A generalization of Eisenstein-Schönemann Irreducibility Criterion*

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Abstract. One of the results generalizing Eisenstein Irreducibility Criterion states that if $\phi(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is a polynomial with coefficients from the ring of integers such that a_s is not divisible by a prime p for some $s \leqslant n$, each a_i is divisible by p for $0 \leqslant i \leqslant s-1$ and a_0 is not divisible by p^2, then $\phi(x)$ has an irreducible factor of degree at least s over the field of rational numbers. We have observed that if $\phi(x)$ is as above, then it has an irreducible factor $g(x)$ of degree s over the ring of p-adic integers such that $g(x)$ is an Eisenstein polynomial with respect to p. In this paper, we prove an analogue of the above result for a wider class of polynomials which will extend the classical Schönemann Irreducibility Criterion as well as Generalized Schönemann Irreducibility Criterion and yields irreducibility criteria by Akira, Panaitopol and Stefănescu (cf. J. Number Theory, 25 (1987) 107-111).

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1. Introduction.

The classical Schönemann Irreducibility Criterion states that if $f(x)$ is a monic polynomial with coefficients from the ring $\mathbb{Z}$ of integers which is irreducible modulo a prime number p and if $g(x)$ belonging to $\mathbb{Z}[x]$ is a polynomial of the form $g(x) = f(x)^s + pM(x)$ where $M(x)$ belonging to $\mathbb{Z}[x]$ has degree less than that of $g(x)$ and is relatively prime to $f(x)$ modulo p, then $g(x)$ is irreducible over the field $\mathbb{Q}$ of rational numbers. Observe that if $g(x)$ is as above, then the $f(x)$-expansion of $g(x)$ obtained on dividing it by successive powers of $f(x)$ given by

$$g(x) = \sum_{i=0}^{s} A_i(x)f(x)^i,\ deg\ A_i(x) < deg f(x),$$

satisfies (i) $A_s(x) = 1$, (ii) p divides the content of each polynomial $A_i(x)$ for $0 \leqslant i \leqslant s-1$ and (iii) p^2 does not divide the content of $A_0(x)$. Conversely, it is clear that any polynomial $g(x)$ belonging to $\mathbb{Z}[x]$, whose $f(x)$-expansion has the above three properties, satisfies the conditions of Schönemann Irreducibility Criterion. The above observation led to the extension of this criterion to polynomials with coefficients in arbitrary valued fields (see [6]). In 2008, Ron Brown [3] gave a simple proof of the most general version of the criterion which will be stated after introducing some notations.

Throughout v is a Krull valuation of a field K, i.e., $v : K \to \Gamma \cup \{\infty\}$ where Γ is a totally ordered additively written abelian group such that $v(a) = \infty \Leftrightarrow a = 0$, $v(ab) = v(a) + v(b)$ and $v(a + b) \geqslant \min\{v(a), v(b)\}$ for all a, b in K. We shall denote by v^x the Gaussian valuation of the field $K(x)$ of rational functions in an indeterminate x which extends the valuation v of K and is defined on $K[x]$ by

$$v^x(\sum_i a_i x^i) = \min_i\{v(a_i)\},\ a_i \in K.$$

For an element ξ in the valuation ring R_v of v with maximal ideal $\mathcal{M}_v$, $\bar{\xi}$ will denote its v-residue, i.e., the image of ξ under the canonical homomorphism from R_v onto $R_v/\mathcal{M}_v$. For $f(x)$ belonging to $R_v[x]$, $\bar{f}(x)$ will stand for the polynomial over $R_v/\mathcal{M}_v$ obtained by replacing each coefficient of $f(x)$ by its v-residue.

With the above notations, the following theorem holds (see [3, Lemma 4]).

Theorem 1.A. *Let v be a Krull valuation of a field K with value group Γ and valuation ring R_v. Let $f(x)$ belonging to $R_v[x]$ be a monic polynomial of degree m such that $\bar{f}(x)$ is irreducible over the residue field of v. Assume that $g(x)$*

belonging to $R_v[x]$ is a monic polynomial whose $f(x)$-expansion $\sum_{i=0}^{n} A_i(x)f(x)^i$ satisfies (i) $A_0(x) \neq 0$, $A_n(x) = 1$, (ii) $\frac{v^x(A_i(x))}{n-i} \geqslant \frac{v^x(A_0(x))}{n} > 0$ for $0 \leqslant i \leqslant n-1$ and (iii) $v^x(A_0(x)) \notin d\Gamma$ for any number $d > 1$ dividing n. Then $g(x)$ is irreducible over K.

