TAME KEY POLYNOMIALS

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Abstract. We introduce a new method of constructing complete sequences of key polynomials for simple extensions of tame fields. In our approach the key polynomials are taken to be the minimal polynomials over the base field of suitably constructed elements in its algebraic closure, with the extensions generated by them forming an increasing chain. In the case of algebraic extensions, we generalize the results to countably generated infinite tame extensions over henselian but not necessarily tame fields. In the case of transcendental extensions, we demonstrate the central role that is played by the implicit constant fields, which reveals the tight connection with the algebraic case.

1. Introduction

In this paper we will work with (Krull) valuations on fields and their extensions to rational function fields. Note that we *always identify equivalent valuations*. For basic information on valued fields and for notation, see Section 2. The value group $v(L^\times)$ of a valued field (L,v) will be denoted by vL, and its residue field by Lv. The value of an element a will be denoted by va, and its residue by av.

Take a valued field (K,v). It is an important task to describe, analyse and classify all extensions of the valuation v from K to the rational function field $K(x)$. In order to be able to compute the value of every element of $K(x)$ with respect to v, it suffices to be able to compute the value of all polynomials in x, that is, we only have to deal with the polynomial ring $K[x]$. Indeed, if $f,g\in K[x]$, then necessarily, $v\frac{f}{g} = vf - vg$. We know the values of all elements in K. If in addition we know the value vx, then everything would be easy if for every polynomial

$$f(x) \;=\; \sum_{i=0}^{n} c_i x^i \in K[x] \tag{1}$$

Date: January 8, 2023.

2010 *Mathematics Subject Classification.* 12J20, 13A18, 12J25.

Key words and phrases. Valuation, minimal pairs, key polynomials, pseudo Cauchy sequences, valuation transcendental extensions, tame fields.

Preliminary versions of this paper were written when the first author was a Post-Doctoral fellow (supported by the Post-Doctoral Fellowship of the National Board of Higher Mathematics, India) in IISER Mohali. The second author is supported by Opus grant 2017/25/B/ST1/01815 from the National Science Centre of Poland.

the following equation would hold:

$$vf(x) \;=\; \min_{0\le i\le n} vc_i + ivx\,. \tag{2}$$

We can define valuations on $K(x)$ in this way by choosing vx to be any element in some ordered abelian group which contains vK. If we choose $vx = 0$, we obtain the **Gauß valuation**.

But what if Equation (2) does not always hold? Then there are *polynomials of unexpected value*, the value of which is larger than the minimum of the values of its monomials. This observation has led to the theory of *key polynomials*, on which by now an abundant number of articles are available. Several of them present *complete sequences of key polynomials* in order to describe, or construct, all extensions.

In the present paper, we add a new aspect. When working over tame fields (K, v), we are able to prove stronger results than those for the case of general valued base fields. An algebraic extension $(L|K, v)$ is called **tame** if it is **unibranched**, i.e., the extension of v from K to L is unique, and every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(TE1) the ramification index $(vE : vK)$ is not divisible by $\operatorname{char} Kv$.
(TE2) the residue field extension $Ev|Kv$ is separable.
(TE3) the extension $(E|K, v)$ is **defectless**, i.e.,

$$[E : K] \;=\; (vE : vK)[Ev : Kv]\,.$$

A henselian field (K, v) is called a **tame field** if its algebraic closure $\widetilde{K}$ with the unique extension of the valuation is a tame extension. The Lemma of Ostrowski (see [6, 12]) shows that every henselian valued field of residue characteristic 0 is a tame field.

For the formulation of our main theorems we will need some more definitions. For an arbitrary extension $(K(x)|K, v)$, we set

$$v(x - K) \;:=\; \{v(x - c) \mid c \in K\}\,.$$

A transcendental extension $(K(x)|K, v)$ is **valuation algebraic** if $vK(x)/vK$ is a torsion group and the residue field extension $K(x)v|Kv$ is algebraic; otherwise, it is called **valuation transcendental**. In [9] we introduced the notion of an extension $(K(x)|K, v)$ being weakly pure; here we will give a simpler, but equivalent, definition (for the equivalence, see Section 2.2). We say that the extension $(K(x)|K, v)$ is **weakly pure (in** x**)** if there is $a \in K$ such that $v(x - a) = \max v(x - \widetilde{K})$, or x is limit of a pseudo Cauchy sequence in (K, v) of transcendental type. For background on pseudo Cauchy sequences, see [8].

Given an arbitrary simple extension $(K(x)|K, v)$ and a polynomial $f \in K[X]$, then we define

$$\delta(f) \;:=\; \max\{v(x - a) \mid a \text{ is a root of } f\}\,, \tag{3}$$

with $\delta(f) = -\infty$ if $f \in K$. A root a of f such that $\delta(f) = v(x - a)$ is said to be a **maximal root of** f. A monic polynomial $Q(X) \in K[X]$ is said to be a **key**

polynomial for $(K(x)|K,v)$ if

$$\deg f < \deg Q \;\Rightarrow\; \delta(f) < \delta(Q) \text{ for all } f \in K[X]\,.$$

Further, a sequence $(Q_i)_{i\in S}$ of key polynomials is said to be **complete** if for every non-constant polynomial $f \in K[X]$ there exists $i \in S$ such that $\deg Q_i \le \deg f$ and $\delta(f) \le \delta(Q_i)$.

Given any extension $(F|K,v)$, we take $\mathrm{IC}\,(F|K,v)$ to be the relative algebraic closure of K in a fixed henselization of (F,v) and call it the **implicit constant field** of $(F|K,v)$. Since the henselization F^h of any valued field (F,v) is unique up to valuation preserving isomorphism over F, the implicit constant field is unique up to valuation preserving isomorphism over K. In [9] the notion of "weakly pure" is instrumental in constructing an extension of v from K to the rational function field $K(x)$ such that $\mathrm{IC}\,(F|K,v)$ is equal to any given countably generated separable-algebraic extension of K.

The following is our main theorem for the case of simple transcendental extensions:

Theorem 1. *Let (K,v) be a tame valued field and $(K(x)|K,v)$ a transcendental extension. Then there exist monic irreducible polynomials $\{Q_i\}_{i\in S}$ and $\{Q_\nu\}_{\nu\in\Omega}$ over K, where S is an initial segment of $\mathbb{N}$, $\Omega=\emptyset$ or $\Omega=\{\nu|\nu<\lambda\}$ for some limit ordinal λ, having the following properties:*

(i) $\{Q_i\}_{i\in S}\cup\{Q_\nu\}_{\nu\in\Omega}$ *forms a complete sequence of key polynomials for* $(K(x)|K,v)$*,*

(ii) $\deg Q_1 = 1$*,*

(iii) there exist unique maximal roots a_i of Q_i and z_ν of Q_ν ,

(iv) $K(a_{i-1}) \subsetneq K(a_i)$ *and* $\deg Q_{i-1} < \deg Q_i$ *for* $1 < i \in S$*,*

(v) $v(x-a_i) > v(x-a_{i-1})$ *for* $1 < i \in S$*,*

(vi) $v(x-a_i) = \max v(x-K(a_i))$ *if $i \in S$ is not its last element or* $\Omega=\emptyset$*,*

(vii) $v(x-a_n) = \max v(x-\widetilde{K})$ *if* $S=\{1,\ldots,n\}$ *and* $\Omega=\emptyset$*,*

(viii) if $\Omega\neq\emptyset$, then S is finite, and if n is its last element, then $(z_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence of transcendental type in $(K(a_n),v)$ and we have that $\deg Q_n = \deg Q_\nu$ *for all* $\nu\in\Omega$*,*

(ix) $\mathrm{IC}\,(K(x)|K,v) = K(a_i \mid i\in S)$*, which is equal to $K(a_n)$ if* $S=\{1,\ldots,n\}$*,*

(x) for each $k\in S$, $\{Q_i\}_{1\le i\le k}$ forms a complete sequence of key polynomials for $(K(a_k)|K,v)$*.*

With $L := \mathrm{IC}\,(K(x)|K,v)$, the extension $(L(x)|L,v)$ is weakly pure. The extension $(K(x)|K,v)$ is valuation algebraic if and only if $S=\mathbb{N}$ or $\Omega\neq\emptyset$. In both cases, the extension $(L(x)|L,v)$ is immediate. In the case of $S=\mathbb{N}$, $(a_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence in (L,v) of transcendental type with x as its limit. The same holds for $(z_\nu)_{\nu<\lambda}$ if $\Omega\neq\emptyset$.

From assertion (ix) we conclude:

Corollary 2. *Let (K,v) be a tame field and $(K(x)|K,v)$ a transcendental extension. Then* $\mathrm{IC}\,(K(x)|K,v)$ *is a countably generated separable-algebraic extension of K.*

Instead of Theorem 1, we will prove the following generalization:

Theorem 3. *Let (K,v) be a henselian valued field and $(K(x)|K,v)$ a transcendental extension. Assume that there is a tame extension $(L'|K,v)$ such that for some extension of v from $K(x)$ to $L'(x)$, the extension $(L'(x)|L',v)$ is weakly pure. Then the assertions of Theorem 1 hold, and* $\mathrm{IC}\,(K(x)|K,v)\subseteq L'$.

This theorem indeed implies Theorem 1 since if (K,v) is a tame field, then the extension $(\widetilde{K}|K,v)$ is tame, and by [11, Proposition 5.2] the extension $(\widetilde{K}(x)|\widetilde{K},v)$ is always weakly pure.

We will construct the key polynomials $\{Q_i\}_{i\in S}$ by first constructing the sequence $\{a_i\}_{i\in S}$ and then taking Q_i to be the minimal polynomial of a_i over K. For this purpose, we revisit the notion of *homogeneous sequences* that was introduced in [9] (see also [2]). We develop a stronger version, which we call *key sequences*, in Section 3. This will be done simultaneously for transcendental and algebraic simple extensions. The latter leads to our main theorem for the algebraic case:

Theorem 4. *Let (K,v) be a henselian valued field and $(K(x)|K,v)$ a tame algebraic extension. Then there exist monic irreducible polynomials $\{Q_i\}_{i\in S}$ over K, where $S=\{1,\ldots,n\}$ for some $n\in\mathbb{N}$, having the following properties:*

(i) $\{Q_i\}_{i\in S}$ *forms a complete sequence of key polynomials for $(K(x)|K,v)$,*

(ii) $\deg Q_1 = 1$,

(iii) there exist unique maximal roots a_i of Q_i ,

(iv) $K(a_{i-1})\subsetneq K(a_i)$ *and* $\deg Q_{i-1}<\deg Q_i$ *for $1<i\in S$,*

(v) $v(x-a_i)>v(x-a_{i-1})$ *for $1<i\in S$,*

(vi) $v(x-a_i)=\max v(x-K(a_i))$ *for all $i\in S$,*

(vii) $a_n = x$.

Further, also assertion (x) of Theorem 1 holds.

In the case of a transcendental extension $(K(x)|K,v)$, the implicit constant field $L:=\mathrm{IC}\,(K(x)|K,v)$ can be an infinite extension of K (cf. [9, Proposition 3.16]). In the situation of Theorem 1, it is generated by the elements a_i , $i\in\mathbb{N}$, and assertion (x) shows that for every $k\in\mathbb{N}$, $\{Q_i\}_{1\leq i\leq k}$ forms a complete sequence of key polynomials for $(K(a_k)|K,v)$. So we are tempted to state that $\{Q_i\}_{i\in S}$ forms a complete sequence of key polynomials for $(L|K,v)$. However, so far our definition of key polynomials does not cover cases of extensions that are not simple. In order to address the case of implicit constant fields that are infinite extensions of the base field, and more generally, countably generated algebraic extensions, we generalize our definition in the following way. For the value defined in (3) we will now write $\delta_x(f)$. A sequence $(Q_i)_{i\in S}$ of monic irreducible polynomials $Q_i(X)\in K[X]$, where

S is an initial segment of $\mathbb{N}$, will be said to be a **strongly complete sequence of key polynomials for** $(L|K,v)$ if the sequence $(\deg Q_i)_{i\in S}$ is strictly increasing and there are roots a_i of Q_i such that the following conditions hold:

(SCKP1) $L = K(a_i \mid i \in S)$,

(SCKP2) for all $k \in S$, $(Q_i)_{i\leq k}$ is a complete sequence of key polynomials for $(K(a_k)|K,v)$.

