ON EISENSTEIN-DUMAS AND GENERALIZED SCHÖNEMANN POLYNOMIALS

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ABSTRACT. Let v be a valuation of a field K having value group $\mathbb{Z}$. It is known that a polynomial $x^n + a_{n-1}x^{n-1} + \ldots + a_0$ satisfying $\frac{v(a_i)}{n-i} \geqslant \frac{v(a_0)}{n} > 0$ with $v(a_0)$ coprime to n, is irreducible over K. Such a polynomial is referred to as an Eisenstein-Dumas polynomial with respect to v. In this paper, we give necessary and sufficient conditions so that some translate $g(x+a)$ of a given polynomial $g(x)$ belonging to $K[x]$ is an Eisenstein-Dumas polynomial with respect to v. In fact an analogous problem is dealt with for a wider class of polynomials, viz. Generalized Schönemann polynomials with coefficients over valued fields of arbitrary rank.

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1. INTRODUCTION

The classical Schönemann Irreducibility Criterion states that if $f(x)$ is a monic polynomial with coefficients from the ring $\mathbb{Z}$ of integers which is irreducible modulo a prime number p and if $g(x)$ belonging to $\mathbb{Z}[x]$ is a polynomial of the form $g(x) = f(x)^e + pM(x)$ where $M(x)$ belonging to $\mathbb{Z}[x]$ is relatively prime to $f(x)$ modulo p and the degree of $M(x)$ is less than the degree of $g(x)$, then $g(x)$ is irreducible over the field $\mathbb{Q}$ of rational numbers. Such a polynomial is referred to as a Schönemann polynomial with respect to p and $f(x)$. It can be easily seen that if $g(x)$ is as above, then the $f(x)$-expansion of $g(x)$ obtained on dividing it by successive powers of $f(x)$ given by

$$g(x) = \sum_{i=0}^{e} g_i(x)f(x)^i, \ \ deg\ g_i(x) < deg f(x),$$

satisfies (i) $g_e(x) = 1$, (ii) p divides the content of each polynomial $g_i(x)$ for $0 \leqslant i \leqslant e-1$ and (iii) p^2 does not divide the content of $g_0(x)$. Clearly any polynomial $g(x)$ belonging to $\mathbb{Z}[x]$ whose $f(x)$-expansion satisfies the above three properties is a Schönemann polynomial with respect to p and $f(x)$. Note that a monic polynomial is an Eisenstein polynomial with respect to a prime p if and only if it is a Schönemann polynomial with respect to p and $f(x) = x$.

The Schönemann Irreducibility Criterion has been extended to polynomials with coefficients over valued fields in several ways (cf. Khanduja and Saha, 1997; Ribenboim, 1999, Chapter 4, D; Brown, 2008). In 2008, Ron Brown gave a generalization of the Schönemann Irreducibility Criterion for polynomials with coefficients in a valued field (K, v) of arbitrary rank, which will be stated after introducing some notations.

We shall denote by v^x the Gaussian valuation of the field $K(x)$ of rational functions in an indeterminate x which extends the valuation v of K and is defined on $K[x]$ by

$$v^x(\sum_i a_i x^i) = \min_i\{v(a_i) |\ a_i \in K\}.$$

For an element ξ in the valuation ring R_v of v with maximal ideal $\mathcal{M}_v$, $\bar{\xi}$ will denote its v-residue, i.e., the image of ξ under the canonical homomorphism from R_v onto $R_v/\mathcal{M}_v$. For $f(x)$ belonging to $R_v[x]$, $\bar{f}(x)$ will stand for the polynomial over $R_v/\mathcal{M}_v$ obtained by replacing each coefficient of $f(x)$ by its v-residue. The following result of Ron Brown which generalizes the Schönemann Irreducibility Criterion is proved in section 3 (see Lemma 3.1).

Theorem A. *Let v be a valuation of arbitrary rank of a field K with value group G and valuation ring R_v having maximal ideal $\mathcal{M}_v$. Let $f(x)$ belonging to $R_v[x]$ be a monic polynomial of degree m such that $\bar{f}(x)$ is irreducible over $R_v/\mathcal{M}_v$. Assume that $g(x) \in R_v[x]$ is a monic polynomial whose $f(x)$-expansion $f(x)^e + \sum_{i=0}^{e-1} g_i(x)f(x)^i$ satisfies $\frac{v^x(g_i(x))}{e-i} \geqslant \frac{v^x(g_0(x))}{e} > 0$ for $0 \leqslant i \leqslant e-1$ and $v^x(g_0(x)) \notin dG$ for any number $d > 1$ dividing e. Then $g(x)$ is irreducible over K.*

A polynomial satisfying the hypothesis of Theorem A will be referred to as a Generalized Schönemann polynomial with respect to v and $f(x)$. In the particular case when $f(x) = x$, it will be called an Eisenstein-Dumas polynomial with respect to v. When v is a discrete valuation with value group $\mathbb{Z}$, then a monic polynomial $\sum_{i=0}^{e} a_i x^i$ is an Eisenstein-Dumas polynomial with respect to v if $\frac{v(a_i)}{e-i} \geqslant \frac{v(a_0)}{e}$ and $v(a_0)$ is coprime to e. Thus Theorem A extends the usual Eisenstein-Dumas Irreducibility Criterion[1].