A polynomial $g(x)$ satisfying conditions (i),(ii), (iii) of the above theorem will be referred to as a Generalized Schönemann polynomial with respect to v and $f(x)$. Note that in case v is a discrete valuation of K with value group $\mathbb{Z}$, then condition (iii) of Theorem 1.A says that $v^x(A_0(x))$ and n are coprime. Hence in this case, it is immediate from the above theorem that a polynomial $g(x)$ having $f(x)$-expansion $f(x)^n + \sum_{i=0}^{n-1} A_i(x)f(x)^i$ with $v^x(A_0(x)) = 1$, $v^x(A_i(x)) > 0$ for $0 \leqslant i \leqslant n-1$, is irreducible over K; such a polynomial is called a Schönemann polynomial with respect to the discrete valuation v and $f(x)$; in the particular case when $f(x) = x$, it will be referred to as an Eisenstein polynomial with respect to v.

In this paper, our aim is to extend Theorem 1.A. Precisely stated, we prove

Theorem 1.1. *Let v be a henselian Krull valuation of a field K with value group Γ and valuation ring R_v having maximal ideal $\mathcal{M}_v$. Let $f(x)$ belonging to $R_v[x]$ be a monic polynomial of degree m such that $\bar{f}(x)$ is irreducible over $R_v/\mathcal{M}_v$ and $\phi(x)$ belonging to $R_v[x]$be a monic polynomial having $f(x)$-expansion $\sum_{i=0}^{n} A_i(x)f(x)^i$ with $A_0(x) \neq 0$. Assume that there exists $s \leqslant n$ such that (i) $v^x(A_s(x)) = 0$, (ii) $\frac{v^x(A_i(x))}{s-i} \geqslant \frac{v^x(A_0(x))}{s} > 0$ for $0 \leqslant i \leqslant s-1$ and (iii) $v^x(A_0(x)) \notin d\Gamma$ for any number $d > 1$ dividing s. Then $\phi(x)$ has an irreducible factor $g(x)$ of degree sm over K such that $g(x)$ is a Generalized Schönemann polynomial with respect to v and $f(x)$; moreover the $f(x)$-expansion of $g(x) = f(x)^s + B_{s-1}(x)f(x)^{s-1} + \ldots + B_0(x)$ satisfies $v^x(B_0(x)) = v^x(A_0(x))$.*

It is immediate from the above theorem that if $f(x)$ is as in this theorem, then a monic polynomial $\phi(x)$ belonging to $R_v[x]$ with $f(x)$-expansion $\sum_{i=0}^{n} A_i(x)f(x)^i$ satisfying conditions (ii) and (iii) of Theorem 1.A but not satisfying (i), must be reducible over K.

The following corollaries will be deduced from Theorem 1.1. Corollary 1.2 extends Schönemann Irreducibility Criterion [10, §3.1, Theorem D]. Corollary 1.3 proves an irreducibility criterion due to Panaitopol and Stefănescu by a different method (cf.[8]). Corollary 1.4 extends Akira's criterion (cf. [1]).

Corollary 1.2. *Let v be a discrete valuation of K with value group $\mathbb{Z}$ and π be an element of K with $v(\pi) = 1$. Let $f(x), m$ be as in Theorem 1.1. Let $F(x)$ belonging to $R_v[x]$ be a monic polynomial having $f(x)$-expansion $\sum_{i=0}^{n} A_i(x)f(x)^i$. Assume that there exists $s \leqslant n$ such that π does not divide the content of $A_s(x)$, π divides the content of each $A_i(x), 0 \leqslant i \leqslant s-1$ and π^2 does not divide the content of $A_0(x)$. Then $F(x)$ has an irreducible factor of degree sm over the completion $(\hat{K}, \hat{v})$ of (K, v) which is a Schönemann polynomial with respect to $\hat{v}$ and $f(x)$.*

As usual for polynomials g and h, $R(g, h)$ will stand for the resultant of g and h.

