On invariants and strict systems of irreducible polynomials over henselian valued fields*

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ABSTRACT. Let $g(x)$ be a monic irreducible defectless polynomial over a henselian valued field (K, v), i.e., $K(\theta)$ is a defectless extension of (K, v) for any root θ of $g(x)$. It is known that a complete distinguished chain for θ with respect to (K, v) gives rise to several invariants associated with $g(x)$. Recently Ron Brown studied certain invariants of defectless polynomials by introducing strict systems of polynomial extensions. In this paper, the authors establish a one-to-one correspondence between strict systems of polynomial extensions and conjugacy classes of complete distinguished chains. This correspondence leads to a simple interpretation of various results proved for strict systems. The authors give new characterizations of an invariant γ_g introduced by Brown.

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1. INTRODUCTION

Throughout v is a henselian valuation of arbitrary rank of a field K and $\tilde{v}$ is the unique prolongation of v to the algebraic closure $\widetilde{K}$ of K. For a subfield L of $\widetilde{K}$, $G(L)$, $R(L)$ will stand respectively for the value group and the residue field of the valuation of L obtained by restricting $\tilde{v}$. A finite extension $(K', v')/(K, v)$ (or briefly K'/K) is called defectless if $[K' : K] = ef$, where e, f are the index of ramification and the residual degree of v'/v. An irreducible polynomial $h(x)$ in $K[x]$ will be referred to as defectless if $K(\alpha)$ is a defectless extension of (K, v) for any root α of $h(x)$. Popescu, Zaharescu (Popescu and Zaharescu, 1995), Ota (Ota, 1999), Aghigh and Khanduja (Aghigh and Khanduja, 2002, 2005) have shown that one can associate several invariants to a defectless polynomial by means of complete distinguished chains defined below.

Recall that a pair (θ, α) of elements of $\widetilde{K}$ is called a distinguished pair (more precisely a (K, v)-distinguished pair) if the following three conditions are satisfied: (i) $[K(\theta) : K] > [K(\alpha) : K]$; (ii) $\tilde{v}(\theta - \beta) \leqslant \tilde{v}(\theta - \alpha)$ for every β in $\widetilde{K}$ with $[K(\beta) : K] < [K(\theta) : K]$; (iii) whenever β belonging to $\widetilde{K}$ is such that $[K(\beta) : K] < [K(\alpha) : K]$, then $\tilde{v}(\theta - \beta) < \tilde{v}(\theta - \alpha)$.

Distinguished pairs give rise to distinguished chains in a natural manner. A chain $\theta = \theta_0, \theta_1, \ldots, \theta_n$ of elements of $\widetilde{K}$ will be called a complete distinguished chain for θ (with respect to (K, v)) if (θ_i, θ_{i+1}) is a (K, v)-distinguished pair for $0 \leqslant i \leqslant n-1$ and θ_n belongs to K. In 2005, Aghigh and Khanduja (Aghigh and Khanduja, 2005) characterized those elements θ of $\widetilde{K} \setminus K$ for which there exists a complete distinguished chain with θ as the first element. Indeed they proved

Theorem 1.A. *An element θ belonging to $\widetilde{K} \setminus K$ has a complete distinguished chain with respect to a henselian valued field (K, v) if and only if $K(\theta)$ is a defectless extension of (K, v).*

Two complete distinguished chains $\theta = \theta_0, \theta_1, \ldots, \theta_n$ and $\theta = \theta'_0, \theta'_1, \ldots, \theta'_n$ for θ are said to lie in the same conjugacy class if θ'_i is a K-conjugate of θ_i for $1 \leqslant i \leqslant n$. Note that the invariants given by the following theorem proved in (Aghigh and Khanduja, 2005) are the same for all complete distinguished chains of θ which lie in the same conjugacy class and hence are invariants of the minimal polynomial of θ over K.