It is not a priori clear that in the settings we considered so far, every complete sequence of key polynomials for the corresponding algebraic extensions is also strongly complete. However, assertion (x) of Theorems 1 and 4 implies:

Proposition 5. *1) If $\{Q_i\}_{i\in S}$ is a sequence of key polynomials for $(K(x)|K,v)$, then for every $k \in S$, $\{Q_i\}_{1\leq i\leq k}$ is a strongly complete sequence of key polynomials for $(K(a_k)|K,v)$.*

2) In the setting of Theorems 1 and 3, with $L = \mathrm{IC}\,(K(x)|K,v)$, we have that $(Q_i)_{i\in S}$ is a strongly complete sequence of key polynomials for $(L|K,v)$.

3) In the setting of Theorem 4, $(Q_i)_{i\in S}$ is a strongly complete sequence of key polynomials for $(K(x)|K,v)$.

The following is a generalization of Theorem 4 to the case of infinite algebraic extensions:

Theorem 6. *Take a henselian field (K,v). For every countably generated tame extension $(L|K,v)$ there exists a strongly complete sequence of key polynomials $(Q_i)_{i\in S}$ such $\deg Q_1 = 1$ and also the following holds: if a_i , $i \in S$, are the roots of the polynomials Q_i that satisfy (SCKP1) and (SCKP2), then the following hold:*

1) assertions (iv) and (x) of Theorem 4,

2) assertions (v) and (vi) of Theorem 4, with a_j in place of x for any $j \in S$, $j > i$,

3) each a_i is the unique maximal root of Q_i in the following sense: if $i \leq k \in S$ and $a_i' \neq a_i$ is another root of Q_i, then $\delta_{a_k}(Q_i) = v(a_k - a_i) > v(a_k - a_i')$.

2. Preliminaries

2.1. Tame and defectless extensions and fields, and the set $v(x-K)$.

A valued field is called **algebraically maximal** if it does not admit nontrivial immediate algebraic extensions.

Lemma 7. *Take a valued field (K,v) and any extension of v to $\widetilde{K}$.*

1) Every algebraic extension of a tame field is again a tame field.

2) A unibranched extension $(L|K,v)$ is tame if and only if $(L^h|K^h,v)$ is tame.

3) If (K,v) is henselian and (K_i,v), $i \in I$, are tame extensions of (K,v), then so is their compositum, i.e., the smallest extension that contains all K_i .

4) Assume that $(L_1|L,v)$ and $(L_2|L_1,v)$ are algebraic extensions. Then $(L_2|L,v)$ is a tame extension if and only if $(L_2|L_1,v)$ and $(L_1|L,v)$ are tame extensions.

5) Tame fields are algebraically maximal.

Proof. 1): This follows from [10, part (b) of Lemma 2.17].

2): Since the extension $(L|K,v)$ is unibranched, $L|K$ is linearly disjoint from the extension $K^h|K$ by [3, Lemma 2.1], and the same holds for every subextension $E|K$. Since $E^h = E.K^h$ and henselizations are immediate extensions, (TE1) and (TE2) hold for E and K if and only they hold for E^h and K^h in place of E and K, respectively. Since $[E:K] = [E.K^h : K^h] = [E^h : K^h]$, the same is true for (TE3).

3): By [10, part (b) of Lemma 2.13], the absolute ramification field K^r of a henselian field (K,v) is its unique maximal tame extension. Hence if (K_i,v), $i \in I$, are tame extensions of (K,v), then they are all contained in K^r and so is their compositum, which consequently is also a tame extension of (K,v).

4): For henselian fields, this is [10, part (a) of Lemma 2.13]. In the general case, $(L_2|L,v)$ is unibranched if and only if $(L_2|L_1,v)$ and $(L_1|L,v)$ are, so we can use part 2) of our lemma to reduce to the henselian case.

5): This follows from [10, Theorem 3.2]. □

Lemma 8. *Take any extension $(K(x)|K,v)$.*

1) If the extension $(K(x)|K,v)$ is immediate, then $v(x-K)$ has no largest element.

2) If $v(x-K)$ has a largest element and $v(x-y) > v(x-K)$, then the extension $(K(y)|K,v)$ is not immediate.

3) If $v(x-K)$ has no largest element, then x is the limit of a pseudo Cauchy sequence in (K,v) without a limit in K.

Proof. 1): This is [11, part 2) of Lemma 2.9].

2): By [11, part 4) of Lemma 2.9], $v(y-K) = v(x-K)$, hence also $v(y-K)$ has no largest element. Now part 1) of our lemma shows that $(K(y)|K,v)$ is not immediate.

3): The proof is a straightforward adaptation of the proof of [8, Theorem 1]. □

Lemma 9. *Suppose that in some valued field extension of (K,v), x is the pseudo limit of a pseudo Cauchy sequence in (K,v) of transcendental type. Then $(K(x)|K,v)$ is immediate and x is transcendental over K.*

Proof. Assume that $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in (K,v) of transcendental type. Then by [8, Theorem 2] there is an immediate extension w of v to the rational function field $K(y)$ such that y becomes a pseudo limit of $(a_\nu)_{\nu<\lambda}$; moreover, if also x is a pseudo limit of $(a_\nu)_{\nu<\lambda}$ in $(K(x),v)$, then $x \mapsto y$ induces a valuation preserving isomorphism from $K(x)$ onto $K(y)$ over K. Hence, $(K(x)|K,v)$ is immediate and x is transcendental over K. □

Lemma 10. *1) If $(K(b)|K,v)$ is a unibranched defectless algebraic extension, then $v(b-K)$ has a maximal element.*

2) Every pseudo Cauchy sequence in an algebraically maximal field without a limit in that field is of transcendental type.

3) Assume that (K,v) is an algebraically maximal field and $v(x-K)$ has no largest element. Then x is the limit of a pseudo Cauchy sequence of transcendental type in (K,v).

Proof. 1): This is [4, part a) of Lemma 7].

2): By [8, Theorem 3], a pseudo Cauchy sequence of algebraic type in a field (K,v) without a limit in K gives rise to a nontrivial immediate algebraic extension of (K,v), so (K,v) cannot be algebraically maximal.

3): This follows from part 2) together with part 3) of Lemma 8. □

Since tame extensions are unibranched and defectless by definition, and tame fields are algebraically maximal by part 5) of Lemma 7, we obtain:

Corollary 11. *1) If $(K(b)|K,v)$ is a tame extension, then $v(b-K)$ has a maximal element.*

2) Every pseudo Cauchy sequence in a tame field without a limit in that field is of transcendental type.

2.2. **Weakly pure extensions.**

The following shows that our definition of "weakly pure extension" given in the Introduction coincides with the definition given in [9]:

Lemma 12. *Take a valued field (K,v) and an extension of v from K to the rational function field $K(x)$. Then for $a\in K$, the following are equivalent:*

a) $v(x-a)=\max v(x-\widetilde{K})$,

b) $v(x-a)$ is non-torsion over vK or for some $d\in K$ and $e\in\mathbb{N}$, $vd(x-a)^{\mathrm{e}}=0$ and $d(x-a)^{\mathrm{e}}v$ is transcendental over Kv.

Proof. By [11, part 5) of Lemma 2.8], $v(x-a)$ is the maximal element of $v(x-\widetilde{K})$ if and only if $v(x-a)\notin v\widetilde{K}$ or $v(x-a)\in v\widetilde{K}$ and $\tilde{d}(x-a)v\notin \widetilde{K}v$ for every $\tilde{d}\in\widetilde{K}$ such that $v\tilde{d}(x-a)=0$. Note that since $v\widetilde{K}$ is the divisible hull of vK, $v(x-a)\notin v\widetilde{K}$ if and only if $v(x-a)$ is non-torsion over vK.

Assume that a) holds. If $v(x-a)$ is non-torsion over vK, then b) holds. If $v(x-a)\in v\widetilde{K}$, we proceed as follows. We choose $e\in\mathbb{N}$ such that $ev(x-a)\in vK$. Then there is some $d\in K$ such that $vd=-ev(x-a)=v(x-a)^{\mathrm{e}}$. We pick $\tilde{d}\in\widetilde{K}$ such that $\tilde{d}^e=d$. It follows that $v\tilde{d}(x-a)=0$, and from what we said above we know that $\tilde{d}(x-a)v$ is transcendental over $\widetilde{K}v=\widetilde{Kv}$. Therefore, also $(\tilde{d}(x-a))^e=d(x-a)^ev$ is transcendental over Kv, and we have shown that b) holds.

Now assume that b) holds. If $v(x-a)$ is non-torsion over vK, then $v(x-a) \notin v\widetilde{K}$, and by the equivalence stated above, a) holds. If for some $d \in K$ and $e \in \mathbb{N}$, $vd(x-a)^{\mathrm{e}} = 0$ and $d(x-a)^{\mathrm{e}}v$ is transcendental over Kv, then we proceed as follows. Let $\tilde{d} \in \widetilde{K}$ be such that $v\tilde{d}(x-a) = 0$. Then $v\tilde{d}^e = -v(x-a)^e = vd$, so that $v\tilde{d}^e d^{-1} = 0$ and $0 \neq \tilde{d}^e d^{-1} v \in \tilde{K}v$. As $d(x-a)^{\mathrm{e}}v$ is transcendental over Kv, so are $(\tilde{d}^e d^{-1})v \cdot d(x-a)^{\mathrm{e}}v = \tilde{d}^e(x-a)^{\mathrm{e}}v = (\tilde{d}(x-a)v)^e$ and $\tilde{d}(x-a)v$. That is, $\tilde{d}(x-a)v \notin \widetilde{K}v$ and again by the above equivalence, a) holds. □

The following is [9, Lemma 3.7]:

Lemma 13. *Assume that the extension $(L(x)|L,v)$ is weakly pure. If we take any extension of v to $\widetilde{L(x)}$ and take L^h to be the henselization of L in $(\widetilde{L(x)},v)$, then L^h is the implicit constant field of this extension:*

$$L^h \;=\; \mathrm{IC}\,(L(x)|L,v)\,.$$

Lemma 14. *Assume that $(L|K,v)$ is a tame extension, x is transcendental over L, and we have an extension $(L(x)|L,v)$ which is weakly pure in x.*

1) If $K'|K$ is a subextension of $L|K$ and $v(x-K')$ has no maximal element, then x is the limit of a pseudo Cauchy sequence of transcendental type in (K',v).

2) Assume that v is extended to $\widetilde{K}(x)$. Then every maximal element $v(x-a)$ of $v(x-L)$ is also a maximal element of $v(x-\widetilde{K})$.

3) We have that

$$\mathrm{IC}\,(K(x)|K,v) \;\subseteq\; \mathrm{IC}\,(L(x)|L,v) \;=\; L^h\,.$$

4) If $(L'|L,v)$ is an algebraic extension, then for every extension of v from $L(x)$ to $L'(x)$, also $(L'(x)|L',v)$ is weakly pure in x.