In this paper, we first investigate when a translate $g(x+a)$ of a given polynomial $g(x)$ belonging to $K[x]$ having a root θ is an Eisenstein-Dumas polynomial with respect to an arbitrary henselian valuation v of a field K. It is shown that $g(x+a)$ is such a polynomial if and only if $K(\theta)/K$ is a totally ramified extension and (θ, a) is a (K, v)-distinguished pair as defined below. In particular, it is deduced that if some translate of a polynomial $g(x) = x^e + a_{e-1}x^{e-1} + \ldots + a_0$ is an Eisenstein-Dumas polynomial with respect to v with e not divisible by the characteristic of the residue field of v, then the polynomial $g(x - \frac{a_{e-1}}{e})$ is an Eisenstein-Dumas polynomial with respect to v. This generalizes a result of M. Juras proved in 2006 (cf. Juras, 2006).

We also deal with the following more general problem related to Theorem A.

Let $g(x)$ belonging to $R_v[x]$ be a monic polynomial over a henselian valued field (K, v) of arbitrary rank with $\bar{g}(x) = \phi(x)^e$ where $\phi(x)$ is an irreducible polynomial over $R_v/\mathcal{M}_v$ and θ is a root of $g(x)$. What are necessary and sufficient conditions so that $g(x)$ is a Generalized Schönemann polynomial with respect to v and some polynomial $f(x) \in R_v[x]$ with $\bar{f}(x) = \phi(x)$?

[1]**Eisenstein-Dumas Irreducibility Criterion.** *Let $g(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_o$ be a polynomial with coefficients in $\mathbb{Z}$. Suppose there exists a prime p whose exact power p^{r_i} dividing a_i (where $r_i = \infty$ if $a_i = 0$), satisfy $r_n = 0$, $(r_i/n - i) \geqslant (r_0/n)$ for $0 \leqslant i \leqslant n-1$ and r_0, n are coprime. Then $g(x)$ is irreducible over $\mathbb{Q}$.*

Our results are proved using saturated distinguished chains which will be defined after introducing some notations.

In what follows, v is a henselian valuation of arbitrary rank of a field K and $\tilde{v}$ is the unique prolongation of v to the algebraic closure $\widetilde{K}$ of K with value group $\widetilde{G}$. By the degree of an element α in $\widetilde{K}$, we shall mean the degree of the extension $K(\alpha)/K$ and shall denote it by $deg\ \alpha$. For an element ξ in the valuation ring of $\tilde{v}$, $\bar{\xi}$ will stand for its $\tilde{v}$-residue and for a subfield L of $\widetilde{K}$, $\overline{L}$ and $G(L)$ will denote respectively the residue field and the value group of the valuation of L obtained by restricting $\tilde{v}$. When there is no chance of confusion, we shall write $\tilde{v}(\alpha)$ as $v(\alpha)$ for α belonging to $\widetilde{K}$.

A finite extension $(K', v')/(K, v)$ is called defectless if $[K' : K] = ef$, where e and f are the index of ramification and the residual degree of v'/v.

Recall that a pair (θ, α) of elements of $\widetilde{K}$ is called a distinguished pair (more precisely (K, v)-distinguished pair) if the following three conditions are satisfied: (i) $deg\ \theta > deg\ \alpha$, (ii) $\tilde{v}(\theta - \beta) \leqslant \tilde{v}(\theta - \alpha)$ for every β in $\widetilde{K}$ with $deg\ \beta < deg\ \theta$, (iii) if $\gamma \in \widetilde{K}$ and $deg\ \gamma < deg\ \alpha$, then $\tilde{v}(\theta - \gamma) < \tilde{v}(\theta - \alpha)$.

Distinguished pairs give rise to distinguished chains in a natural manner. A chain $\theta = \theta_0, \theta_1, \ldots, \theta_r$ of elements of $\widetilde{K}$ will be called a saturated distinguished chain for θ if (θ_i, θ_{i+1}) is a distinguished pair for $0 \leqslant i \leqslant r-1$ and $\theta_r \in K$. Popescu and Zaharescu (cf. Popescu and Zaharescu, 1995) were the first to introduce the notion of distinguished chains. In 1995, they proved the existence of a saturated distinguished chain for each θ belonging to $\widetilde{K} \setminus K$ in case (K, v) is a complete discrete rank one valued field. In 2005, Aghigh and Khanduja (cf. Aghigh and Khanduja, 2005) proved that if (K, v) is a henselian valued field of arbitrary rank, then an element θ belonging to $\widetilde{K} \setminus K$ has a saturated distinguished chain with respect to v if and only if $K(\theta)$ is a defectless extension of (K, v). A saturated distinguished chain for θ gives rise to several invariants associated with θ, some of which are given by Theorem B stated below which is proved in (cf. Aghigh and Khanduja, 2005).