Corollary 1.3. *Let (K, v) and π be as in Corollary 1.2. Let $F(x)$ belonging to $R_v[x]$ be a monic polynomial such that $F(x) = f_1(x)^{m_1} \cdots f_r(x)^{m_r} + \pi g(x)$ where $f_1(x), \ldots, f_r(x), g(x)$ belong to $R_v[x]$, each $f_i(x)$ is monic with $\bar{f}_i(x)$ irreducible over the residue field of v, $\bar{f}_i(x) \neq \bar{f}_j(x)$ for $i \neq j$ and $\bar{f}_i \nmid \bar{g}$ if $m_i \geqslant 2$. Let $h = f_1^{m_1} \cdots f_r^{m_r}$. If for every non-trivial factorization $h = h_1h_2$ with h_1, h_2 belonging to $R_v[x], h_1, h_2$ coprime, there is an index $j \in \{1, 2\}$ and $i \in \{1, 2, \cdots, r\}$ such that f_i divides h_j and for all divisors d of $R(f_i, g)$, $R(\bar{f}_i, \overline{h/h_j}) \neq \bar{d}$, then the polynomial $F(x)$ is irreducible in $K[x]$.*

Corollary 1.4. *Let $(K, v), \pi$ be as above and $F(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ be a polynomial over R_v satisfying the following conditions for an index $s \leqslant n - 1$.*
(i) $\pi | a_i$ for $0 \leqslant i \leqslant s - 1$, $\pi^2 \nmid a_0, \pi \nmid a_s$.
(ii) The polynomial $x^{n-s} + \bar{a}_{n-1}x^{n-s-1} + \ldots + \bar{a}_s$ is irreducible over the residue field of v.
(iii) $\bar{d} \neq \bar{a}_s$ for any divisor d of a_0 in R_v.
Then $F(x)$ is irreducible over K.

Indeed our method of proof can be easily carried over (using henselisation of (K, v) instead of its completion) to prove the analogues of Corollaries 1.3, 1.4 when v is a Krull valuation of K of arbitrary rank having value group Γ with $v(\pi)$ as the smallest positive element of Γ.

2. Notations, definitions and some preliminary results.

Recall that the resultant $R(G,H)$ of two polynomials $G(x)$ and $H(x)$ with $G(x)$ monic, equals $\prod_{j=1}^{q} H(y_j)$ where q is the degree of $G(x)$ and $y_1, y_2, \ldots, y_q$ are the roots of $G(x)$.

As usual, for non-negative elements λ, μ in a totally ordered abelian group Γ, we write $\lambda \gg 0$ if $\lambda > 0$ and $\lambda \gg \mu > 0$ if $\lambda > \mu$ and $\lambda - \mu$ does not belong to the largest convex subgroup of Γ not containing μ.

The following well-known lemma proved in [9] which is equivalent to Hensel's Lemma will be used in the sequel.

Rychlik's Lemma. Let v be a henselian Krull valuation of a field K. Let $F(x), G(x), H(x)$ belonging to $R_v[x]$ be non-zero polynomials such that
(i) deg $G(x) > 0$, deg $F(x) =$ deg $G(x)+$deg $H(x)$, $G(x)$ is monic, $v^x(F(x)) = 0$ and $F(x)$ and $H(x)$ have the same leading coefficient.
(ii) $v(R(G,H)) = \rho < \infty$.
(iii) $v^x(F(x) - G(x)H(x)) \gg 2\rho$.
Then there exist polynomials $g(x)$ and $h(x)$ belonging to $R_v[x]$ such that
a) $v^x(G(x) - g(x)) > \rho$, $v^x(H(x) - h(x)) > \rho$.
b) deg $g(x)=$ deg $G(x)$, deg $h(x)=$ deg $H(X)$, $g(x)$ is monic.
c) $F(x) = g(x)h(x)$.

In what follows, v is a henselian Krull valuation of a field K with value group Γ, valuation ring R_v and $\tilde{v}$ is the unique prolongation of v to the algebraic closure $\widetilde{K}$ of K with value group $\widetilde{\Gamma}$. By the degree of an element $\alpha \in \widetilde{K}$, we shall mean the degree of the extension $K(\alpha)/K$ which will be denoted by $\deg \alpha$. A pair (α, δ) belonging to $\widetilde{K} \times \widetilde{\Gamma}$ is said to be a minimal pair (more precisely (K,v)-minimal pair) if whenever β belongs to $\widetilde{K}$ with $\deg \beta < \deg \alpha$, then $\tilde{v}(\alpha - \beta) < \delta$. For example if $f(x)$ is a monic polynomial with coefficients in R_v such that $\bar{f}(x)$ is irreducible over the residue field k_v of v and α is a root of $f(x)$, then (α, δ) is a (K,v)-minimal pair for each positive δ in $\widetilde{\Gamma}$, because whenever β belonging to $\widetilde{K}$ has degree less than $m = \deg f(x)$, then $\tilde{v}(\alpha - \beta) \leq 0$, for otherwise $\bar{\alpha} = \bar{\beta}$, which in view of the fundamental inequality [4, Theorem 3.3.4] would lead to $[K(\beta) : K] \geq [k_v(\bar{\beta}) : k_v] = m$.