Proof. 1): If $K'|K$ is a subextension of $L|K$ and $v(x-K')$ has no maximal element, then by part 3) of Lemma 8, x is the limit of a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ in (K',v) without a limit in K'. Suppose that it is of algebraic type. As it is also a pseudo Cauchy sequence in (L,v) with x as its limit, it must have a limit y in L. Indeed, otherwise $v(x-L)$ would not have a maximum, so by our assumption on the extension $(L(x)|L,v)$, x would have to be the limit of a pseudo Cauchy sequence of transcendental type in (L,v). However, x cannot be simultaneously the limit of a pseudo Cauchy sequence of transcendental type in (L,v) and a pseudo Cauchy sequence of algebraic type in (L,v) without a limit in L (as follows from Theorems 3 and 4 of [8] or from the classification of immediate approximation types in [11]).

Now we have that $v(y-K')$ has no maximum, which is a contradiction to part 1) of Corollary 11 since by part 4) of Lemma 7 the extension $(L|K',v)$ is tame.

2): If $v(x-L)$ has a maximal element $v(x-a)$ with $a \in L$, then x is not the limit of a pseudo Cauchy sequence in (L,v) without a limit in L. Hence by our assumption on the extension $(L(x)|L,v)$, the set $v(x-\widetilde{K})$ must have a maximal element $v(x-a')$

with $a' \in L$. Then $v(x-a') \geq v(x-a) \geq v(x-a')$, so the values are equal. This proves our assertion.

3): This holds since $\mathrm{IC}\,(K(x)|K,v) = K(x)^h \cap \widetilde{K} \subseteq L(x)^h \cap \widetilde{K} = L(x)^h \cap \widetilde{L}$, using also Lemma 13.

4): If there is $a \in L$ such that $v(x-a) = \max v(x-\widetilde{L})$, then our assertion is trivially true since $L \subset L'$ and $\widetilde{L} = \widetilde{L}'$. Now assume that x is the limit of a pseudo Cauchy sequence $(c_\nu)_{\nu<\lambda}$ of transcendental type in (L,v). Then $(c_\nu)_{\nu<\lambda}$ is also a pseudo Cauchy sequence in (L',v). Suppose it were of algebraic type. Then by [8, Theorem 3] there would exist an algebraic extension $(L'(y),v)$ of (L',v) such that y is a limit of $(c_\nu)_{\nu<\lambda}$. However, y is also algebraic over L, which leads to a contradiction, as follows from [8, Theorem 4]. □

2.3. Krasner constant and Krasner's Lemma.

Take any valued field (K,v) and choose some extension of v from K to its algebraic closure $\widetilde{K}$. If $a \in \widetilde{K} \setminus K$ is not purely inseparable over K, then the **Krasner constant of a over K** is defined as:

$$\mathrm{Kras}(a,K) \;:=\; \max\{v(\tau a - \sigma a) \mid \sigma,\tau \in \mathrm{Gal}\,K \;\text{ and }\; \tau a \neq \sigma a\} \;\in\; v\widetilde{K}\,.$$

If $a \in K$, then we set $\mathrm{Kras}(a,K) := va$. We note:

Lemma 15. *1) The definition of* $\mathrm{Kras}(a,K)$ *does not depend on the chosen extension of v from K to $\widetilde{K}$.*
2) If the extension $(K(a)|K,v)$ is unibranched, then

$$\mathrm{Kras}(a,K) \;=\; \max\{v(a-\sigma a) \mid \sigma \in \mathrm{Gal}\,K \;\text{ and }\; a \neq \sigma a\}$$

and for all $\sigma \in \mathrm{Gal}\,K$ such that $a \neq \sigma a$,

$$va \;\leq\; v(a-\sigma a) \;\leq\; \mathrm{Kras}(a,K)\,.$$

Proof. 1): Every other extension of v from K to $\widetilde{K}$ is of the form $v\rho$ for some $\rho \in \mathrm{Gal}\,K$, and

$$\begin{aligned}
\mathrm{Kras}(a,K) &= \max\{v(\tau a - \sigma a) \mid \sigma,\tau \in \mathrm{Gal}\,K \;\text{ and }\; \tau a \neq \sigma a\} \\
&= \max\{v(\rho\tau a - \rho\sigma a) \mid \sigma,\tau \in \mathrm{Gal}\,K \;\text{ and }\; \rho\tau a \neq \rho\sigma a\} \\
&= \max\{v\rho(\tau a - \sigma a) \mid \sigma,\tau \in \mathrm{Gal}\,K \;\text{ and }\; \tau a \neq \sigma a\}\,.
\end{aligned}$$

2): For a unibranched extension $(K(a)|K,v)$ and every $\tau \in \mathrm{Gal}\,K$ we have that $v = v\tau$ on $K(a)$, whence

$$\begin{aligned}
\mathrm{Kras}(a,K) &= \max\{v(\tau a - \sigma a) \mid \sigma,\tau \in \mathrm{Gal}\,K \;\text{ and }\; \tau a \neq \sigma a\} \\
&= \max\{v\tau(a - \tau^{-1}\sigma a) \mid \sigma,\tau \in \mathrm{Gal}\,K \;\text{ and }\; a \neq \tau^{-1}\sigma a\} \\
&= \max\{v(a-\sigma a) \mid \sigma \in \mathrm{Gal}\,K \;\text{ and }\; a \neq \sigma a\}\,,
\end{aligned}$$

and the inequality $va \leq v(a-\sigma a)$ follows from the fact that $va = v\sigma a$. □

Lemma 16. *If $a \in \widetilde{K}$ and $(L(a)|K(a),v)$ is any valued field extension, then*

$$\mathrm{Kras}(a,L) \;\leq\; \mathrm{Kras}(a,K)\,. \tag{4}$$

Proof. Since a is separable-algebraic over K, it is also separable-algebraic over L. If $\sigma \in \mathrm{Gal}\,L$, then $\sigma|_{K^{\mathrm{sep}}} \in \mathrm{Gal}\,K$; therefore,

$$\begin{aligned} &\{v(\tau a - \sigma a) \mid \sigma,\tau \in \mathrm{Gal}\,L \text{ and } \tau a \neq \sigma a\} \\ &\qquad\subseteq\;\; \{v(\tau a - \sigma a) \mid \sigma,\tau \in \mathrm{Gal}\,K \text{ and } \tau a \neq \sigma a\}\,. \end{aligned}$$

This implies inequality (4). □

We will employ the following variant of Krasner's Lemma:

Proposition 17. *Take $K(a)|K$ to be a separable-algebraic extension, and $(K(a,b),v)$ to be any valued field extension of $(K(a),v)$ such that*

$$v(b-a) \;>\; \mathrm{Kras}(a,K)\,. \tag{5}$$

Then every henselization of $(K(b),v)$ contains a.

Proof. Take any extension of v from $K(a,b)$ to $\widetilde{K(b)}$, and take $K(b)^h$ to be the henselization of $(K(b),v)$ with respect to this extension. Then by Lemma 16 applied to $L = K(b)^h$, $\mathrm{Kras}(a,K(b)^h) \leq \mathrm{Kras}(a,K)$. Hence by assumption, $v(b-a) > \mathrm{Kras}(a,K(b)^h)$. We will show that a is fixed by every automorphism $\rho \in \mathrm{Gal}\,K(b)^h$; since a is also separable-algebraic over $K(b)^h$, this will yield that $a \in K(b)^h$. Note that $\rho b = b$ because of $\rho \in \mathrm{Gal}\,K(b)^h$.

Since $(K(b)^h,v)$ is henselian, using the assumption we may compute:

$$v(b-\rho a) \;=\; v\rho(b-a) \;=\; v(b-a) \;>\; \mathrm{Kras}(a,K)\,.$$

In view of Lemma 16, it follows that

$$\begin{aligned} v(a-\rho a) \;&\geq\; \min\{v(b-a),v(b-\rho a)\} \;>\; \mathrm{Kras}(a,K) \;\geq\; \mathrm{Kras}(a,K(b)^h) \\ &\geq\; \max\{v(a-\sigma a) \mid \sigma \in \mathrm{Gal}\,K \text{ and } a \neq \sigma a\}\,, \end{aligned}$$

which yields that $a = \rho a$. □

3. Key sequences

3.1. Homogeneous approximations.

In this section we will lay out an improved version of the theory of homogeneous elements and approximations that was introduced in [9] (cf. also [2]). In contrast to those articles, we will work exclusively with what there was called *strongly homogenous elements*, and we will strengthen the definition of *homogeneous approximations* accordingly. However, in order to simplify notation, we will drop the word "strongly".

We will say that an element a is **homogeneous over** (K,v) (or just **over** K when it is clear which valuation we refer to) if $a\in K^{\rm sep}$, the extension $(K(a)|K,v)$ is unibranched, and

$$va \;=\; \mathrm{Kras}(a,K)\;.$$

Note that if $a\in K$, then $\mathrm{Kras}(a,K)=va$ by definition, so a is homogeneous over K.

Take a second element b in some algebraically closed valued field extension (L,v) of (K,v). We will say that a is a **homogeneous approximation of** b **over** K if a is homogeneous over K and $v(b-a)>vb$. From this it follows that $va=vb$ and $v(b-a)>\mathrm{Kras}(a,K)$.

Remark 18. We note that our present use of "homogeneous approximation" is almost the same as in the definition of homogeneous sequences in [9, 2], except that now all homogeneous approximations are taken over K; this is a stronger condition.

Lemma 19. *1) Assume that a is homogeneous over K. Then*

$$va=v(\tau a-\sigma a)\ \text{ for all } \sigma,\tau\in\mathrm{Gal}\,K \text{ such that } \sigma a\neq\tau a\,, \tag{6}$$

and if $(L(a)|K(a),v)$ is any valued field extension, then a is also homogeneous over L.

2) Take elements a,b,b' in some algebraically closed valued field extension (L,v) of (K,v). If a is a homogeneous approximation of b over K and if $v(b-b')\geq v(b-a)$, then a is also a homogeneous approximation of b' over K.

Proof. 1): Assume that a is homogeneous over K. Then $v\tau a=va=v\sigma a$ since the extension $(K(a)|K,v)$ is unibranched, whence $va\leq v(\tau a-\sigma a)\leq\mathrm{Kras}(a,K)=va$ for all $\sigma,\tau\in\mathrm{Gal}\,K$ such that $\sigma a\neq\tau a$. Hence equality holds everywhere, which proves (6). If $(L(a)|K(a),v)$ is any valued field extension, then $a\in L^{\rm sep}$, $v\sigma a= v\sigma|_{K^{\rm sep}}a=va$ for all $\sigma\in\mathrm{Gal}\,L$, and for every $\sigma,\tau\in\mathrm{Gal}\,L$ with $\tau a\neq\sigma a$, we have that

$$v(\tau a-\sigma a)\;=\;v(\tau|_{K^{\rm sep}}\,a-\sigma|_{K^{\rm sep}}\,a)\;=\;va\,,$$

whence $\mathrm{Kras}(a,L)=va$.

2): By assumption we have that $v(b'-a)\geq\min\{v(b-b'),v(b-a)\}=v(b-a)> vb=vb'$. This yields the assertion, since a is homogeneous over K. □

The following important property of homogeneous approximations is a consequence of Proposition 17:

Lemma 20. *If a is a homogeneous approximation of b over K, then a lies in the henselization of $K(b)$ w.r.t. every extension of the valuation v from $K(a,b)$ to $\widetilde{K(b)}$.*

The following gives a crucial criterion for an element to be homogeneous over K:

Lemma 21. *Suppose that $a\in\widetilde{K}$ and that there is some extension of v from K to $K(a)$ such that if e is the least positive integer for which $\mathrm{e}va\in vK$, then*

a) e is not divisible by $\mathrm{char}\,Kv$, *b) there exists some $c\in K$ such that $vca^{\mathrm{e}}=0$, $ca^{\mathrm{e}}v$ is separable-algebraic over Kv, and the degree of ca^{e} over K is equal to the degree f of $ca^{\mathrm{e}}v$ over Kv.*

Then $(K(a)|K,v)$ is a separable unibranched extension of degree $\mathrm{e}\cdot\mathrm{f}$, and a is homogeneous over K.

