bijection of [1] requires a choice of a set of representatives A for vF and a "display" Υ of F [1, page 475]. For these we make the obvious choices: let $A = \{t^\gamma : \gamma \in vF\}$ and let $\Upsilon : \mathbb{R}(t^{vF}, c) \longrightarrow W$ be the map taking each $\sum_{i \in I} b_i t^i$ to $\{b_i t^i (1 + \mathfrak{p}) : i \in I\}$ where of course each b_i is in $\mathbb{R}$ and I is a well-ordered subset of vF. Here, W denotes the set of well-ordered subsets of $F^\times/(1 + \mathfrak{p})$, where $\mathfrak{p}$ is the maximal ideal of the valuation v, and where we write $a(1 + \mathfrak{p}) < b(1 + \mathfrak{p})$ if and only if $v(a) < v(b)$.

We identify the residue class field of v with $\mathbb{R}$. Since (F, σ) is ultracomplete, σ is its only $\mathbb{R}$-place and in particular it is the only $\mathbb{R}$-place associated with v. The $\mathbb{R}$-places on $F(x)$ must extend σ and they therefore correspond bijectively with the equivalence classes of totally ramified extensions of v to $F(x)$ (i.e., those with residue class field $\mathbb{R}$). By [1, Corollary 4.3], the set of equivalence classes of extensions of v to $F(x)$ is bijective with the set of "signatures" over $(\mathbb{R}, vF)$ of infinite degree (in the sense of [1, Definition (7.3)]), and by [1, Supplement 4.2(C)] the set of equivalence classes of totally ramified extensions is bijective with the set, call it $\mathcal{B}$, of those "signatures" which are represented by "presignatures" of the form $\langle \theta_i, q_i \rangle_{i<n}$ (all in the sense of [1, Definition 7.3]) where $\theta_i \in \mathbb{R}$ for all $i < n$; we shall henceforth call these $\mathbb{R}$-*signatures* to distinguish them from the signatures of this paper.

We now define a bijection from $\mathcal{S}$ to $\mathcal{B}$. Using the language and notation of Definition and Notation 4.1 (and setting $a_m = t^{\alpha_m}$), for any $S = \langle q_i, \theta_i \rangle_{i<n} \in \mathcal{S}$ and any $m < n$ we set

$$M_{\langle \theta_i, q_i \rangle_{i<m}} = \tau(a_m b_m g^{\sigma_m} g^{\gamma_m})$$

where $a_m \in A, b_m \in A, \sigma_m \in J_m$, and $\gamma_m \in J_m$ are (uniquely) determined by the equations

$$v(a_m) + \sum_{j<m} \sigma_m(j) q_j = -\left(v(b_m) + \sum_{j<m} \gamma_m(j) q_j \right) = e_m q_m; \qquad (4.4)$$

and where g is a generating sequence for, and τ is an $\mathbb{R}$-place (whose valuation is) associated with, the $\mathbb{R}$-signature $\langle \theta_i, q_i \rangle_{i<m}$ (cf. [1, Definition 7.5]). The key point here is that $M_{\langle \theta_i, q_i \rangle_{i<m}}$ is independent of the choice of τ since $a_m b_m g^{\sigma_m} g^{\gamma_m}$ is a product of polynomials of degree less than that of $\langle \theta_i, q_i \rangle_{i<m}$ [1, Lemma F of Section 8]. We define $\Delta : \mathcal{S} \longrightarrow \mathcal{B}$ by setting

$$\Delta(\langle q_i, \theta_i \rangle_{i<n}) = \text{ the equivalence class of } \langle \theta'_i, q_i \rangle_{i<n}$$

where we inductively define for all $m < n$

$$\theta'_m = -\theta_m M_{\langle \theta'_i, q_i \rangle_{i<m}},$$

so in particular $\theta'_0 = -\theta_0 \tau(a_0 b_0)$ (starting the induction).

We define $\Phi : \mathcal{S} \longrightarrow \mathcal{M}(F(x))$ by letting $\Phi(S)$ be the unique $\mathbb{R}$-place associated with the $\mathbb{R}$-signature $\Delta(S)$ for each $S \in \mathcal{S}$.

Our next tasks are to verify that Δ (and hence Φ) is a bijection, and to show for any $S \in \mathcal{S}$, that the associated sequence of polynomials of S is exactly the generating sequence of $\Delta(S)$. The first of these tasks is accomplished by constructing the inverse of Δ. By Lemma 3.1 each $\mathbb{R}$-signature has a unique representative $\langle \theta_i, q_i \rangle_{i<n}$ such that for all $i < n$ we have $(\theta_i, q_i) \in \mathbb{R} \times \Gamma$, and $\langle q_i, \theta_i \rangle_{i<n} \in \mathcal{S}$. If $T \in \mathcal{B}$ is represented by such a presignature $\langle \theta_i, q_i \rangle_{i<n}$, then we let $\Delta'(T) = \langle q_m, -\theta_m / M_{\langle \theta_i, q_i \rangle_{i<m}} \rangle_{m<n}$. Moreover, if we write $\Delta\Delta'(T) = \langle \theta''_i, q_i \rangle_{i<n}$, then we must have $\theta_m = \theta''_m$ for all $m < n$, so that $\Delta\Delta'$ is the identity map. After all, by induction on m we have

$$\theta''_m = -(-\theta_m / M_{\langle \theta_i, q_i \rangle_{i<m}}) M_{\langle \theta''_i, q_i \rangle_{i<m}} = \theta_m.$$

(Note that $\theta_0'' = \theta_0\tau(a_0b_0)/\tau(a_0b_0) = \theta_0$.) The argument that $\Delta'\Delta$ is the identity is similar, but doesn't require induction. Thus Δ is indeed bijective.

Now suppose that $S = \langle q_i, \theta_i\rangle_{i<n} \in \mathcal{S}$, and $g = \langle g_i\rangle_{i<n}$ is the generating sequence for $\Delta(S) = \langle \theta_i', q_i\rangle_{i<n}$ and τ is the $\mathbb{R}$-place associated with $\Delta(S)$. We suppose (inductively) that for some $r < n$ the sequence $\langle g_i\rangle_{i<r}$ is an initial segment of the associated sequence of polynomials of S. We show now that g_r is also in the associated sequence of polynomials of S so that this sequence is in fact g. First suppose that we can write $r = m + 1$. Pick $a_m, b_m, \sigma_m, \gamma_m$ as in Equation 4.4. Then for some $\theta \in \mathbb{R}$ we have $g_r = g_m^{e_m} + \theta a_m g^{\sigma_m}$. By [1, Definition (7.5)(v)] we have $\theta_m' = \tau(b_m g^{\gamma_m} g_m^{e_m})$, so that by the definition of Δ

$$\theta = \tau\left(\frac{g_{m+1} - g_m^{e_m}}{a_m g^{\sigma_m}}\right) = \tau\left(\frac{-\theta_m'}{a_m g^{\sigma_m} b_m g^{\gamma_m}}\right) = \theta_m;$$

hence g_r is in the associated sequence of polynomials of S.

On the other hand, if r is a limit ordinal, then g_r (described in [1, Definition 7.5(iii)]) is also the r-th element of the associated sequence of polynomials of S (defined in formula 4.3). The crucial point here is that by our choice of the set of representatives A for vF and the display Υ we see that in the notation of display 4.3, for each $\sigma \in J_k$ we have that $\Upsilon(\sum \theta_j t^{\alpha_j}) = \cup\Upsilon(\theta_j t^{\alpha_j})$.

We now check that Φ satisfies the conditions of the theorem. Condition (B) follows from the Supplement of [1, (4.2)]. Part (C) follows from the fundamental lemma [1, (3.5), as generalized on page 479]. Necessity in part (A) of the theorem follows from the fact that S is associated with v_τ. Sufficiency follows from the fundamental lemma [1, (3.5)]. When the length n of S is a limit ordinal this is obvious. Suppose that $n = m+1$. Then $\Phi(\tau) \triangleright S_m$ and so $\langle g_i\rangle_{i<n}$ is a generating sequence for v_τ and $v_\tau(g_m) = q_m \neq \mathbb{Q}vF$, so $\Phi(\tau) = S$ [1, Definition 7.5(iv)]. Part (D) follows from the discussion of algebraic extensions in [1, page 465, including formula (15)]. The uniqueness of the map Φ is immediate from part (A).