Theorem B. *Let (K, v) and $(\widetilde{K}, \tilde{v})$ be as above. Let $\theta = \theta_0, \theta_1, \ldots, \theta_r$ and $\theta = \eta_0, \eta_1, \ldots, \eta_s$ be two saturated distinguished chains for an element θ belonging to $\widetilde{K} \setminus K$, then $r = s$ and $[K(\theta_i) : K] = [K(\eta_i) : K]$, $G(K(\theta_i)) = G(K(\eta_i))$, $\overline{K(\theta_i)} = \overline{K(\eta_i)}$ for $1 \leqslant i \leqslant r$. Further $G(K(\theta_{i+1})) \subseteq G(K(\theta_i))$, $\overline{K(\theta_{i+1})} \subseteq \overline{K(\theta_i)}$ for $0 \leqslant i \leqslant r-1$.*

In this paper, we prove

Theorem 1.1. *Let v be a henselian valuation of arbitrary rank of a field K with value group G. Let $g(x)$ belonging to $R_v[x]$ be a monic polynomial of degree e having a root θ. Then for an element a of K, $g(x+a)$ is an Eisenstein-Dumas polynomial with respect to v if and only if (θ, a) is a distinguished pair and $K(\theta)/K$ is a totally ramified extension of degree e.*

The following result which generalizes a result of M. Juras will be quickly deduced from the above theorem.

Theorem 1.2. *Let $g(x) = \sum_{i=0}^{e} a_i x^i$ be a monic polynomial with coefficients in a henselian valued field (K, v). Suppose that the characteristic of the residue field of v does not divide e. If there exists an element b belonging to K such that $g(x+b)$ is an Eisenstein-Dumas polynomial with respect to v, then so is $g(x - \frac{a_{e-1}}{e})$.*

Note that if $g(x)$ belonging to $R_v[x]$ is a monic polynomial such that $\bar{g}(x)$ is irreducible over $R_v/\mathcal{M}_v$, then for any non-zero α in $\mathcal{M}_v$, $g(x)$ is a Generalized Schönemann polynomial with respect to $f(x) = g(x) - \alpha$ and v. Therefore to deal with the second problem mentioned after Theorem A, we may assume that $\bar{g}(x) = \phi(x)^e$ with $\phi(x)$ irreducible over $R_v/\mathcal{M}_v$ and $e > 1$. When *deg* $\phi(x) = 1$ then the problem referred to above is already solved in Theorem 1.1 because $g(x+a)$ is an Eisenstein-Dumas polynomial with respect to v if and only if $g(x)$ is a Generalized Schönemann polynomial with respect to v and $x - a$. Setting aside these two cases, we shall prove

Theorem 1.3. *Let v be a henselian valuation of arbitrary rank of a field K with value group G and $f(x)$ belonging to $R_v[x]$ be a monic polynomial of degree $m > 1$ with $\bar{f}(x)$ irreducible over the residue field of v. Let $g(x) \in K[x]$ be a Generalized Schönemann polynomial with respect to v and $f(x)$ having $f(x)$-expansion $f(x)^e + \sum_{i=0}^{e-1} g_i(x) f(x)^i$ with $e > 1$. Let θ be a root of $g(x)$. Then for some suitable root θ_1 of $f(x)$, θ has a saturated distinguished chain $\theta = \theta_0, \theta_1, \theta_2$ of length 2 with $G(K(\theta_1)) = G$, $\overline{K(\theta)} = \overline{K}(\bar{\theta})$ and $[G(K(\theta)) : G] = e$.*

The converse of the above result is also true as asserted by the following theorem.

Theorem 1.4. *Let (K, v) be as in the above theorem. Let $g(x)$ belonging to $R_v[x]$ be a monic polynomial such that $\bar{g}(x) = \phi(x)^e$, $e > 1$ where $\phi(x)$ is an irreducible polynomial over $R_v/\mathcal{M}_v$ of degree $m > 1$. Suppose that a root θ of $g(x)$ has a saturated distinguished chain $\theta = \theta_0,\ \theta_1,\ \theta_2$ of length 2 with $G(K(\theta_1)) = G$, $\overline{K(\theta)} = \overline{K}(\bar{\theta})$ and $[G(K(\theta)) : G] = e$. Then $g(x)$ is a Generalized Schönemann polynomial with respect to v and $f(x)$, where $f(x)$ is the minimal polynomial of θ_1 over K.*