Proof. We have

$$\begin{aligned}\mathrm{e}\cdot\mathrm{f} &\geq [K(a):K(a^{\mathrm{e}})]\cdot[K(a^{\mathrm{e}}):K] \;=\; [K(a):K]\\ &\geq (vK(a):vK)\cdot[K(a)v:Kv] \;\geq\; \mathrm{e}\cdot\mathrm{f}\,.\end{aligned}$$

Hence equality holds everywhere, and we find that $[K(a):K]=\mathrm{e}\cdot\mathrm{f}$, $(vK(a):vK)=\mathrm{e}$ and $[K(a)v:Kv]=\mathrm{f}$. By the fundamental inequality, this implies that the extension $(K(a)|K,v)$ is unibranched. By assumption a), the extension $K(a)|K(a^{\mathrm{e}})$ is separable. By assumption b), the residue field extension $K(a^{\mathrm{e}})v|Kv$ is separable of degree $[K(a^{\mathrm{e}}):K]$, which shows that $K(a^{\mathrm{e}})|K$ must be separable. Altogether, we find that $K(a)|K$ is separable.

In order to show that a is homogeneous over K, we may assume that $a\notin K$. Take $\sigma\in\operatorname{Gal}K$ with $\sigma a\neq a$ and set $\eta:=\sigma a/a\neq 1$. If $\sigma a^{\mathrm{e}}\neq a^{\mathrm{e}}$, then $\sigma ca^{\mathrm{e}}=c\sigma a^{\mathrm{e}}\neq ca^{\mathrm{e}}$ and by hypothesis, their residues are also distinct, so the residue of $\sigma a^{\mathrm{e}}/a^{\mathrm{e}}=\eta^{\mathrm{e}}$ is not 1. It then follows that the residue of η is not 1. If $\sigma a^{\mathrm{e}}=a^{\mathrm{e}}$, then η is an e-th root of unity. Since e is not divisible by the residue characteristic, it again follows that the residue of η is not equal to 1. Hence in both cases, we obtain that $v(\eta-1)=0$, which shows that $v(\sigma a-a)=va$. We have now proved our lemma. □

Lemma 22. *Assume that b is an element in some algebraically closed valued field extension (L,v) of (K,v). Assume further that there are $\mathrm{e}\in\mathbb{N}$ not divisible by $\operatorname{char}Kv$ and $c\in K$ such that $vcb^{\mathrm{e}}=0$ and $cb^{\mathrm{e}}v$ is separable-algebraic of degree f over Kv. Then we can find a homogeneous approximation $a\in\tilde{K}$ of b over K, of degree $\mathrm{e}\cdot\mathrm{f}$ over K.*

If $\tilde{a}\in\tilde{K}$ satisfies $v(b-\tilde{a})>vb$, then $[K(\tilde{a}):K]\geq\mathrm{e}\cdot\mathrm{f}$; if in addition $\tilde{a}\in K(a)$, then $\tilde{a}$ is also a homogeneous approximation of b over K, $K(\tilde{a})=K(a)$, and $\operatorname{Kras}(\tilde{a},K)=\operatorname{Kras}(a,K)$.

Proof. Take a monic polynomial g over K with v-integral coefficients whose reduction modulo v is the minimal polynomial of $cb^{\mathrm{e}}v$ over Kv. Then let $a_0\in\widetilde{K}$ be the root of g whose residue is $cb^{\mathrm{e}}v$. The degree of a_0 over K is the same as that of $cb^{\mathrm{e}}v$ over Kv. We have that $v(\frac{a_0}{cb^{\mathrm{e}}}-1)>0$. So there exists $a_1\in\widetilde{K}$ with residue 1 and such that $a_1^{\mathrm{e}}=\frac{a_0}{cb^{\mathrm{e}}}$. Then for $a:=a_1b$, we find that $v(b-a)=vb+v(a_1-1)>vb$ and $ca^{\mathrm{e}}=a_0$, hence $a\in\tilde{K}$. It follows that $va=vb$ and $ca^{\mathrm{e}}v=cb^{\mathrm{e}}v$. By the foregoing lemma, this shows that a is homogeneous over K.

Now assume that also $\tilde{a}\in\tilde{K}$ satisfies $v(b-\tilde{a})>vb$. Then $v\tilde{a}=vb$, whence $(vK(\tilde{a}):vK)\geq\mathrm{e}$. Further, $v(cb^{\mathrm{e}}-c\tilde{a}^{\mathrm{e}})>0$ and thus, $c\tilde{a}^{\mathrm{e}}v=cb^{\mathrm{e}}v$ and $[K(\tilde{a})v:Kv]\geq\mathrm{f}$. Therefore, $[K(\tilde{a}):K]\geq\mathrm{e}\cdot\mathrm{f}$. We note that

$$v(\tilde{a}-a)\;\geq\;\min\{v(b-a),v(b-\tilde{a})\}\;>\;vb\;=\;va\;=\;v\tilde{a}\,.$$

Finally, assume in addition that $\tilde{a} \in K(a)$. Then

$$\mathrm{e} \cdot \mathrm{f} \;\leq\; [K(\tilde{a}) : K] \;\leq\; [K(a) : K] \;=\; \mathrm{e} \cdot \mathrm{f}\;,$$

showing that $K(\tilde{a}) = K(a)$. Thus for every $\sigma \in \operatorname{Gal} K$ we have that $\sigma\tilde{a} \neq \tilde{a}$ if and only if $\sigma a \neq a$. If this is the case, then

$$v(\tilde{a} - \sigma\tilde{a}) \;=\; v(\tilde{a} - a + a - \sigma a + \sigma a - \sigma\tilde{a})\;.$$

As $(K(a)|K, v)$ is unibranched and $K(\tilde{a}) = K(a)$, we know that

$$v(\sigma a - \sigma\tilde{a}) \;=\; v\sigma(a - \tilde{a}) \;=\; v(a - \tilde{a}) \;>\; va \;=\; \mathrm{Kras}(a, K) \;\geq\; v(a - \sigma a)\;.$$

Consequently, $v(\tilde{a} - \sigma\tilde{a}) = v(a - \sigma a)$ and therefore, $\mathrm{Kras}(\tilde{a}, K) = \mathrm{Kras}(a, K) = va = v\tilde{a}$. This proves that also a is homogeneous over K. □

3.2. **Key sequences.**

We will work with a variant of the notion of "homogeneous sequence" which was introduced in [9], and the stronger notion of "key sequence". Let $(K(x)|K, v)$ be any extension of valued fields. We fix an extension of v to $\widetilde{K(x)}$.

Let S be an initial segment of $\mathbb{N}$, that is, $S = \mathbb{N}$ or $S = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. A sequence

$$\mathfrak{S} \;:=\; (a_i)_{i \in S}$$

of elements in $\widetilde{K}$ will be called a **key sequence for** $(K(x)|K, v)$ if

(KS1) $a_1 \in K$,

(KS2) if $1 < i \in S$, then $K(a_i) = K(\tilde{a}_i)$ and $v(x - a_i) \geq v(x - \tilde{a}_i)$ for some $\tilde{a}_i$ such that $\tilde{a}_i - a_{i-1}$ is a homogeneous approximation of $x - a_{i-1}$ over K,

(KS3) $v(x - a_i) = \max v(x - K(a_i))$, unless $v(x - K(a_i))$ has no largest element, in which case i is the last element of S.

Further, $\mathfrak{S}$ will be called a **pre-complete key sequence** if in addition to the above, $S = \mathbb{N}$ or the following conditions are satisfied:

(CKS1) if $x \in \widetilde{K}$, then S is finite, and if n is its last element, then $a_n = x$,

(CKS2) if x is transcendental over K and $S = \{1, \ldots, n\}$, then either $v(x - a_n) = \max v(x - \widetilde{K})$, or x is limit of a pseudo Cauchy sequence of transcendental type in $(K(a_n), v)$,

and it will be called **complete** if the second case in (CKS2) does not appear.

We call S the **support** of the sequence $\mathfrak{S}$. We set

$$K_{\mathfrak{S}} \;:=\; K(a_i \mid i \in S)\;.$$

If $\mathfrak{S}$ is the empty sequence, then $K_{\mathfrak{S}} = K$.

From the above definitions, the following is obvious:

Lemma 23. *Assume that $S' \subseteq S$ are initial segments of $\mathbb{N}$. If $(a_i)_{i \in S}$ is a key sequence for $(K(x)|K, v)$, then so is $(a_i)_{i \in S'}$.*

From the definition of the key sequence $(a_i)_{i\in S}$, we obtain:

Lemma 24. *Take a key sequence $(a_i)_{i\in S}$. Then the following statements hold:*
1) For every $i \in S$, $a_i \in K^{\rm sep}$.
2) For every $i \in S$,

$$v(x-a_i) \; > \; v(x-a_{i-1}) \quad \text{if } 1 < i \in S \tag{7}$$

and

$$a_i \;\notin\; K(a_{i-1})\,. \tag{8}$$

3) If $i,j \in S$ with $i < j$, then

$$v(x-a_j) \; > \; v(x-a_i) \; = \; v(a_{i+1}-a_i)\,. \tag{9}$$

If $S = \mathbb{N}$, then $(a_i)_{i\in S}$ is a pseudo Cauchy sequence in $K_{\mathfrak{G}}$ with pseudo limit x.

Proof. 1): We proceed by induction on $i \in S$. By (KS1), $a_1 \in K \subseteq K^{\rm sep}$. Assume that $1 < i \in S$ and we have already shown that $a_{i-1} \in K^{\rm sep}$. By (KS2), $\tilde{a}_i - a_{i-1} \in K^{\rm sep}$, whence $\tilde{a}_i \in K^{\rm sep}$ and $a_i \in K(\tilde{a}_i) \subseteq K^{\rm sep}$.
2): By (KS2), for $1 < i \in S$ we have:

$$v(x-a_i) \;\geq\; v(x-\tilde{a}_i) \;=\; v(x-a_{i-1}-(\tilde{a}_i-a_{i-1})) \;>\; v(x-a_{i-1})\,.$$

This proves (7). Assertion (8) follows from (KS3) since $v(x-a_i) > v(x-a_{i-1}) = \max v(x-K(a_{i-1}))$.
3): Take $i,j \in S$ with $1 \leq i < j$. Then (9) follows by induction from (7). Now assume that $S = \mathbb{N}$. Then for all $k > j > i \geq 1$,

$$v(x-a_k) \;>\; v(x-a_j) \;>\; v(x-a_i)$$

and therefore,

$$\begin{aligned} v(a_k-a_j) \;&=\; \min\{v(x-a_k), v(x-a_j)\} \;=\; v(x-a_j) \\ &>\; v(x-a_i) \;=\; \min\{v(x-a_j), v(x-a_i)\} \;=\; v(a_j-a_i)\,. \end{aligned}$$

This shows that $(a_i)_{i\in S}$ is a pseudo Cauchy sequence. The equality in (9) shows that x is a pseudo limit of this sequence. □

Let us also observe the following:

Lemma 25. *Let x, z be elements in some valued field extension of (K,v) that contains $\tilde{K}$. Take a key sequence $(a_i)_{i\in S}$ for $(K(x)|K,v)$. Then the following assertions hold:*
1) Assume that $v(x-z) > v(x-a_i)$ for all $i \in S$. Then $(a_i)_{i\in S}$ is also a key sequence for $(K(z)|K,v)$.
2) Take $k \in S$. Then $(a_i)_{i\leq k}$ is a complete key sequence for $(K(a_k)|K,v)$.

