Some irreducibility results for truncated binomial expansions*

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Abstract. For positive integers $n > k$, let $P_{n,k}(x) = \sum_{j=0}^{k} \binom{n}{j} x^j$ be the polynomial obtained by truncating the binomial expansion of $(1+x)^n$ at the k^{th} stage. These polynomials arose in the investigation of Schubert calculus in Grassmannians. In this paper, the authors prove the irreducibility of $P_{n,k}(x)$ over the field of rational numbers when $2 \leqslant 2k \leqslant n < (k+1)^3$.

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1. Introduction

For positive integers k and n with $k \leqslant n-1$, let $P_{n,k}(x)$ denote the polynomial $\sum_{j=0}^{k} \binom{n}{j} x^j$, where $\binom{n}{j} = \frac{n!}{j!\ (n-j)!}$. In 2007, Filaseta, Kumchev and Pasechnik considered the problem of irreducibility of $P_{n,k}(x)$ over the field $\mathbb{Q}$ of rational numbers. This problem arose during the 2004 MSRI program on "topological aspects of real algebraic geometry" in the work of Inna Scherbak [6]. These polynomials have also arisen in the context of work of Iossif V. Ostrovskii [3]. In the case $k = 2$, $P_{n,k}(x)$ has negative discriminant and hence is irreducible over $\mathbb{Q}$. In fact it is already known that $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ for all $n \leqslant 100$, $k+2 \leqslant n$ (cf. [2, p.455]). In [2], Filaseta et al. pointed out that when $k = n-1$, then $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ if and only if n is a prime number. They also proved that for any fixed integer $k \geqslant 3$, there exists an integer n_0 depending on k such that $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ for every $n \geqslant n_0$. So there are indications that $P_{n,k}(x)$ is irreducible for every n, k with $3 \leqslant k \leqslant n-2$.

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In this paper, we prove the irreducibility of $P_{n,k}(x)$ for all n, k such that $2 \leqslant 2k \leqslant n < (k+1)^3$. We consider the irreducibility of the polynomial $P_{n,k}(x-1) = \sum_{j=0}^{k} c_j x^j$, where $c_j = \sum_{i=j}^{k} \binom{n}{i}\binom{i}{j}(-1)^{i-j}$. As in [2], on using the identity

$$\sum_{j=0}^{a}(-1)^j \binom{b}{j} = (-1)^a \binom{b-1}{a}, \quad a < b \quad \text{non-negative integers,}$$

a simple calculation shows that

$$c_j = (-1)^{k-j}\binom{n}{j}\binom{n-j-1}{k-j} = \frac{(-1)^{k-j} \quad n(n-1)\cdots(n-k)}{j!(k-j)!}\,\frac{1}{(n-j)}. \tag{1}$$

In fact we shall prove the irreducibility of $P_{n,k}(x)$ using Newton polygons with respect to primes exceeding k dividing $\binom{n}{k}$ and some results of Erdős, Selfridge, Saradha, Shorey and Laishram regarding such primes (cf. [7], [5]). The same method gives the irreducibility of polynomial

$$F_{n,k}(x) = \sum_{j=0}^{k} a_j c_j x^j, \tag{2}$$

where $a_0, a_1, \ldots, a_k$ are non-zero integers and each a_i has all of its prime factors $\leqslant k$. We prove

Theorem 1.1. *Let k and n be positive integers such that $2k \leqslant n < (k+1)^3$. Then $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$.*

Theorem 1.1 is derived from the following more general result.

Theorem 1.2. *Let k and n be positive integers such that $8 \leqslant 2k \leqslant n < (k+1)^3$ and $F_{n,k}(x)$ be as in (2). Then $F_{n,k}(x)$ is irreducible over $\mathbb{Q}$ except possibly when (n, k) belongs to the set $\{(8,4), (10,5), (12,6), (16,8)\}$.*

It may be pointed out that the polynomial[1] $F_{10,5}(x)$ given by

$$\begin{aligned} F_{10,5}(x) &= 2000.c_5x^5 - 375.c_4x^4 - 9.c_3x^3 + 10.c_2x^2 - 27.c_1x + 25.c_0 \\ &= 2000.252x^5 + 375.1050x^4 - 9.1800x^3 - 10.1575x^2 - 27.700x - 25.126 \end{aligned}$$

has $7x^2 + 7x + 1$ as a factor which shows that $F_{n,k}$ can be reducible over $\mathbb{Q}$.