2. PROOF OF THEOREMS 1.1, 1.2

Proof of Theorem 1.1. Write $g(x + a) = x^e + a_{e-1}x^{e-1} + \ldots + a_0,\ a, a_i \in K$. Suppose first that $g(x+a)$ is an Eisenstein-Dumas polynomial with respect to v. Then it is irreducible over K in view of Theorem A. Since (K, v) is henselian, all roots of $g(x + a)$ have the same v-valuation and hence $v(\theta - a) = \frac{v(a_0)}{e}$. In view of the hypothesis that $g(x + a)$ is an Eisenstein-Dumas polynomial, we have

$$e = [G + \mathbb{Z}\frac{v(a_0)}{e} : G] = [G + \mathbb{Z}v(\theta - a) : G]. \tag{1}$$

To prove that (θ, a) is a distinguished pair, it is to be shown that

$$\max\{v(\theta - \beta)|\ \beta \in \widetilde{K},\ deg\ \beta < deg\ \theta\} = v(\theta - a) = \frac{v(a_0)}{e}. \tag{2}$$

If β is as in (2) and if $v(\theta - \beta) > v(\theta - a)$, then by the strong triangle law,

$$v(\beta - a) = \min\{v(\beta - \theta), v(\theta - a)\} = v(\theta - a)$$

which in view of the fundamental inequality (cf. Engler and Prestel, 2005, Theorem 3.3.4) and (1) implies that $deg\ (\beta - a) \geqslant e$, a contradiction. Therefore (2) holds and (θ, a) is a distinguished pair with $K(\theta)/K$ totally ramified in view of (1).

Conversely suppose that (θ, a) is a distinguished pair and $K(\theta)/K$ is a totally ramified extension of degree e. Note that $v(\theta - a) \geqslant v(\theta) \geqslant 0$. Keeping in mind that (K, v) is henselian and the relation between the roots and coefficients of the the K-irreducible polynomial $g(x + a) = x^e + a_{e-1}x^{e-1} + \ldots + a_0$, we see that $v(a_i) \geqslant (e - i)v(\theta - a) = (\frac{e-i}{e})v(a_0) \geqslant 0$. So $g(x + a)$ is an Eisenstein-Dumas polynomial with respect to v once we show that $s\frac{v(a_0)}{e} \notin G$ for any positive number $s < e$.

Suppose to the contrary there exists a positive number $s < e$ such that

$s\frac{v(a_0)}{e} = sv(\theta - a) \in G$, say $sv(\theta - a) = v(b),\ b \in K$. Since $K(\theta)/K$ is totally ramified, there exists c in K with $v(c) = 0$ such that $\overline{((\theta - a)^s/b)} = \bar{c}$, which implies that

$$v((\theta - a)^s - bc) > v(b). \tag{3}$$

Set $v(\theta - a) = \delta$ and $h(x) = (x - a)^s - bc$. Let w denote the valuation of $\widetilde{K}(x)$ defined on $\widetilde{K}[x]$ by

$$w(\sum_i c_i(x - a)^i) = \min_i \{\tilde{v}(c_i) + i\delta\},\ c_i \in \widetilde{K}.$$

Note that $w(h(x)) = \min\{s\delta, v(bc)\} = v(b)$. This equality will contradict (3) thereby completing the proof of the theorem once we show that

$$v(h(\theta)) = w(h(x)). \tag{4}$$

To verify (4), write $h(x) = \prod_{i=1}^{s}(x - \beta^{(i)})$. Keeping in mind that $h(x)$ belonging to $K[x]$ is a polynomial of degree $s < e$ and the fact that (θ, a) is a distinguished pair, we have $v(\theta - \beta^{(i)}) \leqslant v(\theta - a)$ for $1 \leqslant i \leqslant s$ and hence it can be easily seen that

$$v(\theta - \beta^{(i)}) = \min\{v(\theta - a), v(a - \beta^{(i)})\} = w(x - \beta^{(i)}).$$

On summing over i, the above equation gives (4).

Proof of Theorem 1.2. In view of Theorem 1.1, it is enough to prove that if (θ, b) is a distinguished pair, then so is $(\theta, \frac{-a_{e-1}}{e})$. In fact it suffices to show that

$$v(\theta + \frac{a_{e-1}}{e}) \geqslant v(\theta - b). \tag{5}$$

Let $\theta = \theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(e)}$ denote the K-conjugates of θ. Using the hypothesis $v(e) = 0$, we have

$$v(\theta + \frac{a_{e-1}}{e}) = v(e\theta + a_{e-1}) = v(e\theta - \sum_{i=1}^{e} \theta^{(i)}) = v(\sum_{i=2}^{e}(\theta - \theta^{(i)}));$$

consequently

$$v(\theta + \frac{a_{e-1}}{e}) \geqslant \min_{i \geqslant 2}\{v(\theta - \theta^{(i)})\} = v(\theta - \theta^{(2)})\ (say). \tag{6}$$

Since $b \in K$, $v(\theta - b) = v(\theta^{(2)} - b)$ and hence $v(\theta - \theta^{(2)}) \geqslant v(\theta - b)$ which together with (6) proves (5) and hence the theorem.