Let (K,v), $(\widetilde{K}, \tilde{v})$ be as above and (α, δ) belonging to $\widetilde{K} \times \widetilde{\Gamma}$ be a (K,v)-

minimal pair. The valuation $\tilde{w}_{\alpha,\delta}$ of $\widetilde{K}(x)$ defined on $\widetilde{K}[x]$ by

$$\tilde{w}_{\alpha,\delta}(\sum_i c_i(x-\alpha)^i) = \min_i\{\tilde{v}(c_i) + i\delta\},\ c_i \in \widetilde{K} \tag{1}$$

will be referred to as the valuation with respect to the minimal pair (α, δ). The valuation of $K(x)$ obtained by restricting $\tilde{w}_{\alpha,\delta}$ will be denoted by $w_{\alpha,\delta}$.

The description of $w_{\alpha,\delta}$ is given by the theorem stated below (cf. [2, Theorem 2.1], [5, Theorem 1.4]).

Theorem 2.A. *Let (α,δ) be a (K,v)-minimal pair. If $f(x)$ is the minimal polynomial of α over K, then for any polynomial $F(x)$ belonging to $K[x]$ with $f(x)$-expansion $\sum_i A_i(x)f(x)^i$, we have*

$$w_{\alpha,\delta}(F(x)) = \min_i\{\tilde{v}(A_i(\alpha)) + iw_{\alpha,\delta}(f(x))\}.$$

Let (α,δ) and $w_{\alpha,\delta}$ be as in Theorem 2.A. For any non-zero polynomial $F(x)$ belonging to $K[x]$ with $f(x)$-expansion $\sum_i A_i(x)f(x)^i$, we shall denote by $I_{\alpha,\delta}(F(x)), S_{\alpha,\delta}(F(x))$ respectively the minimum and the maximum integers belonging to the set

$$\{i \mid w_{\alpha,\delta}(F(x)) = \tilde{v}(A_i(\alpha)) + iw_{\alpha,\delta}(f(x))\}.$$

The following already known result will be used in the proof of the theorem (cf. [7, Lemma 2.1]). Its proof is omitted.

Theorem 2.B. *Let (α,δ) be a (K,v)-minimal pair. For any non-zero polynomials $F(x), G(x)$ in $K[x]$, one has*
(a) $I_{\alpha,\delta}(F(x)G(x)) = I_{\alpha,\delta}(F(x)) + I_{\alpha,\delta}(G(x))$,
(b) $S_{\alpha,\delta}(F(x)G(x)) = S_{\alpha,\delta}(F(x)) + S_{\alpha,\delta}(G(x))$.

We now prove a lemma to be used in the sequel.

Lemma 2.1. *Let α be a root of a monic polynomial $f(x)$ belonging to $R_v[x]$ such that $\bar{f}(x)$ is irreducible over the residue field of v. Let (α,δ) be a (K,v)-minimal pair with $\delta > 0$. Then for any polynomial $\psi(x)$ belonging to $R_v[x]$ with $f(x)$-expansion $\sum_i D_i(x)f(x)^i$, one has*

$$w_{\alpha,\delta}(\psi(x)) = \min_i\{v^x(D_i(x)) + iw_{\alpha,\delta}(f(x))\} \geqslant 0.$$

Proof. We first show that for any polynomial $A(x)=\sum a_i x^i$ belonging to $K[x]$ of degree less than deg α, we have

$$\tilde{v}(A(\alpha)) = v^x(A(x)). \tag{2}$$

Clearly (2) needs to be verified when $m = deg\ \alpha > 1$. Now $\bar{\alpha}$ being a root of the irreducible polynomial $\bar{f}(x)$ is non-zero and so $\tilde{v}(\alpha)=0$. If (2) does not hold, then the triangle inequality would imply $\tilde{v}(A(\alpha)) > \min_i\{\tilde{v}(a_i\alpha^i)\} = v(a_j)$ (say), which gives $\overline{\sum_{i=0}^{m-1}\left(\frac{a_i}{a_j}\right)}\bar{\alpha}^i = \bar{0}$ contradicting the fact that the minimal polynomial of $\bar{\alpha}$ over the residue field of v has degree m.