In the course of the proof of Theorem 1.2, we prove the following result which is of independent interest as well.

[1]This example was constructed by the referee.

Theorem 1.3. *Let k, n be integers such that $n \geqslant k+2 \geqslant 4$. Suppose there exists a prime $p > k, p|(n-l)$ with $1 \leqslant l \leqslant k-1$ and $ord_p(n-l) = e$ such that $gcd(e, l) \leqslant 2$ and $gcd(e, k-l) \leqslant 2$. If $l_1 < k/2$ is a positive integer such that $l \notin \{l_1, 2l_1, k-l_1, k-2l_1\}$, then $F_{n,k}(x)$ cannot have a factor of degree l_1 over $\mathbb{Q}$.*

2. Notation and Preliminary Results

For any non-zero integer a, let $v_p(a)$= $\mathrm{ord}_p(a)$ denote the p-adic valuation of a, i.e., the highest power of p dividing a and denote $v_p(0)$ by ∞. Let $g(x) = \sum_{j=0}^{k} a_j x^j$ be a polynomial over $\mathbb{Q}$ with $a_0 a_k \neq 0$. To each term $a_i x^i$, we associate a point $(n-i, v_p(a_i))$ ignoring however the point $(n-i, \infty)$ if $a_i = 0$ and form the set

$$S = \{(0, v_p(a_k)), \ldots, (n-j, v_p(a_j)), \ldots, (k, v_p(a_0))\}.$$

The Newton polygon of $g(x)$ with respect to p is the polygonal path formed by the lower edges along the convex hull of points of S. Slopes of the edges are increasing when calculated from left to right.

We begin with the following well known results (see [1] for Theorem 2.A and [4, 5.1.F] for Theorem 2.B).

Theorem 2.A. *Let p be a prime and $g(x), h(x)$ belong to $\mathbb{Q}[x]$ with $g(0)h(0) \neq 0$ and $u \neq 0$ be the leading coefficient of $g(x)h(x)$. Then the edges of the Newton polygon of $g(x)h(x)$ with respect to p can be formed by constructing a polygonal path beginning at $(0, v_p(u))$ and using the translates of the edges in the Newton polygon of $g(x)$ and $h(x)$ with respect to p taking exactly one translate for each edge. The edges are translated in such a way as to form a polygonal path with slopes of edges increasing.*

Theorem 2.B. *Let $(x_0, y_0), (x_1, y_1), \ldots, (x_r, y_r)$ denote the successive vertices of the Newton polygon of a polynomial $g(x)$ with respect to a prime p. Let $\tilde{v}_p$ denote the unique extension of v_p to the algebraic closure of $\mathbb{Q}_p$, the field of p-adic numbers. Then $g(x)$ factors over $\mathbb{Q}_p$ as $g_1(x)g_2(x)\cdots g_r(x)$ where the degree of $g_i(x)$ is $x_i - x_{i-1}$, $i = 1, 2, \ldots, r$ and all the roots of $g_i(x)$ in the algebraic closure of $\mathbb{Q}_p$ have $\tilde{v}_p$ valuation $\frac{y_i - y_{i-1}}{x_i - x_{i-1}}$. In particular all the roots of an irreducible factor of $g(x)$ over $\mathbb{Q}_p$ will have the same $\tilde{v}_p$ valuation.*

For an integer $\nu > 1$, let $P(\nu)$ denote the greatest prime divisor of ν and let $\pi(\nu)$ denote the number of primes not exceeding ν. As in [5], $\delta(k)$ will denote the integer defined for $k \geqslant 3$ by

$$\delta(k) = \begin{cases} 2, & \text{if } \ 3 \leqslant k \leqslant 6; \\ 1, & \text{if } 7 \leqslant k \leqslant 16; \\ 0, & \text{otherwise.} \end{cases}$$

For numbers n, k and h, $[n, k, h]$ will stand for the set of all pairs $(n, k), (n+1, k), \dots,$ $(n+h-1, k)$. In particular $[n, k, 1] = \{(n, k)\}$.