□

We end this section with an application of the above theorem which will be needed in the next section.

4.5. **Lemma.** *Let $S = \langle q_i, \theta_i\rangle_{i<n}$ be a signature of finite degree whose length n is a limit ordinal. Suppose $\hat{S} = \langle \hat{q}_i, \hat{\theta}_i\rangle_{i<\hat{n}}$ is a signature of infinite degree, $\hat{S} \not\triangleright S$, and $\hat{S} \triangleright S_i$ for some $i < n$ with* $\deg S = \deg S_i$. *Let j be the smallest ordinal for which $(\widehat{q}_j, \widehat{\theta}_j) \neq (q_j, \theta_j)$. Then $\widehat{v}(g_n) = \min(q_j, \widehat{q}_j)$.*

Proof. Note that $j \geq i$ and $g_k = \widehat{g}_k$ for all $k \leq j$. Let $\hat{v}$ and v' denote the valuations associated with the signatures $\widehat{S}$ and S', respectively, where S' is the signature with $S' \triangleright S$ and with n-th term $(\mathrm{cut}(S), 0)$. Then $v'(g_n) = \mathrm{cut}(S) > v'(g_j) = q_j$. Also $g_n - g_j$ has degree less than that of S_i and hence by Theorem 4.3(A and D) we have $\widehat{v}(g_n - g_j) = v'(g_n - g_j) = \min(\mathrm{cut}(S), q_j) = q_j$. Similarly $\widehat{v}(g_n - g_{j+1}) = q_{j+1}$. If $q_j \neq \widehat{q}_j$, then using Theorem 4.3(A) we see that $\widehat{v}(g_n) = \widehat{v}(g_n - g_j + g_j) = \min(\widehat{v}(g_n - g_j), \widehat{v}(g_j)) = \min(q_j, \widehat{q}_j)$. If on the other hand $q_j = \widehat{q}_j$, so that $\theta_j \neq \widehat{\theta}_j$, then $g_{j+1} - \widehat{g}_{j+1}$ has the form $(\theta_j - \widehat{\theta}_j)ag^\sigma$ where $\widehat{v}$ and v' both assign to ag^σ the value q_j. Also $\widehat{q}_{j+1} > \widehat{e}_j\widehat{q}_j = e_j q_j = q_j$, so that $\widehat{v}(g_{j+1}) = \min(\widehat{v}(\widehat{g}_{j+1}), \widehat{v}(g_{j+1} - \widehat{g}_{j+1})) = \min(\widehat{q}_{j+1}, q_j) = q_j$, and $\widehat{v}(g_n) = \min(\widehat{v}(g_n - g_{j+1}), \widehat{v}(g_{j+1})) = \min(q_{j+1}, q_j) = q_j$. □

5. The homeomorphism $\Phi : \mathcal{M}(F(x)) \longrightarrow \mathcal{S}$

We continue to assume in this section that (F, σ) is ultracomplete and that vF is nontrivial. We now put a topology on $\mathcal{S}$ which will make the bijection Φ of Section 4 a homeomorphism. In outline, we present a subbasis for $\mathcal{S}$, and show in Lemma 5.3 that it makes Φ continuous and in Lemma 5.4 that it generates a Hausdorff topology on $\mathcal{S}$. Since $\mathcal{M}(F(x))$ is compact we conclude the following.

5.1. **Theorem.** *Φ is a homeomorphism and $\mathcal{S}$ is compact.*

We now give the subbasis for $\mathcal{S}$ and introduce some notation that will be used throughout the remainder of the paper; to some extent we are just formalizing the use of notation that appeared in earlier sections.

5.2. *Notation.* If we denote a signature by $\widehat{S}$ then we will write $\widehat{S} = \langle \widehat{q}_i, \widehat{\theta}_i \rangle_{i<\widehat{n}}$; set $\widehat{\Gamma}_m = \mathbb{Z} + \sum_{i<m} \mathbb{Z}\widehat{q}_i$ for all $m \leq \widehat{n}$, and set $\widehat{e}_i = (\widehat{\Gamma}_{i+1} : \widehat{\Gamma}_i)$ for all $i < \widehat{n}$. We will also let $\widehat{g} = \langle \widehat{g}_i \rangle_{i<\widehat{n}}$ (or $\widehat{g} = \langle \widehat{g}_i \rangle_{i \leq \widehat{n}+1}$ if $\widehat{S}$ has finite degree) denote the associated sequence of polynomials of $\widehat{S}$ and write $\widehat{g}_{m+1} = \widehat{g}_m^{\widehat{e}_m} + \widehat{\theta}_m \widehat{a}_m \widehat{g}^{\widehat{\sigma}_m}$ (see Section 4 and equation (4.1); here $\widehat{a}_m$ denotes $t^{\widehat{\alpha}_m}$).

We will use similar notation if the circumflex above is replaced by another symbol (or omitted completely). For example if S^* is a signature, we write $S^* = \langle q_i^*, \theta_i^* \rangle_{i<n^*}$, and if S is a signature we continue to use the notation introduced in Section 4, so $S = \langle q_i, \theta_i \rangle_{i<n}$.

Suppose that S is a signature of finite degree. Pick $0 < \delta \in \mathbb{R}$, $0 < \delta' \in \mathbb{Q}vF$, $\beta \in \mathbb{Q}vF$, and $\alpha \in \mathbb{Q}vF \cup \{-\infty\}$ such that $\beta \geq \alpha \geq \mathrm{cut}(S)$. If n is a limit ordinal, we also let $i = i_S$ be the least ordinal with $\deg S = \deg S_i$.

We give $\mathcal{S}$ the coarsest topology such that for all such $S, \delta, \delta', \alpha, \beta$ and i as above the following four sets are open:

$$\mathcal{P}_{S,\delta,\delta'} = \{\widehat{S} \in \mathcal{S} : \widehat{n} > n; \widehat{S} \rhd S_{n-1}; q_{n-1} = \widehat{q}_{n-1};$$
$$| \theta_{n-1} - \widehat{\theta}_{n-1} |< \delta; \text{ if } \theta_{n-1} = \widehat{\theta}_{n-1}, \text{ then } \widehat{q}_n \leq \delta' + e_{n-1}q_{n-1}\}$$

(defined only if n is not a limit ordinal and $n > 0$; $n-1$ denotes here the predecessor of n);

$$\mathcal{D}_{S,\beta,\delta} = \{\widehat{S} \in \mathcal{S} : \widehat{S} \rhd S, \beta \geq \widehat{q}_n, \text{ and if } \beta = \widehat{q}_n \text{ then } |\widehat{\theta}_n| > \delta\}$$
$$\cup \{\widehat{S} \in \mathcal{S} : \widehat{S} \not\rhd S, \widehat{S} \rhd S_i, \text{ and } \widehat{q}_i \geq q_i\}$$

(defined only if n is a limit ordinal);

$$\mathcal{N}_{S,\beta,\delta} = \{\widehat{S} \in \mathcal{S} : \widehat{S} \rhd S; \widehat{q}_n \geq \beta; \text{ and } \widehat{q}_n = \beta \text{ only if } | \widehat{\theta}_n |< \delta\},$$

and

$$\mathcal{N}_{S,\alpha,\beta,\delta} = \{\widehat{S} \in \mathcal{S} : \widehat{S} \rhd S; \beta \geq \widehat{q}_n \geq \alpha; \text{ and } \widehat{q}_n = \beta \text{ only if } | \widehat{\theta}_n |> \delta\}.$$

We now prove Theorem 5.1 by proving the two lemmas promised above.

5.3. **Lemma.** *The map $\Phi : \mathcal{M}(F(x)) \to \mathcal{S}$ is continuous.*

Proof. Suppose that $\gamma \in vF$, $0 < b \in \mathbb{Z}$, and $f \in F[x]$. Then for any $\tau \in \mathcal{M}(F(x))$, we have $\tau(1 + (f^b/t^\gamma)^2) \in (0, \infty)$ if and only if $v_\tau(f) \geq \gamma/b$ and similarly that $\tau(1 + (t^\gamma/f^b)^2) \in (0, \infty)$ if and only if $v_\tau(f) \leq \gamma/b$. Thus we will sometimes let $H(v(f) \geq \gamma/b)$ denote the Harrison set $H(1+(f^b/t^\gamma)^2))$ and similarly for $H(v(f) \leq \gamma/b)$, and also set $H(v(f) = \gamma/b) = H(v(f) \geq \gamma/b) \cap H(v(f) \leq \gamma/b)$. Finally, we will also let $H(v(f) \geq -\infty)$ denote the set of $\mathbb{R}$-places τ with $v_\tau(f) \geq -\infty$; it will be either all of $\mathcal{M}(F(x))$, or the complement of the singleton set consisting of the $\mathbb{R}$-place whose valuation maps x to $-\infty$ (cf., the first paragraph of Section 3).