Theorem 1.B. *Let (K, v) and $(\widetilde{K}, \tilde{v})$ be as above. Let $\theta = \theta_0, \theta_1, \ldots, \theta_n$ and $\theta = \eta_0, \eta_1, \ldots, \eta_m$ be two complete distinguished chains with respect to (K, v) for an element θ belonging to $\widetilde{K} \setminus K$. Then $n = m$, $[K(\theta_i) : K] = [K(\eta_i) : K]$, $G(K(\theta_i)) = G(K(\eta_i))$, $R(K(\theta_i)) = R(K(\eta_i))$ for $1 \leqslant i \leqslant n$. If $f_i(x)$ and $F_i(x)$ denote respectively the minimal polynomials of θ_i, η_i over K, then $\tilde{v}(f_i(\theta_{i-1})) = \tilde{v}(F_i(\eta_{i-1}))$, $1 \leqslant i \leqslant n$.*

In 2008, Ron Brown (Brown, 2009) introduced the notion of a strict system of polynomial extensions. Recently he and J. Merzel studied invariants of defectless polynomials using these strict systems defined below. They also developed some connections between the two approaches and gave several applications of strict systems (cf. Brown and Merzel). In this paper, our aim is to complete the work begun by Brown and Merzel towards establishing the equivalence of the two approaches and to use the results about chains for studying properties of strict systems and vice versa. With the help of this equivalence, we determine explicitly (see Theorem 1.3, Corollary 1.4) the best possible constant λ_g associated to any defectless polynomial $g(x)$ over a henselian valued field (K, v) satisfying the property that whenever $\tilde{v}(g(\beta)) > \lambda_g$, β in $\widetilde{K}$, then some root of $g(x)$ comes sufficiently close to β; in the particular case when $g(x)$ is a tame polynomial, i.e., $K(\theta)$ is a tamely ramified extension of (K, v) for a root θ of $g(x)$, then the above result implies that $K(\beta)$ contains a root of $g(x)$ which yields a result of Brown proved in (Brown, 2009). This invariant λ_g turns out to be equal to the invariant γ_g defined in (Brown, 1972; 2009) and hence gives a new characterization of the invariant. Recall that a finite defectless extension $(K', v')/(K, v)$ is said to be tamely ramified if the residue field of v' is a separable extension of the residue field of v and the ramification index of v'/v is not divisible by the characteristic of the residue field of v.

We shall denote by $\mathbb{Q}G$ a fixed divisible hull of the value group G of v. By an extension w of v to $K[x]$, we mean a mapping

$$w : K[x] \to \mathbb{Q}G \cup \{\infty\}$$

satisfying $w(f+g) \geqslant \min\{w(f), w(g)\}$, $w(fg) = w(f) + w(g)$ for all f, g in $K[x]$, with $w^{-1}(\infty)$ not necessarily the zero ideal. If $w^{-1}(\infty)$ is a non-zero (prime) ideal I, then w gives rise to a Krull valuation w_I of the field $K[x]/I$. We shall denote by K_w the residue field of w_I and by $\tau_w : K[x] \to K_w \cup \{\infty\}$ the associated place.

Definition. Suppose that $n \geqslant 0$. A strict system of polynomial extensions over

(K, v) of length $n+1$ is a finite sequence $(g_0, w_0, \gamma_0), (g_1, w_1, \gamma_1), \ldots, (g_{n+1}, w_{n+1}, \gamma_{n+1})$ where each w_i is an extension of v to $K[x]$ and $\gamma_i \in \mathbb{Q}G \cup \{-\infty\}$ such that the following properties are satisfied:

(A) $g_0 = x - a$, $a \in K$, $\gamma_0 = -\infty$, $w_0(h) = v(h(a))$ for every $h \in K[x]$,

and for $0 \leqslant i \leqslant n$,

(B) $deg\ g_{i+1} > deg\ g_i$, $deg\ g_i$ divides $deg\ g_{i+1}$,

(C) $\gamma_{i+1} = w_i(g_{i+1})$,

(D) $w_{i+1}(g_{i+1}) = \infty$,

(E) the g_i-expansion of g_{i+1} given by $g_{i+1} = g_i^{d_i} + \sum_{r<d_i} A_r g_i^r$ $(deg\ A_r < deg\ g_i)$ satisfies $\frac{w_i(A_r)}{d_i - r} \geqslant \frac{w_i(A_0)}{d_i} > \gamma_i$ for all $r < d_i$,

(F) If e_i is the least positive integer such that $e_i w_i(A_0) \in d_i w_i(K[x])$ and $l_i = d_i/e_i$, then the polynomial

$$Y^{l_i} + \sum_{r<l_i} \tau_{w_i}(s^{l_i - r} A_{e_i r}) Y^r$$

is irreducible over K_{w_i} for all s in $K[x]$ with $w_i(A_0 s^{l_i}) = 0$.