We shall denote by S the union of the sets

$[6, 3, 1], [8, 3, 3], [18, 3, 1], [9, 4, 1], [10, 5, 4], [16, 5, 1], [18, 5, 3], [27, 5, 2], [12, 6, 2], [20, 6, 1],$
$[14, 7, 3], [18, 7, 1], [20, 7, 2], [30, 7, 1], [16, 8, 1], [21, 8, 1], [26, 13, 3], [30, 13, 1], [32, 13, 2],$
$[36, 13, 1], [28, 14, 1], [33, 14, 1], [36, 17, 1]$

and by T the union of the sets

$[38, 19, 3], [42, 19, 1], [40, 20, 1], [94, 47, 3], [100, 47, 1], [96, 48, 1], [144, 71, 2], [145, 72, 1],$
$[146, 73, 3], [156, 73, 1], [148, 74, 1], [162, 79, 1], [166, 83, 1], [172, 83, 1], [190, 83, 1],$
$[192, 83, 1], [178, 89, 1], [190, 89, 1], [192, 89, 1], [210, 103, 2], [212, 103, 2][216, 103, 2],$
$[213, 104, 1], [217, 104, 1], [214, 107, 12], [216, 108, 10], [218, 109, 9], [220, 110, 7]$
$[222, 111, 5], [224, 112, 3], [226, 113, 7], [250, 113, 1], [252, 113, 2], [228, 114, 5], [253, 114, 1],$
$[230, 115, 3], [232, 116, 1], [346, 173, 1], [378, 181, 1], [380, 181, 2], [381, 182, 1], [392, 193, 2],$
$[393, 194, 1], [396, 197, 1], [398, 199, 3], [400, 200, 1], [552, 271, 5], [553, 272, 1], [555, 272, 2],$
$[556, 273, 1], [554, 277, 3], [558, 277, 5], [556, 278, 1], [559, 278, 4], [560, 279, 3], [561, 280, 1],$
$[562, 281, 7], [564, 282, 5], [566, 283, 5], [576, 283, 1], [568, 284, 3], [570, 285, 1], [586, 293, 1].$

With the above notations, we shall use the following theorem due to Laishram and Shorey [5, Theorem 3].

Theorem 2.C. *Let $n \geqslant 2k \geqslant 6$ and $f_1 < f_2 < \cdots < f_\mu$ be integers in $[0, k)$. Assume that the greatest prime factor of $(n - f_1) \dots (n - f_\mu) \leqslant k$. Then $\mu \leqslant k - \left[\frac{3}{4}\pi(k)\right] + 1 - \delta(k)$ unless $(n, k) \in S \cup T$.*

The following corollary is an immediate consequence of Theorem 2.C.

Corollary 2.D. *Let n and k be positive integers with $n \geqslant 2k \geqslant 38$. Then there are at least five distinct terms of the product $n(n-1)\cdots(n-k+1)$ each divisible by a prime exceeding k except when $(n, k) \in T$.*

For the proof of Theorem 1.3, we need the following propositions.

Proposition 2.1. *Let $k \geqslant 6$ and $n > k^2$. Then there exist two distinct terms $n+r$ and $n+s$ of the product $n(n+1)\cdots(n+k-1)$ which are divisible by primes $> k$ exactly to an odd power.*

Proof. Suppose the proposition is false for some n and k with $k \geqslant 6$ and $n > k^2$. Let $\Delta(n,k) = n(n+1)\cdots(n+k-1)$. Thus either $\mathrm{ord}_p(\Delta(n,k))$ is even for all primes $p > k$ or there is exactly one term $n+i$ and a prime $p > k$ such that $\mathrm{ord}_p(\Delta(n,k))$ is odd. The first possibility is excluded since for any positive integer b with $P(b) \leqslant k$, the equation

$$n(n+1)\cdots(n+k-1) = by^2$$

has no solution in positive integers n, k, y when $n > k^2 \geqslant 4^2$ by [7, Theorem A]. We now consider the case when there is exactly a term $n+i$ and a prime $p > k$ such that $\mathrm{ord}_p(\Delta(n,k))$ is odd. Suppose first that $0 < i < k-1$. Removing the term $n+i$ from $\Delta(n,k)$, we see that $n(n+1)\cdots(n+i-1)(n+i+1)\cdots(n+k-1) = b_1y_1^2$ where $P(b_1) \leqslant k$ which is impossible by virtue of [7, Theorem 2[2]].