We will show that each of our subbasic open sets is the image of an open subset of $\mathcal{M}(F(x))$. We continue to use the notation of Notation 5.2. With β and S as in Notation 5.2, we set $e = (\Gamma_S + \mathbb{Z}\beta : \Gamma_S)$ and choose $\mu \in J_n$ and $a \in A$ such that $e\beta = v(a) + \sum_{j<n} \mu(j)q_j$. In each case below we will assume that $\tau \in \mathcal{M}(F(x))$ and set $\widehat{S} = \Phi(\tau)$.

We begin by showing that $\Phi(H) = \mathcal{D}_{S,\beta,\delta}$ where

$$H := H(v(g_i) \geq q_i) \cap H(\delta^{-2} - (g_n^e/ag^\mu)^{-2}).$$

Suppose that $\tau \in H$. By Theorem 4.3(A) if $\widehat{S} \rhd S$, then $v_\tau(g_j) = q_j$ whenever $j < n$, so $v_\tau(ag^\mu) = e\beta$. Also $v_\tau(g_n^e/ag^\mu) \leq 0$, so $v_\tau(g_n) \leq \beta$. If the last inequality is strict, then $\widehat{S} \in \mathcal{D}_{S,\beta,\delta}$, so suppose that

$v_\tau(g_n) = \beta$. Then $\widehat{g}_{n+1} = g_n^e + \widehat{\theta}_n a g^\mu$, so $|\widehat{\theta}_n| = |\tau(g_n^e/ag^\mu)| > \delta$, so again $\widehat{S} \in \mathcal{D}_{S,\beta,\delta}$. Next suppose that $\widehat{S} \not\triangleright S$. By Theorem 4.3(C) and the last sentence of 4.2, since by hypothesis $v_\tau(g_i) \geq q_i$, therefore $\widehat{S} \triangleright S_i$. Thus $g_i = \widehat{g}_i$, so again $\widehat{S} \in \mathcal{D}_{S,\beta,\delta}$. Conversely, suppose now that $\widehat{S} \in \mathcal{D}_{S,\beta,\delta}$. If $\widehat{S} \triangleright S$, then $v_\tau(g_i) = q_i$ and $v_\tau(g_n) = v_\tau(\widehat{g}_n) = \widehat{q}_n \leq \beta$ and if $\beta = \widehat{q}_n$, then

$$|\tau(g_n^e/ag^\mu)| = |\tau((\widehat{g}_{n+1} - \widehat{\theta}_n a g^\mu)/ag^\mu)| = |\widehat{\theta}_n| > \delta,$$

so $\tau \in H$. On the other hand if $\widehat{S} \not\triangleright S$, we still have $\widehat{S} \triangleright S_i$, so $v_\tau(g_i) = \widehat{q}_i \geq q_i$. Also by Lemma 4.5 for some $j < n$ we have $v_\tau(g_n) = \min(q_j, \widehat{q}_j) \leq q_j < \mathrm{cut}(S) < \beta$, so again we have $\tau \in H$.

We will now show that that $\Phi(H) = \mathcal{N}_{S,\beta,\delta}$ where

$$H = H(v(g_n) \geq \beta) \cap H(\delta^2 - (g_n^e/(ag^\mu)^2)).$$

We must show that $\tau \in H$ if and only if $\Phi(\tau) \in \mathcal{N}_{S,\beta,\delta}$. If $\tau \in H$, then $v_\tau(g_n) \geq \beta > \mathrm{cut}(S)$, so by Theorem 4.3(C) we have $\widehat{S} \triangleright S$ and therefore $\widehat{g}_i = g_i$ for all $i \leq n$. (In the application of the theorem let the signature of length $n+1$ with initial segment S and last term $(\mathbb{Q}vF, 0)$ play the role of S and n play the role of k .) Hence $\widehat{q}_n = v_\tau(\widehat{g}_n) = v_\tau(g_n) \geq \beta$. If $v_\tau(\widehat{g}_n) = v_\tau(g_n) = \beta$, then $\widehat{e}_n = e$ and $\widehat{g}_{n+1} = g_n^e + \widehat{\theta}_n a g^\mu$, so

$$|\widehat{\theta}_n| = \left| \tau \left(\frac{\widehat{g}_{n+1}}{ag^\mu} - \frac{g_n^e}{ag^\mu} \right) \right| = |\tau(g_n^e/(ag^\mu))| < \delta$$

since $\tau \in H$. Now suppose that $\Phi(\tau) \in \mathcal{N}_{S,\beta,\delta}$. Then $\widehat{S} \triangleright S$ and hence $\widehat{g}_n = g_n$. Therefore $\beta \leq \widehat{q}_n = v_\tau(\widehat{g}_n) = v_\tau(g_n)$. If $v_\tau(g_n) > \beta$, then $v_\tau(g_n^e/(ag^\mu)) > 0$ so $\tau(\delta^2 - (g_n^e/(ag^\mu)^2)) \in (0, \infty)$. But if $v_\tau(g_n) = \beta$, then $\widehat{e}_n = e$ and $\widehat{g}_{n+1} = g_n^e + \widehat{\theta}_n a g^\mu$, so $\delta > |\widehat{\theta}_n| = |\tau(g_n^e/(ag^\mu))|$ and hence in either case $\tau(\delta^2 - (g_n^e/(ag^\mu))^2) \in (0, \infty)$. Thus $\tau \in H$.

We next show that $\Phi(H) = \mathcal{N}_{S,\alpha,\beta,\delta}$ where now we set

$$H = H(\beta \geq v(g_n)) \cap H(v(g_n) \geq \alpha) \cap H(\delta^{-2} - (g_n^e/(ag^\mu))^{-2}).$$

First suppose that $\tau \in H$. Arguing as in the previous paragraph, we conclude that $\widehat{S} \triangleright S$, so that $g_j = \widehat{g}_j$ for all $j \leq n$. Thus $\widehat{q}_n = v_\tau(\widehat{g}_n) = v_\tau(g_n)$ is between α and β. If $\widehat{q}_n = \beta$, then $e = \widehat{e}_n$ and $\widehat{g}_{n+1} = g_n^e + \widehat{\theta}_n a g^\mu$, so $|\widehat{\theta}_n| = |\tau(g_n^e/(ag^\mu))| > \delta$. Hence $\widehat{S} \in \mathcal{N}_{S,\alpha,\beta,\delta}$. Now suppose that $\widehat{S} \in \mathcal{N}_{S,\alpha,\beta,\delta}$. Arguing as above we deduce that $g_j = \widehat{g}_j$ for all $j \leq n$ and $\beta \geq v_\tau(g_n) \geq \alpha$. If $v_\tau(g_n) < \beta$, then $v_\tau(ag^\mu/g_n^e) > 0$, so $|(\tau(g_n^e/(ag^\mu)))^{-1}| = 0 < \delta^{-1}$. If $v_\tau(g_n) = \beta$, then $\widehat{g}_{n+1} = g_n^e + \widehat{\theta}_n a g^\mu$, so $\delta < |\widehat{\theta}_n| = |\tau(g_n^e/(ag^\mu))|$. Hence $\tau \in H$.