We use Theorem 1.1 to construct examples of totally ramified extensions $K(\theta)/K$ such that no translate of the minimal polynomial of θ over K is an Eisenstein-Dumas polynomial with respect to v.

Notation. For α separable over K of degree > 1, $\omega_K(\alpha)$ will stand for the Krasner's constant defined by

$$\omega_K(\alpha) = \max\{\tilde{v}(\alpha - \alpha') \mid \alpha' \neq \alpha \text{ runs over } K\text{-conjugates of } \alpha\}.$$

Example 2.1. Let K be the field of 2-adic numbers with the usual valuation v_2 given by $v_2(2) = 1$. The prolongation of v_2 to the algebraic closure of K will be denoted by v_2 again. Consider $\theta = 2+2(2^{-1/2})+2^2(2^{-1/2^2})$ and $\theta_1 = 2+2(2^{-1/2})$. It will be shown that $K(\theta) = K(2^{1/4})$ and (θ, θ_1) is a distinguished pair. Note that the Krasner's constant $\omega_K(\theta_1) = 3/2$ and $v_2(\theta - \theta_1) = 7/4 > \omega_K(\theta_1)$. Therefore by Krasner's Lemma (cf. Engler and Prestel, 2005, Theorem 4.1.7), $K(\theta_1) \subseteq K(\theta)$ and hence $2^2(2^{-1/4}) = \theta - \theta_1$ belongs to $K(\theta)$ as asserted. To show that (θ, θ_1) is a distinguished pair, we first verify that whenever γ belonging to $\widetilde{K}$ satisfies $v_2(\theta - \gamma) > v_2(\theta - \theta_1) = 7/4$, then $deg\ \gamma \geqslant 4$. If γ is as above, we have by the strong triangle law

$$v_2(\theta_1 - \gamma) = \min\{v_2(\theta_1 - \theta), v_2(\theta - \gamma)\} = 7/4 > \omega_K(\theta_1) = 3/2.$$

So by Krasner's Lemma, $K(\theta_1) \subseteq K(\gamma)$ and hence $G(K(\gamma))$ contains $v_2(\theta_1 - \gamma) = 7/4$ which implies that $deg\ \gamma \geqslant 4$. Therefore

$$7/4 = v_2(\theta - \theta_1) = \max\{v_2(\theta - \beta) |\ \beta \in \widetilde{K},\ deg\ \beta < 4\}.$$

Also for any $b \in \widetilde{K}$ with $deg\ b < deg\ \theta_1$, we have $b \in K$ and clearly $v_2(\theta_1 - b) \leqslant 1/2 < v_2(\theta - \theta_1)$. So (θ, θ_1) is a distinguished pair. As can be easily checked, θ is a root of $g(x) = x^4 - 8x^3 + 20x^2 - 80x + 4$ which must be irreducible over K. By Theorem 1.1 no translate of $g(x)$ can be an Eisenstein-Dumas polynomial with respect to v_2 because (θ, θ_1) is a distinguished pair with $\theta_1 \notin K$ and consequently (θ, a) cannot be a distinguished pair for any a in K in view of Theorem B. Moreover, if p be a prime number different from 2, then no translate of $g(x)$ can be an Eisenstein-Dumas polynomial with respect to the p-adic valuation v_p, for otherwise in view of Theorem 1.2, $g(x + 2) = x^4 - 4x^2 - 64x - 124$ would be an Eisenstein-Dumas polynomial with respect to v_p, which is clearly impossible.

3. PROOF OF THEOREM 1.3.

We need the following lemma which is proved in (cf. Brown, 2008, Lemma 4). Its proof is omitted.

Lemma 3.1. *Let v, G, $f(x)$ and $g(x)$ be as in Theorem A. Let θ be a root of $g(x)$ and v' be a prolongation of v to $K(\theta)$ with value group G'. Then $v'(f(\theta)) = \frac{v^x(g_0(x))}{e}$, $G' = G + \mathbb{Z}\frac{v^x(g_0(x))}{e}$, the residue field of v' is a simple extension of the residue field of v generated by the v'-residue $\bar{\theta}$ of θ and $g(x)$ is irreducible over K. In particular the index of ramification and the residual degree of v'/v are e and deg $f(x)$ respectively.*

Proof of Theorem 1.3. Since $\bar{\theta}$ is a root of $\bar{g}(x) = \bar{f}(x)^e$ and $\bar{f}(x)$ is an irreducible polynomial of degree $m > 1$, it follows that $v(\theta) = 0$ and there exists a root $\alpha^{(i)}$ of $f(x)$ such that $v(\theta - \alpha^{(i)}) > 0$. Let α be a root of $f(x)$ satisfying

$$0 < v(\theta - \alpha) = \max\{v(\theta - \alpha^{(i)})|\ \alpha^{(i)} \text{ runs over roots of } f(x)\} = \delta\ (say). \quad (7)$$