In 2009, R. Brown and J. Merzel raised the following problem:
Given a strict system $(g_0, w_0, \gamma_0), (g_1, w_1, \gamma_1), \ldots, (g_{n+1}, w_{n+1}, \gamma_{n+1})$ over (K, v), does there exist a root θ_i of g_{n+1-i} such that $\theta_0, \theta_1, \ldots, \theta_{n+1}$ is a complete distinguished chain with respect to (K, v)?

They proved that the answer to the above question is "yes" when $n = 1$ or when each g_i is a tame polynomial (cf. Brown and Merzel). In this paper, we prove that the answer to the above question is always in the affirmative. Indeed our key result is the following.

Theorem 1.1. *Let $(g_0, w_0, \gamma_0), (g_1, w_1, \gamma_1), \ldots, (g_{n+1}, w_{n+1}, \gamma_{n+1})$ be a strict system of polynomial extensions over a henselian valued field (K, v) of arbitrary rank. Then for each i, one can choose a root θ_{n+1-i} of g_i such that $\theta_0, \theta_1, \ldots, \theta_{n+1}$ is a complete distinguished chain with respect to (K, v).*

The corollary stated below which is already proved in (Brown and Merzel) as Theorem 9.3 is an immediate consequence of the above theorem in view of Theorem 1.A.

Corollary 1.2. *If an irreducible polynomial $g(x)$ with coefficients in a henselian valued field (K, v) belongs to a strict system of polynomial extensions (i.e., $g(x) =$*

$g_i(x)$ for some i), then $g(x)$ is a defectless polynomial.

The converse of Theorem 1.1 is given by the following result which is proved in (Brown and Merzel, Theorem 9.1). We omit its proof.

Theorem 1.C. *Let $\theta_0, \theta_1, \ldots, \theta_{n+1}$ be a complete distinguished chain with respect to (K, v) and g_i be the minimal polynomial of θ_{n+1-i} over K, $0 \leqslant i \leqslant n+1$. If w_i denotes the extension of v to $K[x]$ defined for any $q(x)$ in $K[x]$ by $w_i(q(x)) = \tilde{v}(q(\theta_{n+1-i}))$ and $\gamma_0 = -\infty$, $\gamma_{i+1} = w_i(g_{i+1})$, then $(g_0, w_0, \gamma_0), (g_1, w_1, \gamma_1), \ldots,$ $(g_{n+1}, w_{n+1}, \gamma_{n+1})$ is a strict system of polynomial extensions over (K, v).*

The one-to-one correspondence between strict systems of polynomial extensions and conjugacy classes of chains follows immediately from Theorems 1.1 and 1.C. As an application of this correspondence, we shall prove the following theorem which in turn yields Corollaries 1.4, 1.5 proved respectively in (Brown, 2009, Theorem 1; Brown and Merzel, Theorem 7.1) by different methods.

For any $\theta \in \widetilde{K} \setminus K$, $\delta_K(\theta)$ will stand for the main invariant associated with θ defined by

$$\delta_K(\theta) = \sup\{\tilde{v}(\theta - \alpha) \mid \alpha \in \widetilde{K},\ deg\ \alpha < deg\ \theta\}. \tag{1}$$

Theorem 1.3. *Let $K(\theta)$ be a defectless extension of (K, v) and $g(x)$ be the minimal polynomial of θ over K. If $\theta = \theta_0, \theta_1, \ldots, \theta_n$ is a complete distinguished chain for θ, then given any β in $\widetilde{K}$ with $\tilde{v}(g(\beta)) > \tilde{v}(g(\theta_1))$, there exists a K-conjugate θ' of θ such that $\tilde{v}(\theta' - \beta) > \delta_K(\theta)$. Moreover the constant $\lambda_g = \tilde{v}(g(\theta_1))$ depends only on $g(x)$ and is the least element λ of $G(\widetilde{K})$ such that for any β in $\widetilde{K}$ with $\tilde{v}(g(\beta)) > \lambda$, there exists a K-conjugate θ' of θ satisfying $\tilde{v}(\theta' - \beta) > \delta_K(\theta)$.*

It may be pointed out that the above theorem gives a new characterization of the invariant (denoted here by λ_g) introduced by Brown which is denoted by γ_g in (Brown, 1972; 2009). Observe that for any monic defectless polynomial $g(x) = g_{n+1}(x) \in K[x]$, with notations as in Theorems 1.1 and 1.C, in view of part (C) of definition of strict systems, we have

$$\gamma_{n+1} = w_n(g_{n+1}) = \tilde{v}(g_{n+1}(\theta_1)) = \lambda_g,$$

which gives λ_g in terms of strict systems as well as complete distinguished chains.