Finally we suppose that $n > 0$ and that n is not a limit ordinal and show that $\Phi(H) = \mathcal{P}_{S,\delta,\delta'}$ where now (with the obvious notation) H denotes

$$H(v(g_{n-1}) = q_{n-1},\ v(g_n) \le \delta' + e_{n-1}q_{n-1})\ \cap\ H(\delta^2 - (g_n/(bg^\rho))^2)$$

and where we now pick $b \in A$ and $\rho \in J_{n-1}$ with $e_{n-1}q_{n-1} = v(b) + \sum_{j<n-1} \rho(j)q_j$. Suppose that $\widehat{S} = \Phi(\tau) \in \mathcal{P}_{S,\delta,\delta'}$. Since $\widehat{S} \rhd S_{n-1}$ we have $g_j = \widehat{g}_j$ for all $j < n$ and so $v_\tau(g_j) = v_\tau(\widehat{g}_j) = \widehat{q}_j = q_j$ for all $j < n$. Hence

$$g_n = g_{n-1}^{e_{n-1}} + \theta_{n-1}bg^\rho \text{ and } \widehat{g}_n = g_{n-1}^{e_{n-1}} + \widehat{\theta}_{n-1}bg^\rho.$$

Thus

$$v_\tau(g_n) - e_{n-1}q_{n-1} = v_\tau\left(\frac{g_{n-1}^{e_{n-1}}}{bg^\rho} + \theta_{n-1}\right) = v_\tau\left(\frac{\widehat{g}_n}{bg^\rho} - \widehat{\theta}_{n-1} + \theta_{n-1}\right)$$

which equals 0 if $\theta_{n-1} \neq \widehat{\theta}_{n-1}$ since $\widehat{q}_n > e_{n-1}q_{n-1}$, and which is at most δ' if $\theta_{n-1} = \widehat{\theta}_{n-1}$. Moreover

$$|\tau(g_n/(bg^\rho))| = \left|\tau\left(\frac{\widehat{g}_n}{bg^\rho} - \widehat{\theta}_{n-1} + \theta_{n-1}\right)\right| = |-\widehat{\theta}_{n-1} + \theta_{n-1}| < \delta,$$

so $\tau \in H$. Conversely, suppose that $\tau \in H$. Then $v_\tau(g_{n-1}) = q_{n-1} > \mathrm{cut}(S_{n-1})$, so by Theorem 4.3(C), $\widehat{S} \rhd S_{n-1}$ and hence $\widehat{g}_j = g_j$ for all $j \le n-1$. Hence $q_{n-1} = v_\tau(g_{n-1}) = v_\tau(\widehat{g}_{n-1}) = \widehat{q}_{n-1}$. Also

$$|\theta_{n-1} - \widehat{\theta}_{n-1}| = |\tau((g_n - \widehat{g}_n)/(bg^\rho))| = |\tau(g_n/(bg^\rho))| < \delta.$$

Finally, suppose that $\theta_{n-1} = \widehat{\theta}_{n-1}$, so $g_n = \widehat{g}_n$. Then

$$\widehat{q}_n - e_{n-1}q_{n-1} = v_\tau(g_n) - e_{n-1}q_{n-1} \le \delta'.$$

Thus $\Phi(\tau) = \widehat{S} \in \mathcal{P}_{S,\delta,\delta'}$. □

5.4. **Lemma.** *$\mathcal{S}$ is Hausdorff.*

Proof. Suppose that S^* and $\widehat{S}$ are two distinct elements of $\mathcal{S}$. Then there is a least n with $(q_n^*, \theta_n^*) \neq (\widehat{q}_n, \widehat{\theta}_n)$. We will find disjoint open sets $\mathcal{N}^*$ and $\widehat{\mathcal{N}}$ containing S^* and $\widehat{S}$, respectively. Set $S := S_n^* = \widehat{S}_n$.

First suppose that $\widehat{q}_n = q_n^*$, so that $\widehat{\theta}_n \neq \theta_n^*$. Let δ denote a positive rational less than $|\widehat{\theta}_n - \theta_n^*|$ and let δ' denote a positive element of $\mathbb{Q}vF$. At least one of θ_n^* and $\widehat{\theta}_n$ is nonzero, so $\widehat{q}_n = q_n^*$ is in $\mathbb{Q}vF$, so both θ_n^* and $\widehat{\theta}_n$ are nonzero and n^* and $\widehat{n}$ are both larger than $n+1$. Then $\widehat{S}_{n+1} = \langle \widehat{q}_i, \widehat{\theta}_i \rangle_{i\le n}$ and $S_{n+1}^* = \langle q_i^*, \theta_i^* \rangle_{i \le n}$ are distinct signatures of finite degree. If there exists $\widehat{q} \in \mathbb{Q}vF$ with $\widehat{q}_{n+1} > \widehat{q} > \widehat{e}_n\widehat{q}_n$, then set $\widehat{\mathcal{N}} = \mathcal{N}_{\widehat{S}_{n+1},\widehat{q},\delta}$; otherwise we have $\widehat{q}_{n+1} = (\widehat{e}_n\widehat{q}_n)^+$ and we set $\widehat{\mathcal{N}} = \mathcal{P}_{\widehat{S}_{n+1},\delta/2,\delta'}$. Similarly, if there exists $q^* \in \mathbb{Q}vF$ with $q_{n+1}^* > q^* > e_n^*q_n^*$, then set $\mathcal{N}^* = \mathcal{N}_{S^*_{n+1},q^*,\delta}$ and otherwise set $\mathcal{N}^* = \mathcal{P}_{S^*_{n+1},\delta/2,\delta'}$. Then clearly $\widehat{S} \in \widehat{\mathcal{N}}$ and $S^* \in \mathcal{N}^*$. Suppose $S' \in \widehat{\mathcal{N}} \cap \mathcal{N}^*$. If $\widehat{q}$ and q^* both exist then $S' \rhd S^*_{n+1}$ and $S' \rhd \widehat{S}_{n+1}$, so $\widehat{\theta}_n = \theta'_n = \theta^*_n$, a contradiction. If neither exists then $\delta < |\theta^*_n - \widehat{\theta}_n| \leq |\theta^*_n - \theta'_n| + |\theta'_n - \widehat{\theta}_n| \leq \delta$, a contradiction. Thus without loss of generality $\widehat{q}$ exists and q^* does not, so that $\theta'_n = \widehat{\theta}_n$ and $\delta/2 > |\theta'_n - \theta^*_n| = |\widehat{\theta}_n - \theta^*_n| > \delta$, a contradiction. Therefore $\mathcal{N}^*$ and $\widehat{\mathcal{N}}$ are indeed disjoint.

Hence without loss of generality we may assume that $\widehat{q}_n < q^*_n$. Suppose that there exist α and β in $\mathbb{Q}vF$ with $\widehat{q}_n < \alpha < \beta < q^*_n$. Now suppose further that there exists $\delta \in \mathbb{Q}vF$ with $\widehat{q}_n \geq \delta \geq \mathrm{cut}(S)$. Then we set $\widehat{\mathcal{N}} = \mathcal{N}_{S,\delta,\alpha,1}$ so $\widehat{S} \in \widehat{\mathcal{N}}$ and set $\mathcal{N}^* = \mathcal{N}_{S,\beta,1}$, so $S^* \in \mathcal{N}^*$. Just suppose that $S' \in \mathcal{N}^* \cap \widehat{\mathcal{N}}$. Then $\beta \leq q'_n \leq \alpha < \beta$, a contradiction. Hence without loss of generality no such δ exists. Thus either (1) $n = 0$ and $\widehat{q}_0$ is the empty cut, or (2) n is a successor ordinal and $\widehat{q}_n = (e_{n-1}q_{n-1})^+$, or (3) n is a limit ordinal and $\widehat{q}_n = \mathrm{cut}(S)$.

In the first case (1) above, $\widehat{S} = \langle(\emptyset, 0)\rangle \in \widehat{\mathcal{N}} := \widehat{\mathcal{N}}_{\emptyset,-\infty,\alpha,1}$ and $S^* \in \mathcal{N}^* := \widehat{\mathcal{N}}_{\emptyset,\beta,1}$ and if $S' \in \widehat{\mathcal{N}} \cap \mathcal{N}^*$, then $\alpha \geq q'_0 \geq \beta$, a contradiction, so $\widehat{\mathcal{N}}$ and $\mathcal{N}^*$ are indeed disjoint.

In the second case (2) above $S^* \in \mathcal{N}^* := \mathcal{N}_{S,\beta,1}$ and $\widehat{S} \in \widehat{\mathcal{N}} := \mathcal{P}_{S,1,\beta-\alpha}$ (because $\widehat{q}_n = (e_{n-1}q_{n-1})^+ \leq \beta - \alpha + e_{n-1}q_{n-1}$). Further, if $S' \in \widehat{\mathcal{N}} \cap \widehat{\mathcal{N}}^*$, then

$$\beta \leq q'_n \leq (\beta - \alpha) + e_{n-1}q_{n-1} < \beta,$$

a contradiction.