We claim that (θ, α) is a distinguished pair. Observe that if γ belonging to $\widetilde{K}$ is such that $deg\ \gamma < deg\ \alpha$, then $v(\theta - \gamma) < \delta$, for otherwise by the triangle law we would have $v(\alpha - \gamma) > 0$ and hence $\bar{\alpha} = \bar{\gamma}$ which is impossible because

$$m = [\overline{K}(\bar{\alpha}) : \overline{K}] = [\overline{K}(\bar{\gamma}) : \overline{K}] \leqslant [K(\gamma) : K] < m.$$

So to prove the claim, it suffices to show that whenever β belongs to $\widetilde{K}$ with $v(\theta - \beta) > \delta$, then $deg\ \beta \geqslant deg\ \theta$. For proving this inequality, in view of the fundamental inequality and the fact $deg\ \theta = [G(K(\theta)) : G][\overline{K(\theta)} : \overline{K}]$ derived from Lemma 3.1, it is enough to show that

$$G(K(\theta)) \subseteq G(K(\beta)),\ \overline{K(\theta)} \subseteq \overline{K(\beta)}. \quad (8)$$

Let β be an element of $\widetilde{K}$ with $v(\beta - \theta) > \delta$ and $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(m)}$ be the roots of $f(x)$, counted with multiplicities, if any. Write

$$\frac{f(\beta)}{f(\theta)} = \prod_{i=1}^{m}\left(\frac{\beta - \alpha^{(i)}}{\theta - \alpha^{(i)}}\right) = \prod_{i=1}^{m}\left(1 + \frac{\beta - \theta}{\theta - \alpha^{(i)}}\right).$$

Since $v(\theta - \beta) > \delta$ and by (7) $v(\theta - \alpha^{(i)}) \leqslant \delta$ for every i, it follows from the above expression for $f(\beta)/f(\theta)$ that its $\tilde{v}$-residue equals $\bar{1}$ and hence $v(f(\beta)) = v(f(\theta))$. Therefore in view of Lemma 3.1, $G(K(\theta)) = G + \mathbb{Z}v(f(\theta)) \subseteq G(K(\beta))$. Also keeping in mind that $v(\theta - \beta) > \delta > 0$, we have by Lemma 3.1, $\overline{K(\theta)} = \overline{K}(\bar{\theta}) = \overline{K}(\bar{\beta})$ which proves (8) and hence the claim.

Recall that $\bar{\alpha}$ is a root of the irreducible polynomial $\bar{f}(x)$ of degree $m > 1$. So $v(\alpha - 1) = 0$. Also for any β in $\widetilde{K}$ with $deg\ \beta < deg\ \alpha$, we have $v(\alpha - \beta) \leqslant 0$, for otherwise $\bar{\alpha} = \bar{\beta}$ and this in view of the fundamental inequality would imply $[K(\beta) : K] \geqslant [\overline{K}(\bar{\beta}) : \overline{K}] = m$. So $(\alpha, 1)$ is a distinguished pair. Thus we have proved that θ has a saturated distinguished chain $\theta, \alpha, 1$ of length 2. Since $[K(\alpha) : K] = [\overline{K}(\bar{\alpha}) : \overline{K}] = m$, it follows from the fundamental inequality that $G(K(\alpha)) = G$. The other two equalities hold by virtue of Lemma 3.1.

4. PROOF OF THEOREM 1.4.

We retain the notations of the previous sections as well as the assumption that v is a henselian valuation of arbitrary rank of a field K with unique prolongation $\tilde{v}$ to the algebraic closure $\widetilde{K}$ having value group $\widetilde{G}$. Recall that a pair (α, δ) belonging to $\widetilde{K} \times \widetilde{G}$ is said to be a minimal pair (more precisely (K, v)-minimal pair) if whenever β belonging to $\widetilde{K}$ satisfies $\tilde{v}(\alpha - \beta) \geqslant \delta$, then *deg* $\beta \geqslant$ *deg* α. It can be easily seen that if (θ, α) is a distinguished pair and $\delta = \tilde{v}(\theta - \alpha)$, then (α, δ) is a minimal pair.

Let (α, δ) be a (K, v)-minimal pair. The valuation $\tilde{w}_{\alpha,\delta}$ of $\widetilde{K}(x)$ defined on $\widetilde{K}[x]$ by

$$\tilde{w}_{\alpha,\delta}(\sum_i c_i(x-\alpha)^i) = \min_i\{\tilde{v}(c_i) + i\delta\},\ c_i \in \widetilde{K}$$

will be referred to as the valuation defined by the pair (α, δ). The description of $\tilde{w}_{\alpha,\delta}$ on $K[x]$ is given by the already known theorem stated below (cf. Alexandru, Popescu and Zaharescu, 1988; Khanduja, 1992).