Corollary 1.4. *Let the hypothesis be as in Theorem 1.3. Assume in addition that either $K(\theta)$ or $K(\beta)$ is a tamely ramified extension of (K, v). Then $K(\beta)$ contains a root of $g(x)$.*

Corollary 1.5. *If* $(g_0, w_0, \gamma_0), (g_1, w_1, \gamma_1), \ldots, (g_n, w_n, \gamma_n)$ *and* (h_0, w'_0, γ'_0), $(h_1, w'_1, \gamma'_1), \ldots, (h_m, w'_m, \gamma'_m)$ *are two strict systems of polynomial extensions over a henselian valued field* (K, v) *with* $g_n = h_m$, *then* $n = m$ *and for* $0 \leqslant i \leqslant n$, *deg* $g_i =$ *deg* h_i, $w_i(K[x]) = w'_i(K[x])$, *the residue field of* w_i *is equal to the residue field of* w'_i *and* $\gamma_i = \gamma'_i$.

2. SOME PRELIMINARY RESULTS

We retain the notations of the previous section. For an element α of $\widetilde{K}$, *deg* α will stand for the degree of the extension $K(\alpha)/K$. For any ξ in the valuation ring of $\tilde{v}$, $\bar{\xi}$ will denote the $\tilde{v}$-residue of ξ, i.e., the image of ξ under the canonical homomorphism from the valuation ring of $\tilde{v}$ onto its residue field. When there is no chance of confusion, we shall write $\tilde{v}(\alpha)$ as $v(\alpha)$ for α in $\widetilde{K}$.

Lemma 2.D. *Let* η, α *be elements of* $\widetilde{K}$ *such that* $\tilde{v}(\alpha - \eta) > \tilde{v}(\alpha - \beta)$ *for every* β *in* $\widetilde{K}$ *with deg* $\beta <$ *deg* α. *Then for any polynomial* $h(x) \in K[x]$ *of degree less than deg* α, *one has* $\overline{\left(\frac{h(\eta)}{h(\alpha)}\right)} = \bar{1}$.

Proof. Write $h(x) = c\prod_j (x - \beta_j)$. Then $\frac{h(\eta)}{h(\alpha)} = \prod_j \left(\frac{\eta - \beta_j}{\alpha - \beta_j}\right) = \prod_j \left(1 + \frac{\eta - \alpha}{\alpha - \beta_j}\right)$. By hypothesis, $\tilde{v}(\frac{\eta - \alpha}{\alpha - \beta_j}) > 0$. Therefore $\tilde{v}(\frac{h(\eta)}{h(\alpha)} - 1) > 0$ and so $\overline{\left(\frac{h(\eta)}{h(\alpha)}\right)} = \bar{1}$.

The following corollary is an immediate consequence of the above lemma.

Corollary. *Let* η, α *be as in Lemma 2.D. Then* $G(K(\alpha)) \subseteq G(K(\eta))$, $R(K(\alpha)) \subseteq R(K(\eta))$.

The following already known results will be used in the sequel (see (Aghigh and Khanduja, 2005) for Lemma 2.E and (Brown, 2009, Proposition 5) for Proposition 2.F). Their proofs are omitted.

Lemma 2.E. *Let* $f(x)$ *and* $g(x)$ *be two monic irreducible polynomials over a henselian valued field* (K, v) *of degree* m *and* n *respectively such that* $f(\alpha) = g(\beta) = 0$. *Then* $n\tilde{v}(f(\beta)) = m\tilde{v}(g(\alpha))$.

Remark. With notations as in Theorem 1.B, the last assertion of this theorem in view of the above lemma and the fact that *deg* $f_i =$ *deg* F_i for every i, can be rewritten as $\tilde{v}(f_{i-1}(\theta_i)) = \tilde{v}(F_{i-1}(\eta_i))$.

Proposition 2.F. Let $(g_0, w_0, \gamma_0), (g_1, w_1, \gamma_1), \ldots, (g_{n+1}, w_{n+1}, \gamma_{n+1})$ be a strict system of polynomial extensions over (K, v). Suppose that w is an extension of

v to $K[x]$ with $w(g_{n+1}) > \gamma_{n+1}$. Then for all i with $0 \leqslant i \leqslant n$,
(i) $w(g_i) = w_{i+1}(g_i)$,
(ii) g_{i+1} is irreducible over K and w_{i+1} is the unique extension of v to $K[x]$ with $w_{i+1}(g_{i+1}) = \infty$.