Proof. 1): It follows from part 2) of Lemma 19 that if (KS2) holds, then it also holds with z in place of x. We now show the same for (KS3). Since $v(x-z) > v(x-a_i)$, it follows that $v(z-a_i) = \min\{v(x-a_i), v(x-z)\} = v(x-a_i)$. In order to complete our proof, we need to show that $v(z-y) \leq v(z-a_i)$ for $y \in K(a_i)$. Suppose otherwise. Then $v(x-y) \geq \min\{v(x-z), v(z-y)\} > v(x-a_i) = \max v(x-K(a_i))$, contradiction. This shows that (KS3) also holds with z in place of x.

2): By Lemma 23, $(a_i)_{i<k}$ is a key sequence for $(K(x)|K,v)$. Since $v(x-a_k) > v(x-a_i)$ for $i<k$, part 1) shows that it is also a key sequence for $(K(a_k)|K,v)$. Next, we show that (KS2) also holds for a_k in place of x and k in place of i. Since $(a_i)_{i\in S}$ is a key sequence for $(K(x)|K,v)$, we know already that $K(a_k) = K(\tilde{a}_k)$, and it remains to show that $a := \tilde{a}_k - a_{k-1}$ is a homogeneous approximation of $b' := a_k - a_{k-1}$ over K. With $b := x - a_{k-1}$, we compute:

$$v(b-b') \;=\; v(x-a_k) \;\geq\; v(x-\tilde{a}_k) \;=\; v(b-a)\,.$$

Hence the required result follows from part 2) of Lemma 19. Also (KS3) is satisfied for a_k in place of x and k in place of i because $v(a_k-a_k) = \infty = \max v(x-K(a_k))$. This also proves that by definition, $(a_i)_{i\leq k}$ is a complete key sequence for $(K(a_k)|K,v)$. □

What is special about key sequences is described by the following lemma:

Lemma 26. *Assume that $(a_i)_{i\in S}$ is a key sequence for $(K(x)|K,v)$. Then*

$$K_{\mathfrak{S}} \;\subseteq\; K(x)^h\,. \tag{10}$$

For every $n\in S$, $a_1,\ldots,a_n \in K(a_n)^h$. If $S=\{1,\ldots,n\}$, then

$$K_{\mathfrak{S}}^h \;=\; K(a_n)^h\,. \tag{11}$$

Hence if (K,v) is henselian, then $K_{\mathfrak{S}} = K(a_n)$.

Proof. By induction on $i\in S$, we show that $a_i \in K(x)^h$. By (KS1), $a_1 \in K \subseteq K(x)^h$. Assume that $1 < i \in S$ and we have already shown that $a_{i-1} \in K(x)^h$. As $\tilde{a}_i - a_{i-1}$ is a homogeneous approximation of $x - a_{i-1} \in K(x)^h$ over K for $\tilde{a}_i$ as in (KS2), we know from Lemma 20 that

$$\tilde{a}_i - a_{i-1} \;\in\; K(x-a_{i-1})^h \;\subseteq\; K(x)^h$$

and hence also $\tilde{a}_i \in K(x)^h$. This implies that $a_i \in K(\tilde{a}_i) \subset K(x)^h$, which proves (10).

Now all other assertions follow when we replace x by a_n in the above argument, using the fact that by Lemma 23 with $S' = \{1,\ldots,n\}$, $(a_i)_{i\leq n}$ is a key sequence for $(K(a_n)|K,v)$. □

Lemma 27. *Every key sequence $(a_i)_{i\in S}$ has property*

(KS4) $K(a_{i-1})^h \subsetneq K(a_i)^h$ *if $1 < i \in S$.*

Proof. Suppose that $1 < i \in S$ and $K(a_{i-1})^h = K(a_i)^h$. Then $a_i \in K(a_{i-1})^h$ and since $(K(a_{i-1})^h|K(a_{i-1}),v)$ is immediate, by part 1) of Lemma 8 the set $v(a_i - K(a_{i-1}))$ has no maximum. Hence there exists $c \in K(a_{i-1})$ such that $v(a_i - c) > v(a_i - a_{i-1})$. On the other hand, $v(x - a_i) > v(x - a_{i-1})$ by Lemma 24, whence

$$v(a_i - a_{i-1}) \;=\; \min\{v(x-a_i), v(x-a_{i-1})\} \;=\; v(x-a_{i-1})\,.$$

We conclude that

$$v(x-c) \;=\; \min\{v(x-a_i), v(a_i-c)\} \;>\; v(x-a_{i-1})$$

in contradiction to (KS3). This proves our assertion. □

Proposition 28. *Take a henselian field (K,v). Assume that $(a_i)_{i\in S}$ is a key sequence for $(K(x)|K,v)$. Then it has the following additional properties:*

(KS5) $\mathrm{Kras}(a_i,K) = \mathrm{Kras}(a_i - a_{i-1},K) = v(x-a_{i-1})$ *if* $1 < i \in S$,

(KS6) *if $1 < i \in S$ and $z \in \tilde{K}$ such that $v(x-z) > v(x-a_{i-1})$, then $[K(z) : K] \geq [K(a_i) : K]$.*

This proposition follows by induction on $i \in S \setminus \{1\}$ from a slightly more general result:

Lemma 29. *Take a henselian field (K,v). Assume that $i > 1$, $(a_j)_{j<i}$ is a key sequence for $(K(x)|K,v)$ with properties (KS5) and (KS6)), that $\tilde{a}_i - a_{i-1}$ is a homogeneous approximation of $x - a_{i-1}$ over K, and that $a_i \in K(\tilde{a}_i)$ with $v(x-a_i) \geq v(x-\tilde{a}_i)$. Then $K(a_i) = K(\tilde{a}_i)$, and the sequence $(a_j)_{j\leq i}$ has properties (KS5) and (KS6).*

Proof. We will first prove (KS5) for $\tilde{a}_i$ in place of a_i. We start with the case of $i = 2$, where $a_{i-1} = a_1 \in K$ by (KS1). Hence for every $\sigma \in \mathrm{Gal}\,K$,

$$v(\tilde{a}_2 - \sigma\tilde{a}_2) \;=\; v(\tilde{a}_2 - a_1 - \sigma(\tilde{a}_2 - a_1))\,.$$

This implies that $\mathrm{Kras}(\tilde{a}_2,K) = \mathrm{Kras}(\tilde{a}_2 - a_1,K) = v(\tilde{a}_2 - a_1)$.

Now we consider the case of $2 < i \in S$. By the assumption of our lemma, (KS5) holds with $i-1$ in place of i. We wish to compute $v(\tilde{a}_i - \sigma\tilde{a}_i)$ whenever $\sigma \in \mathrm{Gal}\,K$ with $\tilde{a}_i \neq \sigma\tilde{a}_i$. We set $a := \tilde{a}_i - a_{i-1}$. Since $\tilde{a}_i = a_{i-1} + a$, we must have that $a_{i-1} \neq \sigma a_{i-1}$ or $a \neq \sigma a$, and

$$v(\tilde{a}_i - \sigma\tilde{a}_i) \;=\; v(a_{i-1} - \sigma a_{i-1} + a - \sigma a)\,.$$

If $a_{i-1} \neq \sigma a_{i-1}$, then

$$v(a_{i-1} - \sigma a_{i-1}) \;\leq\; \mathrm{Kras}(a_{i-1},K) \;=\; v(x-a_{i-2}) \;<\; v(x-a_{i-1})\,,$$

where we have used (7) for the last inequality. If $a \neq \sigma a$, then $v(a-\sigma a) = va = v(x-a_{i-1})$ since a is a homogeneous approximation of $x - a_{i-1}$ over K. In both cases,

$$v(\tilde{a}_i - \sigma\tilde{a}_i) \;=\; \min\{v(a_{i-1} - \sigma a_{i-1}), v(a-\sigma a)\} \;\leq\; v(x-a_{i-1})\,. \tag{12}$$

On the other hand, as $v(x-\tilde{a}_i) > v(x-a_{i-1}) = \max v(x-K(a_{i-1}))$ by our assumption on $\tilde{a}_i - a_{i-1}$ and (KS3) with $i-1$ in place of i, we have that $\tilde{a}_i \notin K(a_{i-1})$. Consequently, $K(a_{i-1},\tilde{a}_i) = K(a_{i-1},a)$ is a nontrivial extension of $K(a_{i-1})$. It is a separable-algebraic extension of K since by part 1) of Lemma 24, a_{i-1} is separable over K, and the same holds for a as it is homogeneous over K. Hence there is $\sigma \in \operatorname{Gal} K$ such that $a_{i-1} = \sigma a_{i-1}$ and $a \neq \sigma a$, in which case $v(a_i - \sigma a_i) = v(x - a_{i-1})$. Consequently, $\operatorname{Kras}(\tilde{a}_i, K) = v(x-a_{i-1})$.

Now we take $a_i \in K(\tilde{a}_i)$ with $v(x-a_i) \geq v(x-\tilde{a}_i)$. We set $d := a_i - \tilde{a}_i$ and observe that $vd \geq v(x-\tilde{a}_i) > v(x-a_{i-1})$. As (K,v) is henselian, we have that $vd = v\sigma d$ and hence $v(d-\sigma d) > v(x-a_{i-1})$ for all $\sigma \in \operatorname{Gal} K$. Assuming that $\sigma\tilde{a}_i \neq \tilde{a}_i$, we can use (12) to obtain that

$$v(d-\sigma d) \;>\; v(\tilde{a}_i - \sigma\tilde{a}_i)\,,$$

which then yields:

$$v(a_i - \sigma a_i) \;=\; v(\tilde{a}_i - \sigma\tilde{a}_i + d - \sigma d) \;=\; \min\{v(\tilde{a}_i - \sigma\tilde{a}_i), v(d-\sigma d)\} \;=\; v(\tilde{a}_i - \sigma\tilde{a}_i)\,.$$

In particular, $\sigma\tilde{a}_i \neq \tilde{a}_i$ implies that $\sigma a_i \neq a_i$. Since $a_i \in K(\tilde{a}_i)$ and $K(\tilde{a}_i)|K(a_i)$ is a separable extension, we find that $K(a_i) = K(\tilde{a}_i)$. Hence the conditions $\sigma a_i \neq a_i$ and $\sigma\tilde{a}_i \neq \tilde{a}_i$ are equivalent for all σ, and from what we have just shown we find that $v(a_i - \sigma a_i) = v(\tilde{a}_i - \sigma\tilde{a}_i)$ always holds. Therefore,

$$\operatorname{Kras}(a_i, K) \;=\; \operatorname{Kras}(\tilde{a}_i, K) \;=\; v(x-a_{i-1})\,.$$

In order to show that (KS6) holds, assume that $z \in \tilde{K}$ satisfies $v(x-z) > v(x-a_{i-1})$. Since also $v(x-a_i) > v(x-a_{i-1})$, we have that

$$v(z-a_i) \;\geq\; \min\{v(x-z), v(x-a_i)\} \;>\; v(x-a_{i-1}) \;=\; \operatorname{Kras}(a_i,K)\,,$$

where the last equation holds by (KS5). Hence from Proposition 17 it follows that $a_i \in K(z)$, which proves that (KS6) holds. □

Proposition 30. *Take any valued field (K,v) and any extension $(K(x)|K,v)$. Assume that $\mathfrak{S} = (a_i)_{i\in\mathbb{N}}$ is a key sequence for $(K(x)|K,v)$. Then $\mathfrak{S}$ is a complete key sequence for $(K(x)|K,v)$, x is transcendental over K, $(a_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence of transcendental type in $(K_{\mathfrak{S}},v)$ with pseudo limit x, and $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}},v)$ is immediate.*

Proof. $\mathfrak{S}$ is complete as it satisfies (CKS2) because its support is $\mathbb{N}$.