Denote $w_{\alpha,\delta}(f(x))$ by λ. Write $f(x)=\sum_{i=1}^{m} c_i(x-\alpha)^i$, $c_i \in R_v[\alpha]$. Then $\tilde{v}(c_i) \geqslant 0$. Using (1) and the fact that $\delta > 0$, we get

$$\lambda = w_{\alpha,\delta}(f(x)) = \min_{1\leqslant i\leqslant m}\{\tilde{v}(c_i)+i\delta\} \geqslant 0. \tag{3}$$

Keeping in mind that $\psi(x)$ and hence each $D_i(x)$ belongs to $R_v[x]$, it follows immediately from Theorem 2.A, (2) and (3) that

$$w_{\alpha,\delta}(\psi(x)) = \min_i\{\tilde{v}(D_i(\alpha))+i\lambda\} = \min_i\{v^x(D_i(x))+i\lambda\} \geqslant 0.$$

3. Proof of Theorem 1.1

Denote $\frac{v^x(A_0(x)}{s}$ by λ. Fix a root α of $f(x)$. Write $f(x)=\sum_{i=1}^{m} c_i(x-\alpha)^i$, $c_i \in \widetilde{K}$. Determine δ in $\widetilde{\Gamma}$ so that

$$\lambda = \min_{1\leqslant i\leqslant m}\{\tilde{v}(c_i)+i\delta\},\ \ i.e,\quad \delta = \max_{1\leqslant i\leqslant m}\left(\frac{\lambda-\tilde{v}(c_i)}{i}\right).$$

Note that $\delta > 0$ in view of the fact that $c_m = 1$ and $\lambda > 0$ by hypothesis. So (α,δ) is a (K,v)-minimal pair and $w_{\alpha,\delta}(f(x))=\lambda$ by virtue of (1) and the choice of λ. Therefore keeping in mind assumptions (i) and (ii) of the theorem, it follows from Lemma 2.1 that

$$w_{\alpha,\delta}(\phi(x)) = \min_i\{v^x(A_i(x))+i\lambda\} = s\lambda = v^x(A_0(x)). \tag{4}$$

It is immediate from (2) and (4) that

$$I_{\alpha,\delta}(\phi(x)) = 0 \quad \text{and} \quad S_{\alpha,\delta}(\phi(x)) = s. \tag{5}$$

Write $\phi(x)$ as a product $\phi_1(x)\phi_2(x)\cdots\phi_r(x)$ of monic, irreducible polynomials over R_v. We split the proof into two main steps.

Step I. In this step, it will be shown that there exists $j,\ 1 \leqslant j \leqslant r$, such that

$$w_{\alpha,\delta}(\phi_j(x)) = w_{\alpha,\delta}(\phi(x)) \ \ \textit{and} \ \ w_{\alpha,\delta}(\phi_i(x)) = 0 \text{ for each } i \neq j.$$

Applying Theorem 2.B, we have

$$S_{\alpha,\delta}(\phi(x)) = \sum_{i=1}^{r} S_{\alpha,\delta}(\phi_i(x)) \ \ , \ \ I_{\alpha,\delta}(\phi(x)) = \sum_{i=1}^{r} I_{\alpha,\delta}(\phi_i(x)). \tag{6}$$

It now follows from (5) and (6) that there exists j such that

$$S_{\alpha,\delta}(\phi_j(x)) > 0 \ \ , \ \ I_{\alpha,\delta}(\phi_i(x)) = 0 \ \text{ for } \ 1 \leqslant i \leqslant r, \tag{7}$$

which in view of Lemma 2.1 implies that

$$0 \leqslant w_{\alpha,\delta}(\phi_i(x)) \in \Gamma \ \ , \ \ 1 \leqslant i \leqslant r. \tag{8}$$