It remains to consider the case when $i = 0$ or $k-1$. Let Δ' denote the product $(n+1)\cdots(n+k-1)$ or $n(n+1)\cdots(n+k-2)$ according as $i = 0$ or $k-1$. Then Δ' is a product of $k-1$ consecutive integers such that

$$\Delta' = b_2y_2^2 \tag{3}$$

with $P(b_2) \leqslant k$. This is impossible when $P(b_2) \leqslant k-1$ by [7, Theorem A]. It only remains to deal with the situation when $P(b_2) = k$. Then k will be a prime dividing only one term of the product Δ', say k divides $n+j, j \neq i$. We remove the term $n+j$ of the product Δ' and it is clear from (3) that

$$\frac{\Delta'}{n+j} = b_3y_3^2\ , \quad P(b_3) \leqslant k-2. \tag{4}$$

It is immediate from (4) and [7, Theorem 2] that $n+j$ is the first or last term of the product Δ' as $k-1 \geqslant 5$. Thus we see that $\frac{\Delta'}{n+j}$ is the product of $k-2$ consecutive integers. This is impossible by [7, Theorem A].

Proposition 2.2. *Let n, k be positive integers with $n \geqslant k+2 \geqslant 4$ and $F_{n,k}(x)$ be given by (2). Suppose there exists a prime $p > k$ such that $p^e || (n-l)$ for some l, $1 \leqslant l \leqslant k-1$. Let $d =$gcd(e,l) and $d' =$gcd$(e,k-l)$. Then the following hold.*

(i) The edges of the Newton polygon of $F_{n,k}(x)$ with respect to p have slopes $\frac{-e}{k-l}$, $\frac{e}{l}$.

[2] It states that for $n > k^2 \geqslant 5^2$ the equation $n(n+1)\cdots(n+i-1)(n+i+1)\cdots(n+k-1) = by^2$ has no solution in positive integers n, k, b, y with $P(b) \leqslant k$ and $0 < i < k-1$.

(ii) $F_{n,k}(x)$ *has at least two distinct irreducible factors over* $\mathbb{Q}_p$*; one of them has degree a multiple of* $\frac{l}{d}$ *and other has degree a multiple of* $\frac{k-l}{d'}$.
(iii) If $d = d' = 1$*, then* $F_{n,k}(x)$ *factors over* $\mathbb{Q}_p$ *as a product of two distinct irreducible polynomials of degrees* l *and* $k-l$.

Proof. We consider the Newton polygon of $F_{n,k}(x)$ with respect to the prime p. In view of (1), the vertices of the Newton polygon are $(0, e), (k-l, 0), (k, e)$. Thus the Newton polygon has two edges, one from $(0, e)$ to $(k-l, 0)$ and other from $(k-l, 0)$ to (k, e) with respective slopes $\frac{-e}{k-l}$ and $\frac{e}{l}$ proving (i).

Note that equations of the two edges are given by:

$$y - e = \frac{-e}{k-l}\, x \ \text{ and } \ y = \frac{e}{l}\, (x - k + l).$$

On the first edge, the x-coordinates of the lattice points occur at multiples of $\frac{k-l}{d'}$, i.e., when $x = \frac{k-l}{d'}.M$ where $0 \leqslant M \leqslant d'$; on the second edge the x-coordinates of lattice points are given by $k - l + \frac{l}{d}.N$ where $0 \leqslant N \leqslant d$. By Theorem 2.B, all the roots of an irreducible factor of $F_{n,k}(x)$ over $\mathbb{Q}_p$ have the same valuation. Since the slopes of the two edges as shown in (i) are different, we see that the Newton polygon with respect to p of any irreducible factor of $F_{n,k}(x)$ over $\mathbb{Q}_p$ must lie on the first edge or on the second edge. Hence assertion (ii) now follows from Theorem 2.A. Assertion (iii) is an immediate consequence of (ii). The last assertion quickly yields the following result.