Now consider the third case (3). Then $S^* \in \mathcal{N}_{S,\beta,1}$ and $\widehat{S} \in \mathcal{D}_{S,\alpha,1}$. If $S' \in \mathcal{D}_{S,\alpha,1} \cap \mathcal{N}_{S,\beta,1}$, then $\beta > \alpha \geq v'(g_n) = q'_n \geq \beta$, a contradiction. (Here we have let v' denote the valuation associated with S'.)

We may now assume there are no such α and β. But there does exist some $\beta \in \mathbb{Q}vF$ with $\widehat{q}_n \leq \beta \leq q^*_n$. Thus one of the following cases must occur:

(4) $q^*_n = \beta^+$ and $\widehat{q}_n = \beta$, or
(5) $q^*_n = \beta$ and $\widehat{q}_n = \beta^-$, or
(6) $q^*_n = \beta^+$ and $\widehat{q}_n = \beta^-$.

In the first of these cases (labeled (4)) since $\widehat{q}_n \in \mathbb{Q}vF$ therefore $\widehat{\theta}_n \neq 0$. Then $S^* \in \mathcal{N}^* := \mathcal{N}_{S,\beta,|\widehat{\theta}_n|/2}$. Also $\widehat{S} \in \widehat{\mathcal{N}} := \mathcal{N}_{S,\beta,\beta,|\widehat{\theta}_n|/2}$. But if $S' \in \mathcal{N}^* \cap \widehat{\mathcal{N}}$, then $q'_n = \beta$ so $|\widehat{\theta}_n|/2 < |\theta'_n| < |\widehat{\theta}_n|/2$, a contradiction.

Now consider the case labeled (5). Note that $\theta^*_n \neq 0$ since $q^*_n = \beta \in \mathbb{Q}vF$. First suppose that n is not a limit ordinal. Then there exists $\alpha \in \mathbb{Q}vF$ such that $\widehat{q}_n = \beta^- > \alpha > \mathrm{cut}(S)$. It then suffices to take $\widehat{\mathcal{N}} = \mathcal{N}_{S,\alpha,\beta,2|\theta^*_n|}$ and $\mathcal{N}^* = \mathcal{N}_{S,\beta,2|\theta^*_n|}$. Next suppose that n is a limit ordinal. Again set $\mathcal{N}^* = \mathcal{N}_{S,\beta,2|\theta_n^*|}$ and now set $\widehat{\mathcal{N}} = \mathcal{D}_{S,\beta,2|\theta_n^*|}$. Clearly, $\widehat{S} \in \widehat{\mathcal{N}}$ and $S^* \in \mathcal{N}^*$; suppose $S' \in \widehat{\mathcal{N}} \cap \mathcal{N}^*$. Then $q_n' = \beta$ and hence $|2\theta_n^*| < |\theta_n'| < |2\theta_n^*|$, a contradiction.

Finally we consider the last case (6). First note that $S^* \in \mathcal{N}^* := \mathcal{N}_{S,\beta,1}$. If n is not a limit ordinal then again there exists $\alpha \in \mathbb{Q}vF$ with $\widehat{q}_n = \beta^- > \alpha \geq \operatorname{cut}(S)$ and we can take $\widehat{\mathcal{N}} = \mathcal{N}_{S,\alpha,\beta,1}$. If n is a limit ordinal we can take $\widehat{\mathcal{N}} := \mathcal{D}_{S,\beta,1}$. In either case $\widehat{S} \in \widehat{\mathcal{N}}$ and if $S' \in \mathcal{N}^* \cap \widehat{\mathcal{N}}$, then $q_n' = \beta$, and hence $1 > |\theta_n'| > 1$, a contradiction. $\square$

6. Path-connectedness

In this section we continue with the hypothesis that (F, σ) is ultracomplete, and assume in addition that it has countable value group. Our object is to complete the proof of the Main Theorem 1.1 by proving Proposition 2.5.

Suppose that S is a signature of finite degree. We give $\Gamma \times \mathbb{R}$ the lexicographic order and consider its ordered subset

$$A_S = \{(q, \theta) \in \Gamma \times \mathbb{R} : q \geq \operatorname{cut}(S); \theta \leq 0; \text{ and } \theta = 0 \Leftrightarrow q \notin \mathbb{Q}vF\}.$$

For any $(q, \theta) \in \Gamma \times \mathbb{R}$ we set $S(q, \theta) = S \oplus (q, \theta)$ if $q \notin \mathbb{Q}vF$ and $S(q, \theta) = S \oplus (q, \theta) \oplus (((\Gamma_S + \mathbb{Z}q : \Gamma_S)q)^+, 0)$ otherwise. (For sequences A and B we let $A \oplus B$ denote the concatenation of A and B; if $B = \{b\}$ is a singleton, we will often write $A \oplus b$ for $A \oplus B$.) Note that if $(q, \theta) \in A_S$ or $(q, -\theta) \in A_S$, then $S(q, \theta) \in \mathcal{S}$. We will show that A_S is order isomorphic to the interval $[0, 1]$ and that the function $(q, \theta) \mapsto S(q, \theta)$ maps A_S (given the order topology) continuously into $\mathcal{S}$.

We first show that A_S is order isomorphic and therefore homeomorphic (with respect to the order topology) to a closed interval. We note that A_S has smallest element $(\operatorname{cut}(S), 0)$ and largest element $(\mathbb{Q}vF, 0)$, so by [15] we only need to show that A_S has the least upper bound property and that it has a countable order dense subset, i.e., a countable subset such that between any two distinct elements of A_S we can find an element of the subset.

6.1. **Lemma.** *$T := \{(q, \theta) \in A_S : q \in \mathbb{Q}vF, \theta \in \mathbb{Q}\}$ is a countable order dense subset of A_S.*

Proof. Since vF is countable, so is T. Suppose $(q_i, \theta_i) \in A_S$ $(i = 1, 2)$ with $(q_1, \theta_1) < (q_2, \theta_2)$.

Case 1: $q_1 = q_2 = q \in \mathbb{Q}vF$. Then $\theta_1 < \theta_2$; choosing $\theta \in \mathbb{Q}$ between them, we have $(q_1, \theta_1) < (q, \theta) < (q_2, \theta_2)$.

Case 2: Neither q_1 nor q_2 is in $\mathbb{Q}vF$. Then we can find $q \in \mathbb{Q}vF$ between them, and we have $(q_1, \theta_1) < (q, -1) < (q_2, \theta_2)$.

Case 3: $q_1 \in \mathbb{Q}vF$ and $q_2 \notin \mathbb{Q}vF$. Choose $\theta \in (\theta_1, 0)$ so $(q_1, \theta_1) < (q_1, \theta) < (q_2, \theta_2)$.

Case 4: $q_1 \notin \mathbb{Q}vF$ and $q_2 \in \mathbb{Q}vF$. Similar. □

6.2. **Lemma.** *A_S has the least upper bound property.*

Proof. Suppose that $B \subseteq A_S$. We may assume without loss of generality that B is nonempty (since A_S has a minimum) and that it is not finite. Let $\alpha = \{q \in \mathbb{Q}vF : \exists(q', \theta') \in B; q \leq q'\}$. Then α is a cut in $\mathbb{Q}vF$ and $\alpha \geq \text{cut}(S)$ since $B \neq \emptyset$. We note that $(\alpha, 0)$ is necessarily an upper bound for B. (For if not, we'd have some $(q', \theta') \in B$ with $q' > \alpha$, but then we could find $q \in \mathbb{Q}vF$ with $\alpha < q \leq q'$, which implies $q \in \alpha$, a contradiction.)

Case 1: α is not of the form q^+, $q \in \mathbb{Q}vF$. Then we claim that $(\alpha, 0) = \sup(B)$. This holds because if we take any $(q_1, \theta_1) < (\alpha, 0)$ then, as $\alpha \neq q_1^+$, there is some $q \in \mathbb{Q}vF$ with $q_1 < q < \alpha$, but then $q \in \alpha$ and so (q_1, θ_1) is not an upper bound for B. (And, as observed above, $(\alpha, 0)$ is an upper bound for B.)