3. PROOF OF THEOREM 1.1

The following lemma is a consequence of Theorem 10.1 of (Brown and Merzel). For reader's convenience, we give a simple proof of this lemma here.

Lemma 3.1. *Let $(g_0, w_0, \gamma_0), (g_1, w_1, \gamma_1)$ be a strict system of polynomial extensions over a henselian valued field (K, v). If θ and a are roots of $g_1(x), g_0(x)$ respectively, then (θ, a) is a (K, v)-distinguished pair.*

Proof. Write $g_1(x) = (x-a)^d + a_{d-1}(x-a)^{d-1} + \ldots + a_0$. By Proposition 2.F, $g_1(x)$ is irreducible over K. As (K, v) is henselian, we have

$$v(\theta - a) = v(a_0)/d\ ,\quad \frac{v(a_i)}{d-i} \geqslant \frac{v(a_0)}{d}\ ,\ 1 \leqslant i \leqslant d-1. \tag{2}$$

We are required to show that $v(\theta - a) = \sup\{v(\theta - \beta) |\ \beta \in \widetilde{K}\ ,\ deg\ \beta <\ deg\ \theta\}$. For this it is enough to prove that if η belongs to $\widetilde{K}$ and $v(\theta - \eta) > v(\theta - a)$, then $deg\ \eta \geqslant deg\ \theta$. Let η be such that $v(\theta - \eta) > v(\theta - a)$. Then by the strong triangle law, we have

$$v(\eta - a) = \min\{v(\eta - \theta), v(\theta - a)\} = v(\theta - a), \tag{3}$$

$$\overline{\left(\frac{\theta - a}{\eta - a}\right)} = \overline{\left(1 + \frac{\theta - \eta}{\eta - a}\right)} = \bar{1}. \tag{4}$$

Let e be the smallest positive integer such that $ev(\theta - a)$ belongs to the value group G of v, say $ev(\theta - a) = v(h),\ h \in K$. Denote d/e by l. Keeping in mind (2), we have $v\left(\frac{(\theta-a)^i a_i}{h^l}\right) \geqslant 0$; clearly this inequality is strict if $e \nmid i$. So taking the image in the residue field of the equation $\frac{g_1(\theta)}{h^l} = 0$, we see that $\overline{\left(\frac{(\theta-a)^e}{h}\right)}$ satisfies the polynomial $Y^l + \overline{(\frac{a_{d-e}}{h})} Y^{l-1} + \ldots + \overline{(\frac{a_0}{h^l})}$ over the residue field $R(K)$ of v which is an irreducible polynomial by the property (F) of the strict system. Therefore $\overline{\left(\frac{(\theta-a)^e}{h}\right)}$ is algebraic of degree l over $R(K)$. By virtue of (4), $\overline{\left(\frac{(\theta-a)^e}{h}\right)} = \overline{\left(\frac{(\eta-a)^e}{h}\right)}$. Therefore if $e(v'/v),\ f(v'/v)$ denote the ramification index and the residual degree of the valuation v' obtained by restricting $\tilde{v}$ to $K(\eta)$, then $f(v'/v) \geqslant l$; also $e(v'/v) \geqslant e$ in view of (3). It now follows from the fundamental inequality (Engler and Prestel, 2005, Theorem 3.3.4) that $deg\ \eta \geqslant el = d = deg\ \theta$ as desired.

Lemma 3.2. *Let $(g_0, w_0, \gamma_0), (g_1, w_1, \gamma_1), \dots, (g_n, w_n, \gamma_n)$ be a strict system of polynomial extensions over a henselian valued field (K, v) and α, α_1 be roots of g_n, g_{n-1} respectively such that $v(\alpha - \alpha_1) = \delta_K(\alpha)$, with $\delta_K(\alpha)$ defined by (1). If $v(g_n(\beta)) > v(g_n(\alpha_1))$ for some β in $\widetilde{K}$, then there exists a K-conjugate β' of β such that $v(\alpha - \beta') > \delta_K(\alpha)$.*