By Lemma 24, $(a_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence in $K_{\mathfrak{S}}$ with pseudo limit x. Suppose $(a_i)_{i\in\mathbb{N}}$ were of algebraic type. Then by [8, Theorem 3], there would exist some algebraic extension $(K_{\mathfrak{S}}(y)|K_{\mathfrak{S}},v)$ with y a pseudo limit of the sequence. But then $v(x-y) > v(x-a_i)$ for all $i \in \mathbb{N}$ and by part 1) of Lemma 25, $(a_i)_{i\in S}$ is also a key sequence for $(K(y)|K,v)$. Hence by Lemma 26, $K_{\mathfrak{S}} \subset K(y)^h = K^h(y)$. Since y is algebraic over K, the extension $K^h(y)|K^h$ is finite. On the other hand, $K_{\mathfrak{S}}^h|K^h$ is infinite since (KS4) holds. This contradiction shows that the sequence $(a_i)_{i\in\mathbb{N}}$ is of transcendental type. Hence x is transcendental over K and it follows from Lemma 9 that $(K_{\mathfrak{S}}(x)|K_{\mathfrak{S}},v)$ is immediate. □

The following is Theorem 5.9 of [9], reformulated using the notion of "pre-complete key sequence".

Theorem 31. *Assume that $(K(x)|K,v)$ is a transcendental extension and $\mathfrak{S}$ is a pre-complete key sequence for $(K(x)|K,v)$. Then*

$$K^h_{\mathfrak{S}} \;=\; \mathrm{IC}\,(K(x)|K,v)\,.$$

Further, $K_{\mathfrak{S}}v$ is the relative algebraic closure of Kv in $K(x)v$, and the torsion subgroup of $vK(x)/vK_{\mathfrak{S}}$ is finite.

3.3. **Key sequences and key polynomials.**

Proposition 32. *Take any valued field (K,v), an extension $(K(x)|K,v)$, and a key sequence $\mathfrak{S}=(a_i)_{i\in S}$ for $(K(x)|K,v)$ with additional properties (KS5) and (KS6). Let $Q_i\in K[X]$ be the minimal polynomial of a_i over K. Then Q_i is a key polynomial and a_i is the unique maximal root of Q_i.*

Assume in addition that $\mathfrak{S}$ is complete. Then $(Q_i)_{i\in S}$ is a complete sequence of key polynomials for $(K(x)|K,v)$.

Proof. By (KS5) and (7) we know that whenever $a_i'\neq a_i$ is a conjugate of a_i over K, then $v(a_i-a_i')\leq v(x-a_{i-1})<v(x-a_i)$; this implies that $v(x-a_i')=\min\{v(x-a_i),v(a_i-a_i')\}<v(x-a_i)$. Hence a_i is the unique maximal root of Q_i. In particular, $\delta(Q_i)=v(x-a_i)$.

Assume that $f\in K[X]$ such that $\delta(f)\geq\delta(Q_i)=v(x-a_i)$. Take a root z of f such that $v(x-z)=\delta(f)\geq v(x-a_i)>v(x-a_{i-1})$. Then by (KS6), $\deg f\geq [K(z):K]\geq [K(a_i):K]=\deg Q_i$. This proves that Q_i is a key polynomial.

We turn to the last assertion of our proposition. We note that if $S=\{1,\ldots,n\}$ and x is transcendental over K, then by our assumption, $v(x-a_n)=\max v(x-\widetilde{K})$; this also holds if x is algebraic over K, since then by the definition of completeness, $x=a_n$ and thus $v(x-a_n)=\infty$.

We wish to show that for every $f(X)\in K[X]$ there exists $i\in S$ such that $\deg Q_i\leq\deg f$ and $\delta(f)\leq\delta(Q_i)$. Let z be a root of f such that $v(x-z)=\delta(f)$. Take $i\in S$ maximal with $\deg Q_i\leq\deg f$. Suppose that $v(x-z)>v(x-a_i)$. Then by what we have said above, i cannot be the last element of S. By (KS6), $\deg f\geq [K(z):K]\geq [K(a_{i+1}):K]=\deg Q_{i+1}$, which contradicts our choice of i. This shows that $\delta(f)=v(x-z)\leq v(x-a_i)=\delta(Q_i)$. □

3.4. **Key sequences and tameness.**

We wish to characterize the extensions $(K(x)|K,v)$ for which there exist key sequences.

If an element $a\in\widetilde{K}$ satisfies the conditions of Lemma 21, then $(K(a)|K,v)$ is a tame extension. More generally, the following holds.

Proposition 33. *Take any valued field (K, v). If a is homogeneous over K, then $(K(a)|K, v)$ is a tame extension. If (K, v) is henselian and $\mathfrak{S}$ is a key sequence for $(K(x)|K, v)$, then $K_{\mathfrak{S}}$ is a tame extension of K.*

Proof. If $(K(a)|K, v)$ is not a tame extension, then by part 2) of Lemma 7, also $(K(a)^h|K^h, v)$ is not a tame extension. Then a does not lie in the ramification field K^r of the extension $(K^{\mathrm{sep}}|K^h, v)$ since by [10, part (b) of Lemma 2.13], K^r is the unique maximal tame extension of (K^h, v). So there exists an automorphism σ in the ramification group such that $\sigma a \neq a$. But by the definition of the ramification group,

$$\mathrm{Kras}(a, K) \;\geq\; v(\sigma a - a) \;>\; va\,,$$

showing that a is not homogeneous over K.

Now assume that (K, v) is henselian and $\mathfrak{S} = (a_i)_{i\in S}$ is a key sequence for $(K(x)|K, v)$. By induction on $i \in S$, we prove that $(K(a_i)|K, v)$ is a tame extension. This holds for $i = 1$ since $a_1 \in K$. Now assume that we have already shown that $(K(a_{i-1}), v) = (K(\tilde{a}_{i-1}), v)$ is a tame extension of (K, v). Since $\tilde{a}_i - a_{i-1}$ is homogeneous over K, the first part of our proposition shows that $(K(\tilde{a}_i - a_{i-1})|K, v)$ is a tame extension. Hence by parts 3) and 4) of Lemma 7, $(K(a_{i-1}, \tilde{a}_i - a_{i-1}), v)$ and its subfield $(K(a_i), v) = (K(\tilde{a}_i), v)$ are tame extensions of (K, v). As $K_{\mathfrak{S}} = K(a_i \mid i \in S)$, we can again employ part 3) of Lemma 7 to conclude that $(K_{\mathfrak{S}}|K, v)$ is a tame extension. □

We now prove the existence of key sequences under suitable tameness assumptions:

Proposition 34. *Take a henselian field (K, v) and an extension $(K(x)|K, v)$. If $x \in \widetilde{K}$, then assume that the extension $(K(x)|K, v)$ is tame. If x is transcendental over K, then assume that there is a tame extension $(L'|K, v)$ such that for a suitable extension of v from $K(x)$ to $L'(x)$, the extension $(L'(x)|L', v)$ is weakly pure in x. Then there exists a pre-complete key sequence $\mathfrak{S} = (a_i)_{i\in S}$ for $(K(x)|K, v)$ such that $K_{\mathfrak{S}} \subseteq L'$. It is complete if $x \in \widetilde{K}$ or $(L'(x)|L', v)$ is valuation transcendental.*

Proof. If $x \in \widetilde{K}$, then we set $L' = K(x)$; hence in all cases we have that $(L'|K, v)$ is a tame extension.

If $v(x - K)$ has no maximum, then we set $S = \{1\}$ and $a_1 = 0$; otherwise, we choose $a_1 \in K$ such that $v(x - a_1) = \max v(x - K)$. Then (KS1) is satisfied and (KS3) holds for $i = 1$; (KS2) is void for $i = 1$. Note that if $x \in \widetilde{K}$, then a maximum always exists by part 1) of Corollary 11.

Now we assume that $i > 1$ and a key sequence $\mathfrak{S}_{i-1} = (a_j)_{1\leq j\leq i-1}$ for the extension $(K(x)|K, v)$ has already been constructed such that $K_{\mathfrak{S}_{i-1}} \subseteq L'$. By Proposition 28, it has properties (KS5) and (KS6).

If $x \in K(a_{i-1})$, then from (KS3) it follows that $x = a_{i-1}$, i.e., (CKS1) holds for $n = i-1$. In this case, or if (CKS2) holds for $n = i-1$, our construction stops here. Otherwise, we proceed as follows.

First we show that $v(x-K(a_{i-1}))$ has a largest element, which is not the largest element of $v(x-L')$. Assume that x is transcendental over K. If $v(x-K(a_{i-1}))$ would not have a largest element, then x would be the limit of a pseudo Cauchy sequence in $(K(a_{i-1}),v)$ of transcendental type by part 1) of Lemma 14 and consequently, (CKS2) would hold for $n=i-1$. Hence $v(x-K(a_{i-1}))$ has a largest element $v(x-a_{i-1})$. If this would also be the largest element of $v(x-L')$, then by part 2) of Lemma 14 it would also be the largest element of $v(x-\widetilde{K})$ and again, (CKS2) would hold for $n=i-1$. Now assume that x is algebraic over K. As $(L'|K(a_{i-1}),v)$ is a tame extension by part 4) of Lemma 7, we know from part 1) of Corollary 11 that $v(x-K(a_{i-1}))$ has a largest element. However, as $x\in L'\setminus K(a_{i-1})$, this is not the largest element $v(x-x)=\infty$ of $v(x-L')$.

We are going to prove that there is a homogeneous approximation for $x-a_{i-1}$ over K. By what we have shown above, $v(x-a_{i-1})=\max v(x-K(a_{i-1}))$ is not the largest element of $v(x-L')$, so there is some $z\in L'\setminus K(a_{i-1})$ such that $v(x-z)>v(x-a_{i-1})$; in the algebraic case we can choose $z=x$. We have that $v(z-a_{i-1})=v(x-a_{i-1})$. In all cases, $(K(z,a_{i-1})|K,v)$ is a subextension of $(L'|K,v)$, so by part 4) of Lemma 7, it is a tame extension.

We set $b:=z-a_{i-1}$. If e is the smallest natural number such that $\mathrm{e}\,vb\in vK$, then e is not divisible by $\mathrm{char}\,Kv$. Further, if $c\in K$ is such that $vcb^{\mathrm{e}}=0$, then $cb^{\mathrm{e}}v$ is separable-algebraic over Kv. Hence the assumptions of the first part of Lemma 22 are satisfied and we obtain the existence of a homogeneous approximation $a\in\tilde{K}$ of b over K. Also in the transcendental case, a is a homogeneous approximation of $x-a_{i-1}$ over K since

$$v(x-a_{i-1}-a)\;\geq\;\min\{v(z-a_{i-1}-a),v(x-z)\}\;>\;v(z-a_{i-1})\;=\;v(x-a_{i-1})\,.$$

We set $\tilde{a}_i:=a_{i-1}+a$.

Assume that $v(x-K(\tilde{a}_i))$ has a maximum. Then we pick $a_i\in K(\tilde{a}_i)$ such that $v(x-a_i)=\max v(x-K(\tilde{a}_i))$. From Lemma 29 we infer that $K(a_i)=K(\tilde{a}_i)$. If $v(x-K(a_i))$ has no maximum, then we set $a_i:=\tilde{a}_i$. In both cases, (KS2) and (KS3) hold. This shows that $\mathfrak{S}_i:=(a_j)_{1\leq j\leq i}$ a key sequence for $(K(x)|K,v)$. By Lemma 26 and part 3) of Lemma 14, using also that (L',v) is henselian since (K,v) is,

$$K_{\mathfrak{S}_i}\;\subseteq\;K(x)^h\cap\widetilde{K}\;=\;\mathrm{IC}\,(K(x)|K,v)\;\subseteq\;\mathrm{IC}\,(L'(x)|L',v)\;=\;L'\,.$$

This completes the induction step.