Case 2: $\alpha = q^+$ for some $q \in \mathbb{Q}vF$. Then either $(q^+, 0) = \sup B$ or $\theta^* := \sup\{\theta : (q, \theta) \in B\} < 0$. In the latter case, clearly $(q, \theta^*) = \sup B$. □

6.3. **Corollary.** *A_S is order isomorphic to* $[0, 1]$.

The Corollary follows from the previous two lemmas and [15, Theorem 2.30] (either apply the theorem to the interior of A_S or as suggested in the paragraph just before [15, Exercise 2.37, page 40], apply the proof of Theorem 2.30 of [15]).

We set $\mathcal{N}_S = \{S' \in \mathcal{S} : S' \rhd S\}$ and define the function

$$E = E_S : A_S \longrightarrow \mathcal{N}_S$$

by setting $E(q, \theta) = S(q, \theta)$.

6.4. **Theorem.** *Let S be a signature of finite degree and length n. Then $E : A_S \to \mathcal{N}_S$ is continuous.*

Proof. It suffices to show that if $(q, \theta) \in A_S$ and $\widehat{S} := E(q, \theta) \in \mathcal{N}$ where $\mathcal{N}$ has one of the forms $\mathcal{D}_{M,\beta,\delta}$, $\mathcal{P}_{M,\delta,\delta'}, \mathcal{N}_{M,\beta,\delta}$, or $\mathcal{N}_{M,\alpha,\beta,\delta}$, then there exists an open interval I of A_S containing (q, θ) such that $E(I) \subset \mathcal{N}$. For M as above let us write $M = \langle q_i^M, \theta_i^M \rangle_{i<m}$, so of course m is the length of M.

We begin by assuming that $\mathcal{N} = \mathcal{D}_{M,\beta,\delta}$, so m is a limit ordinal and we can let $i = i_M$ (cf. 5.2). Let $(q', \theta') \in A_S$ and let τ' and v' denote the $\mathbb{R}$-place and valuation associated with $S' := E(q', \theta')$.

We first suppose that $\widehat{S} \triangleright M$ and $m = n$. Then $M = S$, $\beta \geq \widehat{q}_n = q$ and if $\beta = q$, then $|\widehat{\theta}_n| > \delta$, so $\theta = \widehat{\theta}_n < -\delta$. Thus

$$(q, \theta) \in I := [(\mathrm{cut}(S), 0), (\beta, -\delta)) \subset A_S.$$

If $(q', \theta') \in I$, then $S' \triangleright S$ and $q'_m = q' \leq \beta$ with equality holding only if $\theta'_m = \theta' < -\delta$, so $|\theta'_m| > \delta$. Hence $S' \in \mathcal{N}$. Thus, $E(I) \subset \mathcal{N}$.

Next suppose that $\widehat{S} \triangleright M$ and $n \neq m$. Since $\widehat{S} \triangleright M$ and m is a limit ordinal, therefore $S \triangleright M$, so $n > m$. We have $\beta \geq \widehat{q}_m = q_m = q'_m$ and if $\beta = q'_m$ then $|\theta'_m| = |\widehat{\theta}_m| > \delta$. Thus $S' \in \mathcal{N}$, so $E(A_S) \subset \mathcal{N}$.

Finally consider the case that $\widehat{S} \not\triangleright M$. Since $\widehat{S} \triangleright M_i$, therefore $S \oplus (q, \theta) \triangleright M_i$. If $n + 1 = i$, then $S = M_n$ and (using the notation of 4.1 and 5.2)

$$((\Gamma_S + \mathbb{Z}q : \Gamma_S)q)^+ = (e_n^M q_n^M)^+ \leq q_i^M \leq \widehat{q}_i = ((\Gamma_S + \mathbb{Z}q : \Gamma_S)q)^+,$$

which contradicts the fact that q_i^M is in $\mathbb{Q}vF$. Thus $i \leq n$. Hence $S \triangleright M_i$ and hence $S' \triangleright M_i$. And further, $S' \not\triangleright M$ since otherwise $\widehat{S} \triangleright S \triangleright M$ since m is a limit ordinal. If $i < n$, then $q'_i = q_i = \widehat{q}_i \geq q_i^M$, so $S' \in \mathcal{N}$. Thus if $i < n$, then we can take $I = A_S$. Hence suppose $i = n$. Then

$$(q, \theta) \in I := \big((q_n^M, \eta), (\mathbb{Q}vF, 0)\big] \subset A_S$$

where we choose $\eta < \theta$. Then if $(q', \theta') \in I$, we have $q'_i = q'_n = q' \geq q_n^M = q_i^M$, so $S' \in \mathcal{N}$. Hence indeed we have $E(I) \subset \mathcal{N}$.

The proof of the theorem in the remaining cases where $\mathcal{N}$ has one of the forms $\mathcal{P}_{M,\delta,\delta'}, \mathcal{N}_{M,\beta,\delta}$, or $\mathcal{N}_{M,\alpha,\beta,\delta}$ is a routine generalization of the proofs of Lemmas 5.5 and 5.6 of [5] (the subasic open set $\mathcal{N}_{M,\delta}$ of that paper corresponds to our subbasic open set $\mathcal{P}_{M,\delta,\delta}$). We will give the proof here only in the case that $\mathcal{N} = \mathcal{P}_{M,\delta,\delta'}$ where the notation in this paper differs the most from that in [5].

We first suppose that $n > m$. Consider any $S^* \in \mathcal{N}_S$. Then $q^*_{m-1} = q_{m-1} = \widehat{q}_{m-1} = q^M_{m-1}$ and similarly, $\theta^*_{m-1} = \theta_{m-1} = \widehat{\theta}_{m-1}$, so $|\theta^*_{m-1} - \theta^M_{m-1}| = |\widehat{\theta}_{m-1} - \theta^M_{m-1}| < \delta$, and if $\theta^*_{m-1} = \theta^M_{m-1}$, then $\widehat{\theta}^* = \theta^M_{m-1}$, so $q^*_m = q_m = \widehat{q}_m \leq \delta' + e^M_{m-1} q^M_{m-1}$. Also $S^* \triangleright S \triangleright M_{m-1}$. Hence $S^* \in \mathcal{N}$. Thus $E(A_S) \subset \mathcal{N}_S \subset \mathcal{N}$, so we may take $I = A_S$.

Therefore we may now assume that $n \leq m$. But $n+2 \geq$ length of $\widehat{S} > m \geq n$. Hence $m = n$ or $m = n + 1$.

First consider the case that $m = n + 1$. Since $\widehat{S} \in \mathcal{N}$, therefore $M_n = S$ and

$$|\theta - \theta_n^M| = |\widehat{\theta}_n - \theta_n^M| < \delta$$

and

$$q_n^M = \widehat{q}_n = q > \mathrm{cut}(S),$$

so

$$(q,\theta) \in I := \big((q_n^M, \theta_n^M - \delta), (q_n^M, \min(\theta_n^M + \delta, \theta/2)\big) \subset A_S$$

(keep in mind that $\theta < 0$). Now if $(q',\theta') \in I$ and $S' = E(q',\theta') = S(q',\theta')$, then $q'_n = q' = q_n^M$ and $|\theta'_n - \theta_n^M| = |\theta' - \theta_n^M| < \delta$. Also $q'_m = (e'q')^+ = (e_n^M q_n^M)^+ < e_n^M q_n^M + \delta'$ (where $e' = (\Gamma_S + \mathbb{Z}q' : \Gamma_S)$). Thus $S' \in \mathcal{N}$, and hence $E(I) \subset \mathcal{N}$.