Proof. Let $h(x)$ denote the minimal polynomial of β over K and w the valuation of $K[x]$ defined for any $q(x)$ belonging to $K[x]$ by $w(q(x)) = v(q(\beta))$. Set $v(g_{n-1}(\alpha)) = \lambda$. By hypothesis, $v(g_n(\beta)) > v(g_n(\alpha_1))$, which in view of Lemma 2.E can be rewritten as

$$v(g_n(\beta)) > v(g_n(\alpha_1)) = \frac{deg\ g_n}{deg\ g_{n-1}} v(g_{n-1}(\alpha)) = \frac{deg\ g_n}{deg\ g_{n-1}} \lambda\ . \tag{5}$$

Now suppose to the contrary that $v(\beta' - \alpha) \leqslant \delta_K(\alpha) = v(\alpha - \alpha_1)$ for every K-conjugate β' of β; then $v(\alpha_1 - \beta') \geqslant \min\{v(\alpha_1 - \alpha), v(\alpha - \beta')\} = v(\alpha - \beta')$; consequently on summing over β', we have

$$v(h(\alpha_1)) \geqslant\ v(h(\alpha)). \tag{6}$$

Note that an application of Lemma 2.E to g_n and h, gives

$$v(g_n(\beta)) = \frac{deg\ g_n}{deg\ h} v(h(\alpha)).$$

In view of (6), the above equation implies that

$$v(g_n(\beta)) \leqslant \frac{deg\ g_n}{deg\ h} v(h(\alpha_1)).$$

By Lemma 2.E, $v(h(\alpha_1)) = \frac{deg\ h}{deg\ g_{n-1}} v(g_{n-1}(\beta))$. Therefore the last inequality now becomes

$$v(g_n(\beta)) \leqslant \frac{deg\ g_n}{deg\ g_{n-1}}\ v(g_{n-1}(\beta)). \tag{7}$$

Recall that α is a root of g_n. In view of Proposition 2.F (ii), $w_n(q(x)) = v(q(\alpha))$ for all $q(x)$ belonging to $K[x]$. Applying this proposition with $i = n-1$, we have $w(g_{n-1}) = w_n(g_{n-1})$, i.e., $v(g_{n-1}(\beta)) = v(g_{n-1}(\alpha)) = \lambda$. Now (7) can be rewritten as

$$v(g_n(\beta)) \leqslant \frac{deg\ g_n}{deg\ g_{n-1}}\ \lambda$$

which contradicts (5) and proves the lemma.

Proof of Theorem 1.1. The result will be proved by induction on the length of the strict system of polynomial extensions. If we have a strict system of length 1, then by Lemma 3.1, we have a corresponding complete distinguished chain (of the type $(\theta, a),\ a \in K$) associated with it. Assume that the result is true for any strict system of length n. Let $(g_0, w_0, \gamma_0), (g_1, w_1, \gamma_1), \ldots, (g_{n+1}, w_{n+1}, \gamma_{n+1})$ be a strict system of length $n+1$ over (K, v). By induction hypothesis, there exists a root α_{n-i} of $g_i,\ 0 \leqslant i \leqslant n$ such that $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n$ is a complete distinguished chain with respect to (K, v). Note that in view of Proposition 2.F(ii), we have

$$w_n(q(x)) = v(q(\alpha)),\quad w_{n-1}(q(x)) = v(q(\alpha_1)),\ q(x) \in K[x]. \tag{8}$$

Choose a root θ of g_{n+1} such that

$$v(\theta - \alpha) = \max\{v(\theta' - \alpha) \mid \theta' \text{ runs over all roots of } g_{n+1}\} = \delta(say).$$

We prove that (θ, α) is a distinguished pair. Firstly it will be shown that

$$\delta > \delta_K(\alpha). \tag{9}$$

For convenience of notation, denote g_{n+1}, g_n by g, f respectively. Let $g(x) = f(x)^d + \sum_{i<d} A_i(x) f(x)^i$ be the $f(x)$-expansion of $g(x)$. By virtue of properties (E), (C) of strict systems and (8), we have

$$\frac{w_n(A_0)}{d} > \gamma_n = w_{n-1}(f) = v(f(\alpha_1)). \tag{10}$$

Using (8) and the facts that $g(\alpha) = A_0(\alpha),\ d = deg\ g/deg\ f$ together with Lemma 2.E, we see that

$$\frac{w_n(A_0)}{d} = \frac{v(A_0(\alpha))}{d} = \frac{deg\ f}{deg\ g}\ v(g(\alpha)) = v(f(\theta)). \tag{11}$$

From (10) and (11), it follows that $v(f(\theta)) > v(f(\alpha_1))$. Applying Lemma 3.2 (with β replaced by θ), it is immediate that there exists a K-conjugate θ' of θ such that $v(\theta' - \alpha) > \delta_K(\alpha)$. This proves (9) in view of the choice of θ.