Corollary 2.3. *If for a pair* (n, k), $n \geqslant k+2$*, there exist terms* $n - l'$, $n - l''$, $1 \leqslant l' < l'' < k$*, divisible respectively by primes* p', p'' *exceeding* k *exactly to the first power such that* $l' + l'' \neq k$*, then* $F_{n,k}(x)$ *is irreducible over* $\mathbb{Q}$.

The following proposition is already known (cf. [2, Lemma 1]). For the sake of reader's convenience, it is proved here.

Proposition 2.4. *Let* n, k *and* $F_{n,k}(x)$ *be as in Proposition 2.2. Let* p *be a prime* $> k$ *and* $e > 0$ *be such that* $p^e || n$*. Then every irreducible factor of* $F_{n,k}(x)$ *over* $\mathbb{Q}_p$ *has degree a multiple of* $\frac{k}{D}$*, where* $D =$*gcd*(e, k).

Proof. The vertices of the Newton polygon of $F_{n,k}(x)$ with respect to p are $(0, e), (k, 0)$. Thus the Newton polygon has only one edge whose equation is given by $y - e = \frac{-e}{k}\, x$. The x-coordinates of the lattice points on this edge occur at multiples of k/D. So arguing as in Proposition 2.2, any irreducible factor of $F_{n,k}(x)$ must have degree a multiple of k/D.

3. Proof of Theorem 1.3

As pointed out in the proof of Proposition 2.2 (with d, d' atmost 2), if (x, y) is a lattice point on the Newton polygon of $F_{n,k}(x)$ with respect to p, then $x \in X = \{0, \frac{k-l}{2}, k-l, k-\frac{l}{2}, k\}$. By Theorems 2.A, 2.B, each irreducible factor of $F_{n,k}(x)$ over $\mathbb{Q}$ must have degree equal to a sum of numbers (may be one of the numbers) from

$$l/2, l/2, (k-l)/2, (k-l)/2;$$

these correspond to possible differences $x_i - x_{i-1}$ in Theorem 2.B, with the actual differences possibly formed from sums of these possible differences. Thus an irreducible factor of $F_{n,k}(x)$ over $\mathbb{Q}$ must have degrees in the set

$$\left\{\frac{l}{2}, l, \frac{k}{2}, \frac{k-l}{2}, k-l, \frac{2k-l}{2}, \frac{k+l}{2}, k\right\}.$$

Given that $l < k$, the elements of this set that can be less than $k/2$ are $l/2, l, (k-l)/2$ and $k-l$. The conditions in Theorem 1.3 imply that l_1 is not among $l/2, l, (k-l)/2$ and $k-l$, so the theorem follows.

4. Proof of Theorem 1.2

With S and T as in Theorem 2.C, we first prove

Lemma 4.1. *For $(n, k) \in S \cup T$, $k \geqslant 4$, $F_{n,k}(x)$ is irreducible over $\mathbb{Q}$ except possibly when (n, k) belongs to the subset S' of S given by $S' = \{(10, 5), (12, 6), (16, 8)\}$.*

Proof. Let S'' denote the subset of S given by $S'' = \{(9, 4), (12, 5), (16, 5), (18, 5), (27, 5)\}$. Observe that if n is divisible by a prime $p > k$ with $\mathrm{ord}_p(n) = 1$, then $x^k F_{n,k}(1/x)$ is an Eisenstein polynomial with respect to p and so $F_{n,k}(x)$ is irreducible over $\mathbb{Q}$. Further if two distinct terms $n - l_1, n - l_2$ of the product $n(n-1)\cdots(n-k+1)$ are divisible by primes p_1 and p_2 exceeding k such that $\mathrm{ord}_{p_i}(n - l_i) = 1$ and $l_1 + l_2 \neq k$, then in view of the above observation and Corollary 2.3, $F_{n,k}(x)$ is irreducible over $\mathbb{Q}$. For each (n, k) belonging to $T \cup (S \setminus S' \cup S'')$ with n not divisible by any prime $> k$ up to the first power, Table 1 at the end of this section indicates two primes p_1 and p_2 satisfying the above property. It can be easily seen that for $(n, k) \in S''$, $F_{9,4}(x)$ is an Eisenstein polynomial with respect to the prime 5, $F_{12,5}(x)$ is Eisenstein with respect to 7, $F_{16,5}(x), F_{27,5}(x)$ are Eisenstein with respect to 11 and $F_{18,5}(x)$ is Eisenstein with respect to 13. Hence the lemma is proved.