We first show that

$$S_{\alpha,\delta}(\phi_j(x)) = s. \tag{9}$$

Let $\sum\limits_{i=0}^{t} B_i(x)f(x)^i$ be the $f(x)$-expansion of $\phi_j(x)$. Denote $S_{\alpha,\delta}(\phi_j(x))$by s_1. In view of (5) and (6), $s_1 \leqslant s$. Using the equality $\tilde{v}(B_i(\alpha)) = v^x(B_i(x))$ proved in (2), we have

$$w_{\alpha,\delta}(\phi_j(x)) = v^x(B_{s_1}(x)) + s_1\lambda. \tag{10}$$

As shown in (8), $w_{\alpha,\delta}(\phi_j(x))$ belongs to Γ and hence by (10), $s_1\lambda \in \Gamma$. The desired equality $s_1 = s$ now follows on recalling that $\lambda = \frac{v^x(A_0(x))}{s}$ and that s is the smallest positive integer for which $s\lambda$ belongs to Γ by assumption (iii) of the theorem.

Keeping in mind that $\phi_j(x)$ and hence $B_s(x)$ belongs to $R_v[x]$, (10) together with (9) implies that

$$w_{\alpha,\delta}(\phi_j(x)) = v^x(B_s(x)) + s\lambda \geqslant s\lambda. \tag{11}$$

Since $w_{\alpha,\delta}(\phi_i(x)) \geqslant 0$ for each i by (8) and $w_{\alpha,\delta}(\phi(x)) = s\lambda$ by (4), it follows from (11) that

$$w_{\alpha,\delta}(\phi_j(x)) = s\lambda \ \ , \ \ v^x(B_s(x)) = 0 \tag{12}$$

and hence the assertion of Step I is proved.

Step II. In this step, it will be shown that $\phi_j(x)$ has degree sm and is a Generalized Schönemann polynomial with respect to v and $f(x)$ which will complete the proof of the theorem.

In view of (9), deg $\phi_j(x) \geqslant sm$. We shall prove that deg $\phi_j(x) = sm$. Suppose to the contrary that deg $\phi_j(x) > sm$. Define polynomials $G(x)$ and $H(x)$ by

$$G(x) = f(x)^s \quad , \quad H(x) = B_t(x)f(x)^{t-s} + B_{t-1}(x)f(x)^{t-s-1} + \ldots + B_s(x). \tag{13}$$

It will be shown that the polynomials $\phi_j(x), G(x)$ and $H(x)$ satisfy the conditions of Rychlik's Lemma. These polynomials clearly satisfy condition (i) of this lemma. We first show that

$$v^x(\phi_j - GH) > 0. \tag{14}$$

In view of (12) and Lemma 2.1, we have

$$s\lambda = w_{\alpha,\delta}(\phi_j(x)) = \min_{0\leqslant i\leqslant s-1} \{v^x(B_i(x)) + i\lambda\}.$$

Consequently

$$v^x(B_i(x)) + i\lambda \geqslant s\lambda \ , \quad 0 \leqslant i \leqslant s-1. \tag{15}$$

Since $I_{\alpha,\delta}(\phi_j(x)) = 0$ by (7), using (2), we see that

$$w_{\alpha,\delta}(\phi_j(x)) = \tilde{v}(B_0(\alpha)) = v^x(B_0(x)).$$

The above equation together with (12) implies that $v^x(B_0(x)) = s\lambda$. Thus (15) can be rewritten as

$$\frac{v^x(B_i(x))}{s-i} \geqslant \frac{v^x(B_0(x))}{s} = \frac{v^x(A_0(x))}{s} > 0 \ \text{ for } \ 0 \leqslant i \leqslant s-1 \tag{16}$$

which implies that $v^x(B_i(x)) > 0$ for $0 \leqslant i \leqslant s-1$; consequently

$$v^x(\phi_j - GH) = v^x(B_{s-1}(x)f(x)^{s-1} + \ldots + B_0(x)) \geqslant \min_{0\leqslant i\leqslant s-1} \{v^x(B_i(x))\} > 0$$

and hence (14) is proved. Keeping in mind (14), conditions (ii) and (iii) of Rychlik's Lemma are verified once we show that

$$v(R(G,H)) = 0. \tag{17}$$

Let $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_m$ denote the roots of $f(x)$. Keeping in mind (13) and the fact that (K, v) is henselian, it is clear that

$$v(R(G,H)) = s\tilde{v}(\prod_{i=1}^{m} H(\alpha_i)) = s\tilde{v}(\prod_{i=1}^{m} B_s(\alpha_i)) = ms\tilde{v}(B_s(\alpha)).$$