Finally, consider the case that $m = n$. Then n is a successor ordinal and $M_{n-1} = S_{n-1}$. Also, $q_{n-1} = \widehat{q}_{n-1} = q_{n-1}^M$ and $\delta > |\widehat{\theta}_{n-1} - \theta_{n-1}^M| = |\theta_{n-1} - \theta_{n-1}^M|$. Let $(q',\theta') \in A_S$ and $S' = E(q',\theta')$. Then $S' \rhd S \rhd M_{n-1}$ and $q'_{n-1} = q_{n-1} = \widehat{q}_{n-1} = q_{n-1}^M$ and similarly $|\theta'_{n-1} - \theta_{n-1}^M| < \delta$. Therefore if $\theta'_{n-1} = \theta_{n-1} \neq \theta_{n-1}^M$, then $S' \in \mathcal{N}$, so $E(A_S) \subset \mathcal{N}$. Hence suppose that $\theta_{n-1} = \theta_{n-1}^M$. By hypothesis $q = \widehat{q}_m \leq \delta' + e_{n-1}^M q_{n-1}^M$, so that

$$(q,\theta) \in I := \big[(\mathrm{cut}(S), 0), (e_{n-1}^M q_{n-1}^M + \delta', \theta/2)\big) \subset A_S.$$

Now suppose that $(q',\theta') \in I$. Then by the choice of I we have $q'_m = q' \leq e_{n-1}^M q_{n-1}^M + \delta'$. Thus $S' \in \mathcal{N}$. And hence $E(I) \subset \mathcal{N}$.

□

We need another class of homeomorphisms. We let Ω denote the first uncountable ordinal. Let $\epsilon = \langle \epsilon_i \rangle_{0 \leq i < \Omega}$ be a sequence with each $\epsilon_i \in \{\pm 1\}$. Define a map $\Psi_\epsilon : \mathcal{S} \to \mathcal{S}$ by the formula

$$\Psi_\epsilon(S') = \langle q'_i, \epsilon_i \theta'_i \rangle_{i < n'} \tag{6.1}$$

for all $S' \in \mathcal{S}$.

6.5. **Lemma.** *Ψ_ϵ is a homeomorphism.*

Proof. Ψ_ϵ is continuous since for all appropriate T, α, β, δ and δ' we have

$$\Psi_\epsilon(\mathcal{P}_{T,\delta,\delta'}) = \mathcal{P}_{\Psi_\epsilon(T),\delta,\delta'},$$
$$\Psi_\epsilon(\mathcal{N}_{T,\beta,\delta}) = \mathcal{N}_{\Psi_\epsilon(T),\beta,\delta},$$
$$\Psi_\epsilon(\mathcal{N}_{T,\alpha,\beta,\delta}) = \mathcal{N}_{\Psi_\epsilon(T),\alpha,\beta,\delta},$$

and

$$\Psi_\epsilon(\mathcal{D}_{T,\beta,\delta}) = \mathcal{D}_{\Psi_\epsilon(T),\beta,\delta}.$$

(In the above formulas we use equation (6.1) to define $\Psi_\epsilon(T)$ even though T is not in $\mathcal{S}$.) Since Ψ_ϵ is its own inverse, therefore Ψ_ϵ is a homeomorphism. □

We now turn more directly toward the proof that $\mathcal{S}$ is path-connected. Our construction of paths will use a map $\rho : \{i : 0 \leq i \leq n\} \longrightarrow [0,1]$ of the type whose existence is asserted in the next lemma.

6.6. **Lemma.** *If $n > 0$ is a countable or finite ordinal number, then there exists a strictly increasing map ρ as above such that $\rho(0) = 0$ and $\rho(n) = 1$ and if $k \leq n$ is a limit ordinal, then $\rho(k)$ is the supremum of $\{\rho(i) : i < k\}$.*

Proof. Let Λ denote the set of ordinal numbers less than or equal to n and let Λ_0 denote the set of elements of Λ which are not limit ordinals (so, for example, $0 \in \Lambda_0$). There is a strictly increasing map $\epsilon_0 : \Lambda \longrightarrow [0,1]$. Define $\epsilon : \Lambda \longrightarrow [0,1]$ by letting ϵ agree with ϵ_0 on Λ_0 and setting $\epsilon(\lambda) = \sup_{\lambda > \delta \in \Lambda_0} \epsilon_0(\delta)$ if $\lambda \in \Lambda$ is a limit ordinal. Then an easy argument by cases shows that ϵ is strictly increasing. It follows that ϵ has the required sup property, and hence the required map ρ can be obtained by composing it with a strictly increasing continuous map $[\epsilon(0), \epsilon(n)] \longrightarrow [0,1]$ taking $\epsilon(0)$ to 0 and $\epsilon(n)$ to 1. □

Recall that the smallest element of $\Gamma \times \mathbb{R}$ is $-\infty = (\emptyset, 0)$. We denote the signature of length one whose only element is $(\emptyset, 0)$ by $S_{-\infty}$.

6.7. **Proposition.** *Let $S \in \mathcal{S}$. Then there is a continuous function $\phi : [0,1] \longrightarrow \mathcal{S}$ such that $\phi(0) = S$ and $\phi(1) = S_{-\infty}$.*

Proof. Let $\epsilon = \langle \epsilon_i \rangle_{0 \leq i < \Omega}$ be a sequence of numbers in $\{\pm 1\}$ (cf. Lemma 6.5) such that $\epsilon_i \theta_i \leq 0$ for all $i < n$. If π_0 were a path from $S_{-\infty}$ to $\Psi_\epsilon(S)$, then $\Psi_\epsilon \circ \pi_0$ would be a path connecting $S_{-\infty}$ to $\Psi_\epsilon \Psi_\epsilon(S) = S$. Thus without loss of generality we may assume that $\theta_i \leq 0$ for all $i < n$.

Pick a map ρ of the type whose existence is asserted in Lemma 6.6. For each $i < n$ let $\mu_i : [\rho(i), \rho(i+1)] \longrightarrow [(\mathrm{cut}(S_i), 0), (q_i, \theta_i)]$ be the constant map if $q_i = \mathrm{cut}(S_i)$ (the codomain is then a singleton) and let it be an order isomorphism otherwise (we can apply Corollary 6.3 here since in this case the codomain is a subinterval of A_{S_i}). Then for each $i < n$ we let $\phi_i = E_{S_i} \circ \mu_i$, so that

$$\phi_i(\rho(i)) = S_i^+ := S_i \oplus (\mathrm{cut}(S_i), 0)$$

and $\phi_i(\rho(i+1)) = S_{i+1}^+$ if $i + 1 < n$ and $\phi_i(\rho(i+1)) = S$ if $i + 1 = n$. We also let ϕ_n be the empty map if n is not a limit ordinal, and let it be the function $\{(1, S)\}$ otherwise. (We are of course identifying each map f with its "graph" $\{(b, f(b)) : b \in \text{domain of } f\}$.) The relation $\phi := \cup_{i \leq n} \phi_i$ is a functional relation with domain $[0,1]$ and range contained in $\mathcal{S}$ such that

$$\phi(0) = \phi_0(0) = S_0 \oplus (\mathrm{cut}(S_0), 0) = S_{-\infty}$$

and $\phi(1) = S$. It remains to prove that ϕ is continuous.

Let λ be any limit ordinal less than or equal to n. By the definition of ϕ it suffices to show that it is left continuous at $\rho(\lambda)$. Let $\mathcal{N}$ be one of our four types of subbasic open sets of $\mathcal{S}$ such that $\widehat{S} := \phi(\rho(\lambda)) \in \mathcal{N}$. For any $k < \lambda$ and $t \in (\rho(k), \rho(\lambda))$ there exists $l \geq k$ with $l < \lambda$ and with $t \in [\rho(l), \rho(l+1))$ (pick l minimal with $\rho(l+1) > t$ and argue that $t \geq \rho(l)$). We can write $\mu_l(t) = (q, \theta) \in A_{S_l}$ where $(q, \theta) < (q_l, \theta_l)$; then $S^* := S_l(q, \theta) = \phi(t)$. It suffices to show that if k is chosen properly, then $\phi(t) = S^* \in \mathcal{N}$ for all $t \in (\rho(k), \rho(\lambda))$.

First consider the case that $\mathcal{N} = \mathcal{D}_{S',\beta,\delta}$.

Subcase 1: $S_\lambda \rhd S'$ and $S_\lambda \neq S'$. Since λ is a limit ordinal, we therefore have $\lambda > n' + 1$. In this case we will will take $k = n' + 1$. For t, l, q, θ and S^* as above, we have $\widehat{S} \rhd S_k \rhd S'$ and $S^* \rhd S_l \rhd S_k \rhd S'$. Thus

$$(q^*_{n'}, \theta^*_{n'}) = (q_{n'}, \theta_{n'}) = (\widehat{q}_{n'}, \widehat{\theta}_{n'}).$$

Since $\widehat{S} \in \mathcal{N}$ and $\widehat{S} \rhd S'$, therefore $\widehat{q}_{n'} \leq \beta$ with equality holding only if $|\widehat{\theta}_{n'}| > \delta$. The last display tells us the same must be true of S^* and hence $S^* \in \mathcal{N}$, as required.