Note that if $deg\ \beta < deg\ \alpha$, then keeping in mind (9), one can verify that $v(\theta - \beta) = v(\alpha - \beta) \leqslant \delta_K(\alpha) < v(\theta - \alpha)$. So to show that (θ, α) is a distinguished pair, it remains to show that whenever η belonging to $\widetilde{K}$ satisfies $v(\theta - \eta) > \delta$, then $deg\ \eta \geqslant deg\ \theta$. Let $\eta \in \widetilde{K}$ be such that $v(\theta - \eta) > \delta$, then by the strong triangle law, $v(\alpha - \eta) = \delta$. If β is any element of $\widetilde{K}$ with $deg\ \beta < deg\ \alpha$, then $v(\alpha - \eta) > v(\alpha - \beta)$ in view of (9). Therefore by corollary to Lemma 2.D, we have

$$G(K(\alpha)) \subseteq G(K(\eta))\ ,\ R(K(\alpha)) \subseteq R(K(\eta)). \tag{12}$$

Let e be the smallest positive integer such that $ev(f(\theta))$ belongs to $G(K(\alpha))$, say $ev(f(\theta)) = v(h(\alpha))$, $h(x) \in K[x]$, $deg\ h(x) < deg\ \alpha$. In view of (11), $v(f(\theta)) = w_n(A_0)/d = v(A_0(\alpha))/d$. So e divides d by virtue of property (F) of the strict system. Set $\xi = f(\theta)^e/h(\alpha)$ and $l = d/e$. We show that $\bar{\xi}$ belongs to $R(K(\eta))$ and that $\bar{\xi}$ is algebraic of degree l over $R(K(\alpha))$.
Write $\frac{f(\eta)}{f(\theta)} = \prod_{\alpha'} \left(\frac{\eta - \alpha'}{\theta - \alpha'}\right) = \prod_{\alpha'} \left(1 + \frac{\eta - \theta}{\theta - \alpha'}\right)$, where α' runs over all roots of f.
Since $v(\theta - \eta) > \delta$ and by our choice $v(\theta - \alpha') \leqslant \delta$, it follows that $\overline{(f(\eta)/f(\theta))} = \bar{1}$ and so

$$v(f(\eta)) = v(f(\theta)), \overline{\left(\frac{f(\eta)^e}{h(\alpha)}\right)} = \overline{\left(\frac{f(\theta)^e}{h(\alpha)}\right)} = \bar{\xi}. \tag{13}$$

We now show that

$$v(A_i(\theta) f(\theta)^i / h(\alpha)^l) \geqslant 0 \text{ for each } i. \tag{14}$$

Since $deg\ A_i(x) < deg\ \alpha$, it follows from (9) that for each root β of $A_i(x)$, we have $v(\alpha - \beta) \leqslant \delta_K(\alpha) < \delta = v(\theta - \alpha)$. So by Lemma 2.D, $v(A_i(\theta)) = v(A_i(\alpha))$. It now follows from Property (E) of the strict system and (8) that

$$\frac{v(A_i(\theta))}{d-i} = \frac{v(A_i(\alpha))}{d-i} = \frac{w_n(A_i)}{d-i} \geqslant \frac{w_n(A_0)}{d} = \frac{v(A_0(\alpha))}{d},$$

which in view of (11) shows that

$$v(A_i(\theta)) + iv(f(\theta)) \geqslant \frac{d-i}{d}\ v(A_0(\alpha)) + iv(f(\theta)) = dv(f(\theta)) = lv(h(\alpha)),$$

proving (14). Note that in view of the fact $v(A_i(\theta)) = v(A_i(\alpha))$, the inequality in (14) will be strict when i is not divisible by e. Therefore taking the image in the residue field of the equation

$$0 = \frac{g(\theta)}{h(\alpha)^l} = \frac{f(\theta)^d}{h(\alpha)^l} + \sum_i \frac{A_i(\theta) f(\theta)^i}{h(\alpha)^l},$$

we see that $\bar{\xi}$ is a root of the polynomial $Y^l + \sum_{j<l} \overline{\left(\frac{A_{ej}(\alpha)}{h(\alpha)^{l-j}}\right)}\ Y^j$ which is irreducible over $R(K(\alpha))$ by property (F) of the strict system. Recall that by