Theorem C. *Let $\tilde{w}_{\alpha,\delta}$ be the valuation of $\widetilde{K}(x)$ defined by a minimal pair (α, δ) and $w_{\alpha,\delta}$ be the valuation of $K(x)$ obtained by restricting $\tilde{w}_{\alpha,\delta}$. Let $f(x)$ be the minimal polynomial of α over K. Then for any polynomial $g(x)$ in $K[x]$ with $f(x)$-expansion $\sum_{i\geqslant 0} g_i(x)f(x)^i$, one has $w_{\alpha,\delta}(g(x)) = \min_i\{\tilde{v}(g_i(\alpha)) + i w_{\alpha,\delta}(f(x))\}$.*

The following result proved in (cf. Aghigh and Khanduja, 2005, Theorem 1.1(i)) will be used in the sequel.

Lemma D. *Let (θ, α) be a (K, v)-distinguished pair and $f(x)$ be the minimal polynomial of α over K. Then $G(K(\theta)) = G(K(\alpha)) + \mathbb{Z}v(f(\theta))$.*

Proof of Theorem 1.4. We divide the proof into three steps.
Step I. Let $f(x)$ be the minimal polynomial of θ_1 over K. In this step, we prove that $\bar{f}(x)$ is irreducible over $\overline{K}$ and $\bar{f}(x) = \phi(x)$. In view of the hypothesis $[\overline{K(\theta)} : \overline{K}] = m$, $[G(K(\theta)) : G] = e$ and the fundamental inequality, it follows that $[K(\theta) : K] \geqslant em$. Since θ is a root of the polynomial $g(x)$ having degree em,

we have $[K(\theta):K] = em$. Note that $v(\theta - \theta_1) > 0$, because if $F(x) \in R_v[x]$ is a monic polynomial with $\overline{F}(x) = \phi(x)$, then there exists a root β of $F(x)$ such that $\bar{\beta} = \bar{\theta}$ which in view of the hypothesis $e > 1$ implies that $v(\theta - \theta_1) \geqslant v(\theta - \beta) > 0$. So the assertion of Step I is proved once we show that

$$[K(\theta_1):K] = [\overline{K}(\bar{\theta}_1):\overline{K}] = m. \tag{9}$$

Recall that $\overline{K(\theta_1)} \subseteq \overline{K(\theta)}$ by Theorem B. Therefore using the hypothesis $\overline{K(\theta)} = \overline{K}(\bar{\theta})$ and the fact $\bar{\theta}_1 = \bar{\theta}$, it follows that $\overline{K(\theta_1)} = \overline{K}(\bar{\theta})$; in particular

$$[\overline{K(\theta_1)}:\overline{K}] = [\overline{K}(\bar{\theta}_1):\overline{K}] = m. \tag{10}$$

Since $K(\theta_1)/K$ is a defectless extension in view of (Aghigh and Khanduja, 2005, Theorem 1.2) and it is given that $G(K(\theta_1)) = G$, we now obtain (9) using (10).

Step II. For simplicity of notation, we shall henceforth denote θ_1 by α. Set $v(\theta - \alpha) = \delta$ and $v(f(\theta)) = \lambda$. Let $g(x) = f(x)^e + \sum\limits_{i=0}^{e-1} g_i(x) f(x)^i$ be the $f(x)$-expansion of $g(x)$. Let $\tilde{w}_{\alpha,\delta}$ be the valuation of $\widetilde{K}(x)$ defined by the minimal pair (α, δ). In this step, we prove that

$$\tilde{w}_{\alpha,\delta}(f(x)) = \lambda \tag{11}$$

and

$$\tilde{w}_{\alpha,\delta}(g(x)) = v^x(g_0(x)) = e\lambda. \tag{12}$$

Write $f(x) = \prod\limits_{i=1}^{m}(x - \alpha^{(i)})$, $g(x) = \prod\limits_{j=1}^{em}(x - \theta^{(j)})$. Using the fact that $v(\theta - \alpha^{(i)}) \leqslant \delta$ and hence $v(\theta - \alpha^{(i)}) = \min\{\delta, v(\alpha - \alpha^{(i)})\}$, we have

$$\tilde{w}_{\alpha,\delta}(f(x)) = \tilde{w}_{\alpha,\delta}(\prod_{i=1}^{m}(x - \alpha^{(i)})) = \sum_{i=1}^{m} \min\{\delta, v(\alpha - \alpha^{(i)})\} = \sum_{i=1}^{m} v(\theta - \alpha^{(i)}) = \lambda$$

which proves (11). Since (K, v) is henselian, for any K-conjugate $\theta^{(j)}$ of θ, there exists a K-conjugate $\alpha^{(i)}$ of α such that $v(\theta^{(j)} - \alpha) = v(\theta - \alpha^{(i)}) \leqslant \delta$; consequently