If the construction stops at some $n\in\mathbb{N}$, then we set $\mathfrak{S}:=\mathfrak{S}_n$. If the construction does not stop (which can only happen in the transcendental case), then we set $\mathfrak{S}:=\bigcup_{i\in\mathbb{N}}\mathfrak{S}_i$; it is straightforward to prove that this is a key sequence for $(K(x)|K,v)$. In both cases, $K_{\mathfrak{S}}\subseteq L'$.

As we have shown above, if the construction stops at some $n\in\mathbb{N}$, then (CKS1) or (CKS2) hold, that is, the key sequence $\mathfrak{S}=\mathfrak{S}_n$ is pre-complete, and it is complete if $x\in\widetilde{K}$, in which case (CKS1) must hold. If the sequence is not complete, then that means that x is limit of a pseudo Cauchy sequence of transcendental type in

$(K(a_n),v)$. But this is also a pseudo Cauchy sequence of transcendental type in (L',v), as shown in the proof of part 4) of Lemma 14. If $(L'(x)|L',v)$ is valuation transcendental, then such a pseudo Cauchy sequence cannot exist, showing that $\mathfrak{S}$ must be complete.

Finally, if the construction does not stop, then the index set S of $\mathfrak{S}$ is $\mathbb{N}$ and $\mathfrak{S}$ is complete by definition. □

Remark 35. It is an open problem whether it can be shown that also $a_i - a_{i-1}$ is a homogeneous approximation of $x - a_{i-1}$ over K. Likewise, one may be tempted to believe that also $\tilde{a}_i$ is homogeneous over K, but this appears to be not the case in general.

We can give the following characterization of elements in tame extensions:

Corollary 36. *An element $y \in \widetilde{K}$ belongs to a tame extension of a henselian field (K,v) if and only if there is a finite key sequence $(a_i)_{1\le i\le k}$ for $(K(y)|K,v)$ such that $y = a_k$.*

Proof. Suppose that such a sequence exists. Then $y = a_k \in K_{\mathfrak{S}}$, and by Proposition 33, $K_{\mathfrak{S}}$ is a tame extension of K.

The converse is part of Proposition 34. □

Corollary 37. *Assume that (K,v) is a henselian field. Then (K,v) is a tame field if and only if for every transcendental extension $(K(x)|K,v)$ there exists a complete key sequence for $(K(x)|K,v)$.*

Proof. The implication "⇒" is part of Proposition 34, for $L' = \widetilde{K}$. The converse follows from [2, Proposition 3.12]. □

3.5. **Proof of the main theorems.**

Proof of Theorem 3: Take a henselian valued field (K,v) and a transcendental extension $(K(x)|K,v)$. Assume that there is a tame extension $(L'|K,v)$ such that for some extension of v from $K(x)$ to $L'(x)$, the extension $(L'(x)|L',v)$ is weakly pure. By Proposition 34 there exists a pre-complete key sequence $\mathfrak{S} = (a_i)_{i\in S}$ for $(K(x)|K,v)$ with $K_{\mathfrak{S}} \subseteq L'$. It satisfies (KS4) by Lemma 27, and (KS5) and (KS6) by Proposition 28.

We take Q_i to be the minimal polynomial of a_i over K. Then by Proposition 32, Q_i is a key polynomial and a_i is its unique maximal root. Assertion (ii) follows from (KS1). Assertion (iv) follows from (KS4). Assertion (v) follows from Lemma 24. Assertion (x) follows from part 2) of Lemma 25 together with Proposition 32.

From Theorem 31 we know that $L := \mathrm{IC}\,(K(x)|K,v) = K_{\mathfrak{S}}^h$. Since (K,v) is henselian, so is $K_{\mathfrak{S}}$, thus $L = K_{\mathfrak{S}}^h = K_{\mathfrak{S}}$. By definition, $K_{\mathfrak{S}} = K(a_i \mid i \in S)$. If $S = \{1,\dots,n\}$, then by Lemma 26, $K_{\mathfrak{S}} = K_{\mathfrak{S}}^h = K(a_n)^h = K(a_n)$. This proves assertion (ix). Further, by part 3) of Lemma 14 and the fact that L' is henselian, $L = \mathrm{IC}\,(K(x)|K,v) \subseteq \mathrm{IC}\,(L'(x)|L',v) = L'$.

Assume that $S=\mathbb{N}$. Then by definition, $\mathfrak{S}$ is a complete key sequence. From Proposition 32 it follows that $(Q_i)_{i\in S}$ is a complete sequence of key polynomials, so we set $\Omega=\emptyset$. Then assertions (i) and (iii) hold, assertion (vi) follows from (KS3), and assertions (vii) and (viii) hold trivially. From Proposition 30 and the equality $L=K_{\mathfrak{S}}$ it follows that the extension $(L(x)|L,v)$ is immediate and weakly pure. Since $K_{\mathfrak{S}}\subseteq\widetilde{K}$, this implies that the extension $(K(x)|K,v)$ is valuation algebraic.

Now assume that $S=\{1,\dots,n\}$. Consequently, as $\mathfrak{S}$ is a pre-complete key sequence and x is transcendental over K, (CKS2) must hold. Assume first that $v(x-a_n)=\max v(x-\widetilde{K})$. Then by Lemma 12, the extension $(K(a_n,x)|K(a_n),v)$ is weakly pure and valuation transcendental. Since $L=K_{\mathfrak{S}}=K(a_n)$, we have actually proved that the extension $(L(x)|L,v)$ is weakly pure and valuation transcendental. Moreover, we know from Proposition 32 that $(Q_i)_{i\in S}$ is a complete sequence of key polynomials. As before we set $\Omega=\emptyset$, so that assertions (i), (iii), (vii) and (viii) hold. Further, assertion (vi) for $i=n$ holds by our assumption, and for $i<n$ follows from (KS3).

Now assume that x is the limit of a pseudo Cauchy sequence, say $(z_\nu)_{\nu<\lambda}$ where λ is some limit ordinal, of transcendental type in $(K(a_n),v)$. Since $L=K(a_n)$, it follows from Lemma 9 that the extension $(L(x)|L,v)$ is weakly pure and immediate. We set $\Omega=\{\nu\mid\nu<\lambda\}$ and take Q_ν to be the minimal polynomial of z_ν over K. Then assertion (vi) follows from (KS3), and assertion (vii) holds because $\Omega\neq\emptyset$. For the proof of assertion (viii) we observe that we have set $\Omega=\emptyset$ when S is not finite (hence equal to $\mathbb{N}$); hence it remains to show that for all $\nu\in\Omega$, $\deg Q_\nu=\deg Q_n=[K(a_n):K]$. Without loss of generality we may assume that $v(x-z_\nu)\geq v(x-a_n)$. Then from Lemma 29 we obtain that $K(z_\nu)=K(a_n)$, which completes our proof of assertion (viii).

In order to prove assertion (iii), we only have to address the polynomials Q_ν for $\nu\in\Omega$. As we have already proved in Proposition 32 that a_n is the unique maximal root of its minimal polynomial over K, and as we have just shown above that a_n may be replaced by z_ν, we find that the same holds for z_ν.

Now we have to show that also assertion (i) holds. We know already from Proposition 32 that each Q_i for $i\in S$ is a key polynomial. To prove the same for each Q_ν, assume that $f\in K[X]$ such that $\delta(f)\geq\delta(Q_\nu)=v(x-z_\nu)$. Take a root z of f such that $v(x-z)=\delta(f)\geq v(x-z_\nu)\geq v(x-a_n)>v(x-a_{n-1})$. Then by (KS6), $\deg f=[K(z):K]\geq[K(a_n):K]=\deg Q_n=\deg Q_\nu$. This proves that Q_ν is a key polynomial.

To show completeness, take any $f(X)\in K[X]$ of positive degree and a root z of f such that $v(x-z)=\delta(f)$. Take $i\in S$ maximal with $\deg Q_i\leq\deg f$. If $i<n$, then as in the proof of Proposition 32 it follows that $\delta(f)=v(x-z)\leq v(x-a_i)=\delta(Q_i)$. Now assume that $i=n$. If $\delta(f)\leq\delta(Q_n)$, then we are done. Hence assume otherwise, so that $v(x-z)>v(x-a_n)$. Since in the present case x is the limit of the pseudo Cauchy sequence $(z_\nu)_{\nu<\lambda}$ of transcendental type in $(K(a_n),v)$ and this cannot have a limit in $K(a_n)$, there must be some $\nu<\lambda$ such that $v(x-z_\nu)>v(x-z)$, whence

$\delta(Q_\nu) > v(x-z) = \delta(f)$. On the other hand, $\deg Q_\nu = \deg Q_n \leq \deg f$. This finishes our proof of the completeness and thus of assertion (i).

We turn to the final assertions of Theorem 1. We have already shown in all cases that the extension $(L(x)|L,v)$ is weakly pure. The extension $(K(x)|K,v)$ is valuation algebraic if and only if the extension $(L(x)|L,v)$ is. This happens precisely if the extension $(L(x)|L,v)$ is immediate, and this is the case if and only if $S=\mathbb{N}$ or Ω is nonempty by our construction. In the case of $S=\mathbb{N}$, we know from Proposition 30 that $(a_i)_{i\in\mathbb{N}}$ is a pseudo Cauchy sequence in (L,v) of transcendental type with x as its limit. □

Proof of Theorem 4: Take a henselian valued field (K,v) and a tame algebraic extension $(K(x)|K,v)$. Then by Proposition 34 there exists a complete key sequence $\mathfrak{S}=(a_i)_{i\in S}$ for $(K(x)|K,v)$. It satisfies (KS4) by Lemma 27, and (KS5) and (KS6) by Proposition 28. Since $(K(x)|K,v)$ is a finite extension, also S must be finite, hence of the form $S=\{1,\dots,n\}$ for some $n\in\mathbb{N}$. As before, we take Q_i to be the minimal polynomial of a_i over K. Then by Proposition 32, $(Q_i)_{i\in S}$ is a complete sequence of key polynomials for $(K(x)|K,v)$, and assertion (iii) holds. Assertion (iv) of Theorem 4 holds by (KS4). Assertion (ii) follows from (KS1). Assertion (v) follows from Lemma 24. Assertion (vi) for $i<n$ follows from (KS3), and for $i=n$ from (CKS1) which also yields assertion (vii). The analogue of assertion (x) of Theorem 1 is proved as in the proof of Theorem 3. □

3.6. **Proof of Theorem 6.**

In view of Theorem 4, we only have to prove Theorem 6 for infinite extensions $L|K$. There is a rather quick way to do this. Since the tame extension is in particular separable-algebraic, we can employ the proof of [9, Theorem 3.16] to construct an immediate extension of v from L to $L(x)$ such that the extension $(L(x)|L,v)$ is weakly pure and $L=\mathrm{IC}\,(K(x)|K,v)$. (Note that for this step it is not needed that $(L|K,v)$ be a tame extension.) Then we can apply Theorem 3 and Proposition 5.

Alternatively, Theorem 6 can be proved by a more direct construction which applies the methods developed for the proof of Theorem 4 to suitable subextensions of increasing finite degree. We choose a sequence $\tilde{y}_j$, $j\in\mathbb{N}$ such that $K(\tilde{y}_j)\subsetneq K(\tilde{y}_{j+1})$ for all j, and $L=\bigcup_{i\in\mathbb{N}} K(\tilde{y}_j)$. Then by part 4) of Lemma 7, all subextensions $K(\tilde{y}_j)|K,v)$ of $(L|K,v)$ and all extensions $K(\tilde{y}_{j+1})|K(\tilde{y}_j),v)$ are tame.