(13), $\bar{\xi}$ belongs to $R(K(\eta))$ which contains $R(K(\alpha))$ in view of (12). So l divides $[R(K(\eta)) : R(K(\alpha))]$. Also in view of (13), $v(f(\eta)) = v(f(\theta))$ and hence by Lagrange's theorem for finite groups, e divides $[G(K(\eta)) : G(K(\alpha))]$. Since $K(\alpha)/K$ is defectless by Theorem 1.A, it follows that *el deg* α divides *deg* η and hence *deg* $\eta \geqslant$ *el deg* $\alpha =$ *deg* θ. This completes the proof of the assertion that (θ, α) is a distinguished pair and hence the theorem.

4. PROOF OF THEOREM 1.3 AND COROLLARIES 1.4, 1.5

Proof of Theorem 1.3. Let g_{n-i} denote the minimal polynomial of θ_i over K so that $g_n = g$. If w_i is defined by $w_i(q(x)) = v(q(\theta_{n-i}))$, $q(x) \in K[x]$ and $\gamma_{i+1} = w_i(g_{i+1})$, $\gamma_0 = -\infty$, then by Theorem 1.C, $\{(g_i, w_i, \gamma_i)\ ,\ 0 \leqslant i \leqslant n\}$ is a strict system of polynomial extensions over (K, v). Applying Lemma 3.2 (with α replaced by θ), we see that there exists a K-conjugate β' of β such that $v(\theta - \beta') > \delta_K(\theta)$. Since (K, v) is henselian, $v(\theta' - \beta) > \delta_K(\theta)$ for some K-conjugate θ' of θ. In view of the remark preceding Proposition 2.F, $v(g(\theta_1))$ is independent of the complete distinguished chain for θ. Note that $\lambda_g = v(g(\theta_1))$ is the smallest constant satisfying the property of the theorem because there does not exist any K-conjugate θ' of θ for which $v(\theta' - \theta_1) > \delta_K(\theta)$.

Proof of Corollary 1.4. We first show that if $K(\beta)/K$ is a tamely ramified extension, then so is $K(\theta)/K$. By Theorem 1.3, there exists a K-conjugate θ' of θ such that

$$v(\theta' - \beta) > \delta_K(\theta). \tag{15}$$

Since $\delta_K(\theta') = \delta_K(\theta)$, it follows from (15) and corollary to Lemma 2.D that $G(K(\theta')) \subseteq G(K(\beta))\ ,\ R(K(\theta')) \subseteq R(K(\beta))$. Thus $K(\theta)/K$ being defectless is tamely ramified.

Let $\omega_K(\theta)$ denote the Krasner's constant defined by

$$\omega_K(\theta) = \max\{v(\theta - \theta')|\ \theta' \neq \theta \text{ runs over all K-conjugates of } \theta\}.$$

Since $K(\theta)/K$ is tamely ramified, by virtue of (Khanduja, 1999, Lemma 2.2), $\delta_K(\theta) \geqslant \omega_K(\theta)$. Using Krasner's Lemma (Engler and Prestel, 2005, Theorem 4.1.7) it now follows from (15) that $K(\theta') \subseteq K(\beta)$.

Proof of Corollary 1.5. Let θ be a root of $g_n = h_m$. By virtue of Theorem 1.1, there exist roots θ_i of g_{n-i} and η_i of h_{m-i} such that $\theta = \theta_0, \theta_1, \ldots, \theta_n$ and $\theta = \eta_0, \eta_1, \ldots, \eta_m$ are complete distinguished chains, so by Theorem 1.B, $n = m$. Recall that by Proposition 2.F, all g_i, h_i are irreducible over K and $w_i(q(x)) =$

$v(q(\theta_{n-i}))$, $w_i'(q(x)) = v(q(\eta_{n-i}))$ for all $q(x)$ in $K[x]$, $0 \leqslant i \leqslant n$. Therefore in view of Theorem 1.B, w_i, w_i' have the same value group and residue field. By the remark preceding Proposition 2.F, we have $v(g_{n-i+1}(\theta_i)) = v(h_{n-i+1}(\eta_i))$, i.e., $\gamma_{n-i+1} = \gamma'_{n-i+1}$.

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