$$\tilde{w}_{\alpha,\delta}(x - \theta^{(j)}) = \min\{\delta, v(\alpha - \theta^{(j)})\} = v(\alpha - \theta^{(j)}),$$

which on summing over j gives

$$\tilde{w}_{\alpha,\delta}(g(x)) = v(g(\alpha)). \tag{13}$$

Recall that in view of Step I, $\bar{f}(x)$ is irreducible over $\overline{K}$ of degree m having $\bar{\theta}$ as a root. So for any polynomial $A(x)=\sum a_i x^i$ belonging to $K[x]$ of degree less than m, we have

$$v(A(\theta)) = v^x(A(x)) = \min_i\{v(a_i)\}, \tag{14}$$

for if the above equality does not hold, then $m > 1$, $v(\theta) = 0$ and hence the triangle law would imply $v(A(\theta)) > \min_i\{v(a_i\theta^i)\} = v(a_j)$ (say) and thus $\sum_{i=0}^{m-1} \overline{(a_i/a_j)}\bar{\theta}^i = \bar{0}$, which is impossible. Keeping in mind the $f(x)$-expansion of $g(x)$ and that $f(\alpha) = 0$, we see that $g(\alpha) = g_0(\alpha)$ and consequently it follows from (14) that $v(g(\alpha)) = v(g_0(\alpha)) = v^x(g_0(x))$ which together with (13) proves the first equality of (12). As $f(x)$, $g(x)$ are irreducible over the henselian valued field (K, v), we have

$$v(f(\theta^{(j)})) = v(f(\theta)),\ 1 \leqslant j \leqslant em,\ v(g(\alpha^{(i)})) = v(g(\alpha)),\ 1 \leqslant i \leqslant m. \tag{15}$$

Keeping in mind that $\prod_{i=1}^{m} g(\alpha^{(i)}) = \pm \prod_{j=1}^{em} f(\theta^{(j)})$, it is clear from (15) that $v(g(\alpha)) = ev(f(\theta)) = e\lambda$, which proves the second equality of (12) in view of (13).

Step III. In this step, we prove that $g(x)$ is a Generalized Schönemann polynomial with respect to v and $f(x)$. By Theorem C, (11) and (12), we have

$$\tilde{w}_{\alpha,\delta}(g(x)) = \min_{0\leqslant i\leqslant e}\{v(g_i(\alpha)) + i\lambda\} = v^x(g_0(x)) = e\lambda. \tag{16}$$

As $v(g_i(\alpha)) = v^x(g_i(x))$, (16) shows that $v^x(g_i(x)) + i\lambda \geqslant e\lambda$ for $0 \leqslant i \leqslant e-1$, i.e.,

$$\frac{v^x(g_i(x))}{e-i} \geqslant \lambda = \frac{v^x(g_0(x))}{e} > 0.$$

Recall that $\lambda = v(f(\theta))$. Since (θ, α) is a distinguished pair, in view of Lemma D and the hypothesis $G(K(\alpha)) = G$, we have

$$G(K(\theta)) = G + \mathbb{Z}\lambda. \tag{17}$$

By hypothesis $[G(K(\theta)) : G] = e$, so it follows from (17) that e is the smallest positive integer for which $e\lambda \in G$. This completes the proof of the theorem.

Example 4.1. Let K be the field of 3-adic numbers with the usual valuation v_3 whose extension to the algebraic closure $\widetilde{K}$ of K will be denoted by $\tilde{v}_3$.

Consider the polynomial $g(x) = x^4 + 14x^2 + 1$ with $\bar{g}(x) = (x^2+1)^2$. It can be easily seen that $\theta = i(2+\sqrt{3})$ is a root of $g(x)$ where $i = \sqrt{-1}$. Since $\bar{\theta} = \overline{2i} \notin \overline{K}$ and $\tilde{v}_3(\theta^2 - 2) = 1/2$, it follows in view of the fundamental inequality that $[K(\theta) : K] = 4$. A simple calculation shows that the Krasner's constant $\omega_K(\theta) = \tilde{v}_3(\theta - 2i) = 1/2$. So by Krasner's Lemma, $\tilde{v}_3(\theta - \beta) \leqslant \frac{1}{2}$ for every $\beta \in \widetilde{K}$ with *deg* $\beta < 4$. Further if for some γ in $\widetilde{K}$, $\tilde{v}_3(\theta - \gamma) = \frac{1}{2}$, then $\bar{\theta} = \bar{\gamma} = \overline{2i}$. Since $\overline{2i} \notin \overline{K}$, we see that $[K(\gamma) : K] \geqslant 2$. Therefore $(\theta, 2i)$ is a distinguished pair. It can be easily seen that $(2i, 0)$ is a distinguished pair and hence $\theta, 2i, 0$ is a saturated distinguished chain for θ satisfying the hypothesis of Theorem 1.4. So $g(x)$ is a Generalized Schönemann polynomial with respect to v_3 and $f(x) = x^2 + 4$.

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