Now we build key sequences for the extensions $(K(\tilde{y}_j)|K,v)$ that satisfy a compatibility condition which will allow us to work with the union over these sequences. We proceed by induction on j. At every step we will adjust the element $\tilde{y}_j$, replacing it by an element y_j such that $K(y_j)=K(\tilde{y}_j)$ and thus in the end, $L=\bigcup_{j\in\mathbb{N}} K(y_j)$. In fact, at every step we will construct a complete key sequence for the extension $(K(y_j)|K,v)$.

We set $y_1=\tilde{y}_1$. From Proposition 34 we obtain a complete key sequence $\mathfrak{S}_1=(a_i)_{1\leq i\leq n_1}$ for the extension $(K(y_1)|K,v)$. Having already chosen a suitable element

$y_j \in L$ with $K(y_j) = K(\tilde{y}_j)$ and constructed a complete key sequence $\mathfrak{S}_j = (a_i)_{1\leq i\leq n_j}$ for the extension $(K(y_j)|K,v)$, we note that $y_j = a_{n_j}$ and proceed as follows. First, we choose $z_j \in K(y_j)$ such that $v(\tilde{y}_{j+1} - z_j) = \max v(\tilde{y}_{j+1} - K(y_j))$. This is possible by part 1) of Corollary 11. Then we choose $c_j \in K$ such that

$$vc_j(\tilde{y}_{j+1} - z_j) \;>\; v(y_j - a_{n_j-1}) \tag{13}$$

and set

$$y_{j+1} \;:=\; y_j + c_j(\tilde{y}_{j+1} - z_j)\,. \tag{14}$$

At this point we note that since $y_j, z_j, c_j \in K(y_j) = K(\tilde{y}_j) \subset K(\tilde{y}_{j+1})$, we have that

$$y_{j+1} \;\in\; K(\tilde{y}_{j+1}) \quad\text{and}\quad \tilde{y}_{j+1} \;=\; c_j^{-1}(y_{j+1} - y_j) + z_j \;\in\; K(y_{j+1}, y_j)\,. \tag{15}$$

We show that

$$v(y_{j+1} - y_j) \;=\; \max v(y_{j+1} - K(y_j))\,. \tag{16}$$

If this were not true, there would be $y'_j \in K(y_j)$ such that $v(y_{j+1}-y'_j) > v(y_{j+1}-y_j)$. By the definition of y_{j+1}, this is equivalent to

$$v(y_j + c_j(\tilde{y}_{j+1} - z_j) - y'_j) \;>\; vc_j(\tilde{y}_{j+1} - z_j)\,,$$

which in turn is equivalent to

$$v(\tilde{y}_{j+1} - (z_j + c_j^{-1}(y'_j - y_j)) \;>\; v(\tilde{y}_{j+1} - z_j)\,,$$

contradicting our choice of z_j.

By construction of y_{j+1},

$$v(y_{j+1} - y_j) \;=\; vc_j(\tilde{y}_{j+1} - z_j) \;>\; v(y_j - a_{n_j-1}) \;\geq\; v(y_j - a_i) \tag{17}$$

for $i < n_j$. Thus, part 1) of Lemma 25 shows that $(a_i)_{i<n_j}$ is a key sequence for $(K(y_{j+1})|K,v)$. As $\tilde{a}_{n_j} - a_{n_j-1}$ is a homogeneous approximation of $y_j - a_{n_j-1}$ over K, we find that

$$v(y_j - \tilde{a}_{n_j}) \;=\; v(y_j - a_{n_j-1} - (\tilde{a}_{n_j} - a_{n_j-1})) \;>\; v(y_j - a_{n_j-1})\,.$$

Further,

$$v(y_{j+1} - a_{n_j-1}) \;=\; \min\{v(y_{j+1} - y_j), v(y_j - a_{n_j-1})\} \;=\; v(y_j - a_{n_j-1})\,.$$

Using this, we compute:

$$\begin{aligned} v(y_{j+1} - a_{n_j-1} - (\tilde{a}_{n_j} - a_{n_j-1})) \;&=\; v(y_{j+1} - \tilde{a}_{n_j}) \;\geq\; \min\{v(y_{j+1} - y_j), v(y_j - \tilde{a}_{n_j})\} \\ &>\; v(y_j - a_{n_j-1}) \;=\; v(y_{j+1} - a_{n_j-1})\,, \end{aligned}$$

shows that (KS2) holds for y_{j+1} in place of x and n_j in place of i. With the same choices for x and i, also (KS3) holds by equation (16). We have now proved that $\mathfrak{S}_j$ is a key sequence for $K(y_{j+1})|K,v)$.

Using the methods of the proof of Proposition 34, we now extend $\mathfrak{S}_j$ to a complete key sequence $\mathfrak{S}_{j+1} = (a_i)_{1\leq i\leq n_{j+1}}$ for $K(y_{j+1})|K,v)$. In particular, we have that $y_{j+1} = a_{n_{j+1}}$ and by (KS4),

$$K(y_j) \;=\; K(a_{n_j}) \;\subset\; K(a_{n_{j+1}}) \;=\; K(y_{j+1})\,,$$

hence by (15), $K(y_{j+1}) = K(\tilde{y}_{j+1})$. This completes our induction. It remains to prove that $\mathfrak{S} := \bigcup_{j\in\mathbb{N}} \mathfrak{S}_j = (a_i)_{i\in\mathbb{N}}$ gives rise to a strongly complete sequence of key polynomials $(Q_i)_{i\in\mathbb{N}}$ for $(L|K,v)$ by taking each Q_i to be the minimal polynomial of the element a_i over K.

In order to show that (SCKP1) holds, we first observe that $L = \bigcup_{j\in\mathbb{N}} K(y_j) = \bigcup_{j\in\mathbb{N}} K(a_{n_j}) \subseteq K(a_i \mid i \in S)$. On the other hand, for every $i \in S$ there is $j \in \mathbb{N}$ such that $i \leq n_j$, and by Lemma 26, $a_i \in K(a_{n_j})$. Hence $L = K(a_i \mid i \in S)$.

In order to show that (SCKP2) holds, pick any $k \in \mathbb{N}$. Choose $j \in \mathbb{N}$ such that $k \leq n_j$. By construction, $(a_i)_{1\leq i\leq n_j}$ is a complete key sequence for the extension $K(a_{n_j}|K,v)$. Hence by part 2) of Lemma 25, $(a_i)_{i\leq k}$ is a complete key sequence for $(K(a_k)|K,v)$. Now Proposition 32 shows that $(Q_i)_{i\leq k}$ is a complete sequence of key polynomials for $(K(a_k)|K,v)$, and that property 3) holds. The validity of assertions (iv), (v), (vi) and (x) is shown as in the proof of Theorem 4. This completes the proof of the theorem. □

4. Construction of extensions

In this section we show how extensions of the valuation from a tame field (K,v) to a transcendental extension $K(x)$ can be constructed by use of implicit function fields, key sequences and pseudo Cauchy sequences. At the same time, we introduce a basic classification of these extensions. The first criterion is whether we want the implicit function field L to be a finite or infinite extension of K. Note that as (K,v) is henselian, the extension of v to every algebraic extension field of K is unique. Using Theorem 1 and Corollary 2 together with Lemma 12, we obtain the following case distinction:

Case A: L is a finite extension of K. As L is supposed to be the implicit function fieldof $(K(x)|K,v)$, we know from Theorem 1 that $(L(x)|L,v)$ is weakly pure. Hence by Lemma 12 there are three possible subcases:

A.1: For some $a \in L$, the value $v(x-a)$ is non-torsion over vK.

A.2: For some $a \in L$ and some $d \in K$ and $e \in \mathbb{N}$, $vd(x-a)^{\mathrm{e}} = 0$ and $d(x-a)^{\mathrm{e}}v$ is transcendental over Kv.

A.3: The element x is limit of a pseudo Cauchy sequence in (L,v) of transcendental type.

Case B: L is an infinite extension of K. For it to be an implicit constant field, it must be countably generated separable-algebraic over K.

We will now discuss the construction methods in all cases in detail. Let us first consider Case A. We take $a \in \widetilde{K}$ such that $L = K(a)$. If we assign to $x-a$ a value in some ordered abelian group containing $v\widetilde{K}$ that is non-torsion over vK (i.e., not contained in $v\widetilde{K}$), then we are in case A.1. If on the other hand we pick $d \in K$ and $e \in \mathbb{N}$ such that $vd(x-a)^{\mathrm{e}} = 0$ and have extended v so that $d(x-a)^{\mathrm{e}}v$ becomes transcendental over Kv, then we are in case A.2. In both cases, for every $b \in \widetilde{K}$ we have that

$$v(x-b) \;=\; \min\{v(x-a), v(b-a)\}\;.$$

Consequently, for each polynomial $f \in \widetilde{K}[X]$, if we write $f(X) = c\prod_{i=1}^{n}(X-b_i)$, then

$$vf(x) \;=\; vc + \sum_{i=1}^{n}\min\{v(x-a), v(b_i-a)\}\;.$$

This shows that in cases A.1 and A.2 the extension of v from K to $K(x)$ is uniquely determined by our choice of a and the information on $x-a$ in the respective cases.

Assume now that we decide for case A.3. We note that every pseudo Cauchy sequence in $(K(a), v)$ without a limit in $K(a)$ must be of transcendental type by part 2) of Corollary 11, as $(K(a), v)$ is a tame field by part 1) of Lemma 7. This implies that it uniquely determines an extension of v from $K(a)$ to $K(x, a)$, and thus determines an extension of v from K to $K(x)$. In order to obtain a construction for case A.3, we must assume that $(K(a), v)$ and hence also (K, v) is not a maximal field. Then we may choose any pseudo Cauchy sequence in $(K(a), v)$ without a limit in $K(a)$ for the construction of the extension.

In all of the above cases, $(L(x)|L, v)$ is weakly pure, and it follows from Lemma 13 that L is the implicit constant field of $L(x)|L, v)$. To obtain that L is also the implicit constant field of $K(x)|K, v)$, we refine our construction as follows. We write $\alpha := \mathrm{Kras}(a, K)$. In cases A.1 and A.2 we may choose the value $v(x-a)$ to be larger that α. In case A.3, provided that (K, v) is not a maximal field, we choose any pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ in $(K(a), v)$ without a limit in $K(a)$. After multiplying all a_ν with an element $c \in K$ of high enough value, we may assume that there is some $\mu < \lambda$ such that $v(a_{\mu+1} - a_\mu) > \alpha$. Now we set $b_\nu := a_\nu - a_\mu + a$. Then also $(b_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in $(K(a), v)$ without a limit in $K(a)$. We extend v from $K(a)$ to $K(x, a)$ by taking x to be a limit of this pseudo Cauchy sequence. Then

$$v(x-a) \;=\; v(x-b_\mu) \;=\; v(b_{\mu+1}-b_\mu) \;=\; v(a_{\mu+1}-a_\mu) \;>\; \alpha\;.$$

In all cases, from Proposition 17 we obtain that $a \in K(x)^h$. Hence,

$$L \;=\; K(a) \;\subseteq\; K(x)^h \cap \tilde{K} \;\subseteq\; \mathrm{IC}\,(K(x)|K, v) \;\subseteq\; \mathrm{IC}\,(L(x)|L, v) \;=\; L\,,$$

which shows that $L = \mathrm{IC}\,(K(x)|K, v)$.

Finally, assume that we are picking a countably generated infinite separable-algebraic extension L of K to be our implicit constant field. In this case, the proof of [9, Proposition 3.16] yields a pseudo Cauchy sequence of transcendental type in (L, v) which determines an extension of v from L to $L(x)$ such that $L = \mathrm{IC}\,(K(x)|K, v)$. In contrast to the previous cases, even if (K, v) is a maximal field, (L, v) is not (see [1, Theorem 1.1]), and there will always be the necessary pseudo Cauchy sequence in (L, v) for the construction of the extension.

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