Lemma 4.2. *For $8 \leqslant n < 5^3$, the polynomial $F_{n,4}(x)$ is irreducible over $\mathbb{Q}$ except when n belongs to the set $U = \{8, 50, 98, 100\}$.*

Proof. As pointed out in the proof of Lemma 4.1, we need to verify the irreducibility of $F_{n,4}(x)$ when n is not divisible by any prime more than 4 exactly with the first power. For such n not exceeding 124 and n not belonging to the set $\{8, 9, 18, 27, 50, 98, 100\}$, Table 2 at the end of this section indicates two terms $n - l', n - l''$, $1 \leqslant l', l'' \leqslant 3$, $l' + l'' \neq 4$ such that $n - l', n - l''$ are divisible by primes p', p'' (respectively) up to the first power only. So the lemma is proved in view of Corollary 2.3 and the fact that $F_{9,4}(x)$, $F_{18,4}(x)$ and $F_{27,4}(x)$ are Eisenstein polynomials with respect to the primes $5, 7$ and 23 respectively.

Proof of Theorem 1.2. We divide the proof into two cases.

Case I. $8 \leqslant 2k \leqslant n < (k+1)^2$. Note that the theorem is already proved in the present case for $k = 4$ by virtue of Lemma 4.2, so it may be assumed that $k \geqslant 5$ here. Applying Theorem 2.C, we see that there exist at least three terms $n - l_i$, $i \in \{1, 2, 3\}$ which are divisible by primes exceeding k exactly up to the first power unless $(n, k) \in S \cup T$. Using Proposition 2.2 (*iii*), $F_{n,k}(x)$ factors over $\mathbb{Q}_{p_i}$ as a product of two non-associate irreducible polynomials of degree l_i and $k - l_i$ for $1 \leqslant i \leqslant 3$. If $F_{n,k}(x)$ were reducible over $\mathbb{Q}$, then $F_{n,k}(x)$ will have a factorization of the type $F_{n,k}(x) = a_k c_k G_i(x) H_i(x)$ where $G_i(x), H_i(x)$ are monic irreducible polynomials belonging to $\mathbb{Q}[x]$ with degrees $k - l_i$, l_i respectively. This is impossible as l_1, l_2 and l_3 are distinct. So the theorem is proved in the present case when (n, k) does not belong to $S \cup T$. When $(n, k) \in (S \setminus S') \cup T$ with $k \geqslant 4$, the irreducibility of $F_{n,k}(x)$ follows from Lemma 4.1.

Case II. $k \geqslant 4$, $(k+1)^2 \leqslant n < (k+1)^3$. In this case, we first show that $F_{n,k}(x)$ cannot factor over $\mathbb{Q}$ as a product of two irreducible polynomials of degree $\frac{k}{2}$ each. For this it is enough to show that there exists $l' \neq k/2$, $0 \leqslant l' \leqslant k - 1$ such that $n - l'$ is divisible by a prime $p' > k$ exactly with the first power. If $l' = 0$, then as pointed out in the opening lines of the proof of Lemma 4.1, $F_{n,k}(x)$ is irreducible over $\mathbb{Q}$. If $l' \geqslant 1$ then by Proposition 2.2 (*iii*), $F_{n,k}(x)$ has two irreducible factors of degree l' and $k - l'$ over $\mathbb{Q}_{p'}$. This leads to a contradiction as $l' \neq k/2$ thereby proving the irreducibility of $F_{n,k}(x)$ over $\mathbb{Q}$. The existence of a term $n - l' \neq n - \frac{k}{2}$, $0 \leqslant l' \leqslant k - 1$, which is divisible by some prime $p' > k$ with $\mathrm{ord}_{p'}(n - l') = 1$ is guaranteed for $k \geqslant 6$ by Proposition 2.1 as $(k+1)^2 \leqslant n < (k+1)^3$ in the present situation. This proves the assertion stated in the opening lines of Case II.