Subcase 2: $S_\lambda = S'$. Thus $\lambda = n'$. In this case we take $k = i' + 1$, where $i' = i_{S'}$ (c.f., the third paragraph of 5.2). For t, l, q, θ and S^* as above, we have $S^* \not\rhd S'$ because otherwise since n' is a limit ordinal we would have $S_l \rhd S' = S_\lambda$, a contradiction. But we do have $S^* \rhd S_{i'+1} = S'_{i'+1}$, so $q^*_{i'} = q'_{i'}$ and hence $S^* \in \mathcal{N}$.

Subcase 3: $S_\lambda \not\rhd S'$. In this case we let $k = i' + 1$. Since n' is a limit ordinal, we must have $\widehat{S} \not\rhd S'$. (Note that if $\lambda = n$, then $\widehat{S} = S_\lambda$.) Thus $\widehat{S} \rhd S'_{i'}$ and $\widehat{q}_{i'} \geq q'_{i'}$. Further, $S^* \rhd S_{i'+1} \rhd S'_{i'}$ and $\widehat{S} \rhd S_{i'+1}$ and therefore $q^*_{i'} = q_{i'} = \widehat{q}_{i'} \geq q'_{i'}$. Therefore $S^* \in \mathcal{N}$.

It remains to consider the case that $\mathcal{N} = \mathcal{P}_{S',\delta,\delta'}, \mathcal{N}_{S',\beta,\delta},$ or $\mathcal{N}_{S',\beta,\alpha,\delta}$. Then $\lambda + 1 \geq \widehat{n} > n'$, so either $\lambda = n'$ or $\lambda > n'$. In the first case we must have $\mathcal{N} \neq \mathcal{P}_{S',\delta,\delta'}$ (since λ is a limit ordinal) and $n > n' = \lambda$ (otherwise, $S \in \mathcal{N}$ so $n > n'$), so

$$\beta \leq \widehat{q}_{n'} = \widehat{q}_\lambda = \mathrm{cut}(S_\lambda) = \mathrm{cut}(S') \leq \beta,$$

contradicting the fact that $\mathrm{cut}(S') \notin \mathbb{Q}vF$. Thus $\lambda > n'$ and hence, since λ is a limit ordinal, $\lambda > n' + 1$. We can then take $k = n' + 1$ and argue essentially as in subcase 1 above.

□

Proposition 2.5 is an immediate corollary of the above Proposition; this completes the proof of the Main Theorem 1.1. Here is a partial converse to the Main Theorem.

6.8. *Remark.* If K is a field with exactly one $\mathbb{R}-$place, then $\mathcal{M}(K(x))$ is path-connected if and only if the value group of the unique $\mathbb{R}-$place on K is countable. Sufficiency here is a (very) special case of our main theorem; we now prove necessity. We will use the notation of this paper, but with F replaced by K. Just suppose that the value group vK is uncountable and that $\mathcal{M}(K(x))$ is path-connected. Then there exists a path $P : [0,1] \longrightarrow M(K(x))$ with $v_{P(0)}(x) = \infty$ (that is, the cut $\mathbb{Q}v(K)$) and with $v_{P(1)}(x) = -\infty$ (i.e., the empty cut). Let γ be in vK. Then $H(v(x) \geq \gamma)$ and $H(v(x) \leq \gamma)$ are open sets with union $\mathcal{M}(K(x))$ containing $P(0)$ and $P(1)$ respectively. Hence their intersection with the image of the path P cannot be empty, i.e., there is a number $t \in [0,1]$ with $v_{P(t)}(x) = \gamma$. Thus the intersection of $H(v_{P(t)}(x) = \gamma)$ with the image of P is a nonempty open subset of the image of P. The collection of all these sets (for various $\gamma \in vK$) is an uncountable set of pairwise disjoint nonempty open subsets of the image of P, a contradiction. Hence $\mathcal{M}K(x))$ is not path-connected.

The above argument shows that the fact that $\mathcal{M}(K)$ is path-connected does not imply that $\mathcal{M}(K(x))$ is path-connected, in contrast with the result of Harmon [8, Theorem 2.12] on connectedness.

7. Some questions

We let (F, σ) denote a field ultracomplete at an $\mathbb{R}$-place σ.

1. Is the countability hypothesis of the Main Theorem 1.1 necessary?

2. We do not know if the space of $\mathbb{R}-$places of the rational function field over $\mathbb{R}$ in countably many variables is path-connected. It is not hard to show that it is an inverse limit of the spaces of $\mathbb{R}-$places of the rational function fields over $\mathbb{R}$ in finitely many variables, and hence by [9, Theorem 117] it is connected.

3. What is the homotopy group of $\mathcal{M}(F(x))$? Same question for $\mathbb{R}(x_1, \cdots, x_n)$.

4. Does the space $\mathcal{M}(\mathbb{R}(x_1, \cdots, x_n))$ contain a disk if $n > 1$?

5. Is the set of $\mathbb{R}$-places of $\mathcal{M}(R(x_i, \cdots, x_n))$ with value group isomorphic to the value group of a particular $\mathbb{R}$-place on $\mathcal{M}(R(x_i, \cdots, x_n))$ dense in $\mathcal{M}(R(x_i, \cdots, x_n))$? This is true if $n = 2$ [5, Theorem 6.3].

References

[1] R. Brown, Valuations, primes and irreducibility in polynomial rings and rational function fields, *Trans. Amer. Math. Soc.* **174** (1972) 451–488.

[2] R. Brown, Extended prime spots and quadratic forms, *Pacific J. Math.* **51** (1974) 379–395.

[3] R. Brown, Real-valued places on the function field of an algebraic curve, *Houston J. Math.* **6** (1980) 227–243.

[4] R. Brown and M. Marshall, The reduced theory of quadratic forms, *Rocky Mountain J. Math* **11** (1981) 161–175.

[5] R. Brown and J. L. Merzel, The space of real places on $\mathbb{R}(x, y)$, *Annales Math. Silesianae* **32** (2018) 99–131.

[6] D. Dubois, Infinite primes and ordered fields, *Dissertationes Math. (Rozprawy Mat.)* **69** (1970) 1–43.
[7] A. J. Engler and A. Prestel, *Valued Fields.* (Springer-Verlag, New York, 2005).
[8] J. Harman, Chains of higher level orderings, *in* "Ordered Fields and Real Algebraic Geometry" (D. W. Dubois and T. Recio, Eds.), *Contemporary Math.* **8** (1982) 141–174.
[9] W. T. Ingram and W. S. Mahavier, *Inverse limits: from continua to chaos* (Springer-Verlag, New York , 2000).
[10] I. Kaplansky, Maximal fields with valuations, *Duke Math. J.* **9** (1941) 303–321.
[11] K. Kuhlmann, The structure of spaces of R-spaces of rational function fields over real closed fields, *Rocky Mountain J. Math.* **46** (2016) 533–557.
[12] K. and F.-V. Kuhlmann, Embedding theorems for spaces of $\mathbb{R}$-places of rational function fields and their products, *Fund. Math.* **218** (2012) 121–149.
[13] T. Y. Lam, *Orderings, valuations and quadratic forms.* CBMS Regional Conf. Ser.Math. **52** Published for the Conf. Board of the Math. Sciences, Washington (1983).
[14] S. MacLane, A construction for absolute values on polynomial rings, *Trans. Amer. Math. Soc.* **40** (1936) 363–395.
[15] J. Rosenstein, *Linear Orderings* (Academic Press, New York, 1982).
[16] P. Ribenboim, *Théorie des Valuations* (Les Presses de l'Université de Montréal, Montreal, 1964).
[17] H.-W. Schülting, Real points and real places, *in* "Ordered Fields and Real Algebraic Geometry" (D. W. Dubois and T. Recio, Eds.), *Contemporary Math.* **8** (1982) 289–295.
[18] S. M. M. Zekavat, Orderings, cuts and formal power series, thesis, University of Saskatchewan, Saskatoon, Saskatchewan, 2000.

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