Using (2) and (12), it now follows from the above equation that $v(R(G,H)) = msv^x(B_s(x)) = 0$ which proves (17). Applying Rychlik's Lemma, we see that $\phi_j(x)$ has a factor $g(x)$ over K of degree equal to that of $G(x)$, i.e., sm, which contradicts the irreducibility of $\phi_j(x)$. This contradiction proves that degree of $\phi_j(x)$ is sm. Moreover $\phi_j(x)$ is a Generalized Schönemann polynomial with respect to v and $f(x)$ in view of (16) and hence the theorem.

4. Proof of Corollaries 1.2, 1.3 and 1.4.

Proof of Corollary 1.2. The hypothesis implies that $v^x(A_i(x)) > 0$ for $0 \leqslant i \leqslant s-1$, $v^x(A_s(x)) = 0$ and $v^x(A_0(x)) = 1$. Therefore by Theorem 1.1, $F(x)$ has an irreducible factor $g(x)$ of degree sm over $\hat{K}$ which is a Generalized Schönemann polynomial with respect to $\hat{v}$ and $f(x)$. The desired result now follows from the last assertion of the theorem and the fact that $v^x(A_0(x)) = 1$.

Proof of Corollary 1.3. Let $(\hat{K}, \hat{v})$ denote the completion of (K, v). The unique prolongation of $\hat{v}$ to the algebraic closure of $\hat{K}$ will be denoted by $\tilde{\hat{v}}$. Claim is that for each $j, 1 \leqslant j \leqslant r$, there exists an irreducible factor $F_j(x)$ belonging to $R_{\hat{v}}[x]$ of $F(x)$ having degree $m_j \deg f_j(x)$ with $\bar{F}_j(x) = \bar{f}_j^{\,m_j}(x)$. Taking into account the degree, this will immediately give that $F(x)$ factors over $R_{\hat{v}}$ as $F(x) = F_1(x) \cdots F_r(x)$.
For convenience of notation, we prove the claim for $j = 1$. Let $\sum\limits_{i \geqslant 0} D_i(x) f_1(x)^i$, $\sum\limits_{i \geqslant 0} E_i(x) f_1(x)^i$ be the f_1-expansions of $f_2^{m_2} \cdots f_r^{m_r}$ and $g(x)$ respectively. Since $\bar{f}_1(x) \nmid \bar{f}_i(x)$ for $i \geqslant 2$, it follows that $\bar{D}_0(x) \neq \bar{0}$, *i.e.*, $v^x(D_0(x)) = 0$. In case $m_1 \geqslant 2$, $\bar{f}_1(x) \nmid \bar{g}(x)$ and so in this case, $v^x(E_0(x)) = 0$. Therefore the f_1-expansion of $F(x) = f_1(x)^{m_1} \cdots f_r(x)^{m_r} + \pi g(x)$ satisfies the hypothesis of Theorem 1.1 with s replaced by m_1 and hence $F(x)$ has a factor $F_1(x)$ belonging to $R_{\hat{v}}[x]$ which is a Generalized Schönemann polynomial with respect to $\hat{v}$ and $f_1(x)$ of degree $m_1 \deg f_1(x)$. This proves the claim.

Suppose to the contrary that $F(x)$ is reducible over K, say $F(x) = G(x)H(x)$ with $G(x), H(x)$ monic polynomials over R_v of positive degree. In view of the claim, we can write $h = h_1h_2$ where $\bar{h}_1(x) = \bar{G}(x)$ and $\bar{h}_2(x) = \bar{H}(x)$. By hypothesis, there exist $j \in \{1,2\}$ and $i \in \{1,2,\ldots,r\}$ such that f_i divides h_j and for all divisors d of $R(f_i,g)$, $R(\bar{f}_i, \overline{h/h_j}) \neq \bar{d}$. Let $\alpha_1,\alpha_2,\ldots,\alpha_t$ denote the roots of $f_i(x)$ counted with multiplicities, if any. After renaming if necessary, assume that f_i divides h_1. Then $\bar{f}_i$ divides $\bar{h}_1 = \bar{G}$ and hence by virtue of the henselian property of $(\hat{K},\hat{v})$, we see that

$$0 < v(R(f_i,G)) = \sum_{k=1}^{t} \tilde{\hat{v}}(G(\alpha_k)) = t\tilde{\hat{v}}(G(\alpha_1)). \tag{18}$$