It only remains to be shown that $F_{n,k}(x)$ cannot have a factor of degree less than $k/2$ over $\mathbb{Q}$. Suppose to the contrary that it has a factor of degree $l_1 < k/2$ over $\mathbb{Q}$. We make some claims.

Claim 1: $P(n) \leqslant k$.
Suppose not. Let p be a prime $> k$ dividing n with exact power $e \geqslant 1$. Then $e \leqslant 2$ since $n < (k+1)^3$. So by Proposition 2.4, every irreducible factor of $F_{n,k}(x)$ over $\mathbb{Q}_p$ has degree a multiple of k or $\frac{k}{2}$ according as $e = 1$ or 2 respectively. This is not possible in view of our supposition.
Claim 2: There are at most four distinct terms in the product $n(n-1)\cdots(n-k+1)$ each of which is divisible by some prime $> k$.
Assume the contrary. Then there is a term $n - l$ with $0 \leqslant l < k$ and a prime $p > k$ with p dividing $(n - l)$ such that $l \notin \{l_1,\ 2l_1,\ k - l_1,\ k - 2l_1\}$ where l_1 is as in the paragraph preceeding Claim I. Note that $l > 0$ in view of Claim 1. Further $e = \mathrm{ord}_p(n-l) \leqslant 2$ implying that $F_{n,k}(x)$ cannot have a factor of degree l_1 over $\mathbb{Q}$ by Theorem 1.3, which contradicts our assumption.
Claim 3: There are at most two distinct terms in the product $n(n-1)\cdots(n-k+1)$ which are divisible by a prime $> \sqrt{n}$.
Suppose not. Let $1 \leqslant l_1' < l_2' < l_3'$ be such that there exist primes $p_i > \sqrt{n}$ dividing $n - l_i'$. Note that $\mathrm{ord}_{p_i}(n - l_i') = 1$ for $i \in \{1,2,3\}$. Since $(k+1)^2 \leqslant n$, in view of Proposition 2.2 (iii), it follows that $F_{n,k}(x)$ factors over $\mathbb{Q}_{p_i}$ as a product of two non-associate irreducible polynomials of degree l_i' and $k - l_i'$, $1 \leqslant i \leqslant 3$. Arguing as in Case I, we get a contradiction because l_1', l_2' and l_3' are distinct.

From Claim 2, Corollary 2.D and Lemma 4.1, it follows that $k \leqslant 18$. Note that for $k = 4$, in view of Lemma 4.2, we have only to consider $n = 50, 98, 100$ as $5^2 \leqslant n < 125$. For each of these values of n, $F_{n,k}(x)$ must be irreducible over $\mathbb{Q}$ by virtue of Claim 1, as $P(n)$ is more than 4. For $k \geqslant 5$, by virtue of Claim 1, we may first restrict to those n for which $P(n) \leqslant k$. Further by Claims 2 and 3, those n can be excluded for which $n(n-1)\cdots(n-k+1)$ has either five terms divisible by a prime $> k$ or three terms divisible by a prime $> \sqrt{n}$. We use *Sage* mathematics software for the above computations. Then we are left with the following pairs (n, k) given by

$$(50,5), (64,5), (100,5), (128,5), (200,5), (50,6).$$

All these pairs satisfy the hypothesis of Corollary 2.3 as is clear from Table 3. This completes the proof of the theorem.

Table 1.

$(n,k) \in [n,k,h] \to Primes$		$(n,k) \in [n,k,h] \to Primes$		$(n,k) \in [n,k,h] \to Primes$	
$[20,5,1]$	$17,19$	$[162,79,1]$	$131,139$	$[346,173,1]$	$293,307$
$[20,6,1]$	$17,19$	$[166,83,1]$	$131,139$	$[378,181,1]$	$293,307$
$[14,7,3]$	$11,13$	$[172,83,1]$	$137,139$	$[380,181,2]$	$293,307$
$[18,7,1]$	$13,17$	$[190,83,1]$	$131,139$	$[381,182,1]$	$293,307$
$[20,7,1]$	$17,19$	$[192,83,1]$	$131,139$	$[392,193,2]$	$293,307$
$[21,7,1]$	$17,19$	$[178,89,1]$	$131,139$	$[393,194,1]$	$293,307$
$[30,7,1]$	$13,29$	$[190,89,1]$	$131,139$	$[396,197,1]$	$293,307$
$[21,8,1]$	$17,19$	$[192,89,1]$	$139,149$	$[398,199,3]$	$293,307$
$[26,13,3]$	$19,23$	$[210,103,1]$	$139,149$	$[400,200,1]$	$283,307$
$[30,13,1]$	$19,23$	$[212,103,2]$	$139,149$	$[552,271,5]$	$421,431$
$[32,13,2]$	$29,31$	$[216,103,2]$	$139,149$	$[553,272,1]$	$421,431$
$[36,13,1]$	$29,31$	$[213,104,1]$	$139,149$	$[555,272,2]$	$421,431$
$[28,14,1]$	$17,19$	$[217,104,1]$	$139,149$	$[556,273,1]$	$421,431$
$[33,14,1]$	$29,31$	$[214,107,12]$	$139,149$	$[554,277,3]$	$421,431$
$[36,17,1]$	$29,31$	$[216,108,10]$	$139,149$	$[558,277,5]$	$421,431$
$[38,19,3]$	$23,29$	$[218,109,9]$	$139,149$	$[556,278,1]$	$421,431$
$[42,19,1]$	$37,41$	$[220,110,7]$	$139,149$	$[559,278,4]$	$421,431$
$[40,20,1]$	$31,37$	$[222,111,5]$	$139,149$	$[560,279,3]$	$421,431$
$[94,47,3]$	$89,83$	$[224,112,3]$	$139,149$	$[561,280,1]$	$421,431$
$[100,47,1]$	$83,89$	$[226,113,7]$	$139,149$	$[562,281,7]$	$409,431$
$[96,48,1]$	$79,83$	$[250,113,1]$	$139,149$	$[564,282,5]$	$409,431$
$[144,71,2]$	$101,103$	$[252,113,2]$	$139,149$	$[566,283,5]$	$421,431$
$[145,72,1]$	$101,103$	$[228,114,5]$	$139,149$	$[576,283,1]$	$421,431$
$[146,73,3]$	$101,103$	$[253,114,1]$	$139,149$	$[568,284,3]$	$419,431$
$[156,73,1]$	$109,113$	$[230,115,3]$	$139,149$	$[570,285,1]$	$421,431$
$[148,74,1]$	$107,113$	$[232,116,1]$	$139,149$	$[586,293,1]$	$421,431$

Table 2.

$n \ \to$	$n-l', n-l''$,	p', p''	$n \ \to$	$n-l', n-l''$,	p', p''	$n \ \to$	$n-l', n-l''$,	p', p''
12	$10,11,$	$5,11$	48	$46,47,$	$23,47$	81	$79,80,$	$79,5$
16	$14,15,$	$7,5$	49	$46,47,$	$23,47$	96	$94,95,$	$47,19$
24	$22,23,$	$11,23$	54	$52,53,$	$13,53$	108	$106,107,$	$53,107$
25	$22,23,$	$11,23$	64	$62,63,$	$31,7$	121	$119,120,$	$17,5$
32	$30,31,$	$5,31$	72	$70,71,$	$5,71$			
36	$34,35,$	$17,5$	75	$73,74,$	$73,37$			

Table 3.

(n,k)	$\rightarrow$	$n-l', n-l''$	(n,k)	$\rightarrow$	$n-l', n-l''$	(n,k)	$\rightarrow$	$n-l', n-l''$
$(50,5)$		$46,47$	$(100,5)$		$97,99$	$(200,5)$		$197,199$
$(64,5)$		$61,63$	$(128,5)$		$126,127$	$(50,6)$		$46,47$

5. Proof of Theorem 1.1

In view of Theorem 1.2., we need to prove the irreducibility of $P_{n,k}(x)$ only when $1 \leqslant k \leqslant 3$ with $2k \leqslant n < (k+1)^3$ or (n,k) belongs to $\{(8,4),(10,5),(12,6),(16,8)\}$. Using Maple, we have verified the irreducibility of $P_{n,k}(x)$ for these values of (n,k).

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