Keeping in mind that $\bar{f}_i$ is irreducible over the residue field of v, it follows from the formula proved in (2) (with v replaced by $\hat{v}$) that $\tilde{\hat{v}}(G(\alpha_1))$ belongs to $\mathbb{Z}$. Consequently (18) implies that $v(\prod_{k=1}^{t} G(\alpha_k)) \geqslant t$ and hence

$$\frac{1}{\pi^t}\prod_{k=1}^{t} G(\alpha_k) \in R_v. \tag{19}$$

Thus we conclude that

$$R(f_i,g) = \prod_{k=1}^{t} g(\alpha_k) = \frac{1}{\pi^t}\prod_{k=1}^{t} F(\alpha_k) = \left(\frac{1}{\pi^t}\prod_{k=1}^{t} G(\alpha_k)\right)\prod_{k=1}^{t} H(\alpha_k)$$

is divisible by the element $\prod_{k=1}^{t} H(\alpha_k) = d$ (say) of R_v in view of (19). This contradicts the hypothesis of Corollary 1.3 because

$$R(\bar{f}_i, \overline{h/h_1}) = R(\bar{f}_i, \bar{H}) = \overline{R(f_i,H)} = \overline{\prod_k H(\alpha_k)} = \bar{d}.$$

Hence $F(x)$ is irreducible over K.

Proof of Corollary 1.4. Applying Corollary 1.2 with $f(x) = x$, we see that $F(x)$ has an irreducible factor $g(x)$ of degree s over the completion $(\hat{K},\hat{v})$ of (K,v), which is an Eisenstein polynomial with respect to $\hat{v}$. Write $F(x) = g(x)h(x)$, where $g(x) = x^s + b_{s-1}x^{s-1} + \ldots + b_0$, $h(x) = x^{n-s} + c_{n-s-1}x^{n-s-1} + \ldots + c_0$. In view of the hypothesis, $\bar{F}(x) = x^s(x^{n-s} + \bar{a}_{n-1}x^{n-s-1} + \ldots + \bar{a}_s)$, so $\bar{h}(x) = x^{n-s} + \bar{a}_{n-1}x^{n-s-1} + \ldots + \bar{a}_s$, which is given to be irreducible over the residue field of v. Hence $h(x)$ is also irreducible over $\hat{K}$. Note that $\bar{c}_0 = \bar{a}_s \neq \bar{0}$ by hypothesis. If $F(x)$ were reducible over K, then $g(x)$ and $h(x)$ being irreducible over $\hat{K}$, would belong to $K[x]$ and consequently the equality $a_0 = b_0c_0$ would contradict assumption (iii) of the corollary for the divisor c_0 belonging to R_v of a_0.

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References.

[1] T. Akira.: Criterion for irreducibility of polynomials. Bull. College Sci. Univ. Ryukyus **33**, 1-4 (1982).
[2] V. Alexandru, N. Popescu and A. Zaharescu.: A theorem of characterization of residual transcendental extension of a valuation. J. Math. Kyoto Univ. **28**, 579-592 (1988).
[3] R. Brown.: Roots of generalized Schönemann polynomials in henselian extension fields. Indian J. Pure Appl. Math. **39**(5), 403-410 (2008).
[4] A. J. Engler and A. Prestel.: Valued Fields. Springer-Verlag, New York (2005).
[5] S. K. Khanduja.: On valuations of $K(x)$. Proc. Edinburgh Math. Soc. **35**, 419-426 (1992).
[6] S. K. Khanduja and J. Saha.: On a generalization of Eisenstein's irreducibility criterion. Mathematika **44**, 37-41 (1997).
[7] S. K. Khanduja and M. Kumar.: Prolongations of valuations to finite extensions. Manuscripta Math. **131** (2), 323-334 (2010).
[8] L. Panaitopol and D. Stefănescu.: A resultant condition for the irreducibility of the polynomials. J. Number Theory **25** (1), 107-111 (1987).
[9] P. Ribenboim.: Equivalent forms of Hensel's Lemma. Expo. Math. **3**, 3-24 (1985).
[10] P. Ribenboim.: The Theory of Classical Valuations. Springer-Verlag, New York (1999).