ON THE COMPUTATION OF KÄHLER DIFFERENTIALS AND CHARACTERIZATIONS OF GALOIS EXTENSIONS WITH INDEPENDENT DEFECT

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Abstract. For important cases of algebraic extensions of valued fields, we develop presentations of the associated Kähler differentials of the extensions of their valuation rings. We compute their annihilators as well as the associated Dedekind differentials. We then apply the results to Galois defect extensions of prime degree. Defects can appear in finite extensions of valued fields of positive residue characteristic and are serious obstructions to several problems in positive characteristic. A classification of defects (dependent vs. independent) has been introduced by the second and the third author. It has been shown that perfectoid fields and deeply ramified fields only admit extensions with independent defect. We give several characterizations of independent defect, using ramification ideals, Kähler differentials and traces of the maximal ideals of valuation rings. All of our results are for arbitrary valuations; in particular, we have no restrictions on their rank or value groups.

1. Introduction

A central goal of this paper is to give for important cases of algebraic extensions of valued fields a presentation of the associated Kähler differentials of the extensions of their valuation rings. This is crucial not only for applications in the present paper, but also in the subsequent paper [6].

By $(L|K,v)$ we denote a field extension $L|K$ where v is a valuation on L and K is endowed with the restriction of v. The valuation ring of v on L will be denoted by $\mathcal{O}_L$, and that on K by $\mathcal{O}_K$. Similarly, $\mathcal{M}_L$ and $\mathcal{M}_K$ denote the unique maximal ideals of $\mathcal{O}_L$ and $\mathcal{O}_K$. The value group of the valued field (L,v) will be denoted by vL, and its residue field by Lv. The value of an element a will be denoted by va, and its residue by av. The **rank** of a valued field (K,v) is the order type of the chain of proper convex subgroups of its value group vK. All of our results are for arbitrary valuations; in particular, we have no restrictions on their rank or value groups. Ranks higher than 1 appear in a natural way when local uniformization, the local

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form of resolution of singularities, is studied. Deeply ramified fields of infinite rank appear in model theoretic investigations of the tilting construction, as presented by Jahnke and Kartas in [12]. Therefore, we do not restrict our computations to rank 1, thereby indicating how Kähler differentials and their annihilators, as well as Dedekind differentials, can be computed in higher rank.

By $\Omega_{B|A}$ we denote the Kähler differentials, i.e., the module of relative differentials, when A is a ring and B is an A-algebra. In Section 4.1, we prove:

Theorem 1.1. *Let $L|K$ be an algebraic field extension of degree n and suppose that A is a normal domain with quotient field K and B is a domain with quotient field L such that $A \subset B$ is an integral extension. Suppose that there exist generators $b_\alpha \in B$ of $L|K$, which are indexed by a totally ordered set S, such that $A[b_\alpha] \subset A[b_\beta]$ if $\alpha \le \beta$ and*

$$\bigcup_{\alpha\in S} A[b_\alpha] \;=\; B.$$

Further suppose that there exist $a_\alpha, a_\beta \in A$ such that $a_\beta \mid a_\alpha$ if $\alpha \le \beta$ and for $\alpha \le \beta$, there exist $c_{\alpha,\beta} \in A$ and expressions

$$b_\alpha \;=\; \frac{a_\alpha}{a_\beta} b_\beta + c_{\alpha,\beta}\,. \tag{1}$$

Let h_α be the minimal polynomial of b_α over K. Let U and V be the B-ideals

$$U \;=\; (a_\alpha \mid \alpha \in S) \text{ and } V = (h'_\alpha(b_\alpha) \mid \alpha \in S).$$

Then we have a B-module isomorphism

$$\Omega_{B|A} \;\cong\; U/UV\,. \tag{2}$$

For the case where $A = \mathcal{O}_K$ and $B = \mathcal{O}_L$, we compute V in (44) and for arbitrary $\gamma \in S$, we obtain a B-module isomorphism

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; U/b^\dagger_\gamma U^n \quad \text{with} \quad b^\dagger_\gamma \;:=\; \frac{h'_\gamma(b_\gamma)}{a_\gamma^{n-1}}\,. \tag{3}$$

We determine the annihilator of U/UV and thus of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ in Proposition 4.2. For information on the Dedekind different $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$, see Theorems 1.6 and 5.4.

We define the defect and defect extensions in Section 2.1. For the construction of a large number of different defect extensions and the role the defect plays in deep open problems in positive characteristic, see [16]. The defect can be understood through jumps in the construction of limit key polynomials and in the construction of pseudo-convergent sequences. A couple of recent references explaining this phenomenon are [25] and [7].

We are generally interested in the study of defect extensions of arbitrary finite degree. As explained in Section 2.1, via ramification theory this can be reduced to the investigation of purely inseparable extensions and of Galois extensions of degree $p = \operatorname{char} Kv > 0$.

We find explicit realizations for the assumptions of Theorem 1.1 for Galois defect extensions of prime degree and deduce the next theorem by combining Theorems 4.5 and 4.6. For the definition of higher ramification groups and ramification ideals, see Section 2.7.

Theorem 1.2. *Let $\mathcal{E} = (L|K,v)$ be a Galois defect extension of prime degree p. If $\mathrm{char}\, K = 0$, then assume that K contains all p-th roots of unity. Then $\mathcal{E}$ has a unique ramification ideal $I_\mathcal{E}$, and there is an $\mathcal{O}_L$-module isomorphism*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \;\cong\; I_\mathcal{E}/I_\mathcal{E}^p\,. \tag{4}$$

In [26], Novacoski and Spivakovsky use the theory of key polynomials to derive a presentation of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ for finite extensions $(L|K,v)$ under the condition $vL = vK$ (which holds for all extensions covered by the above theorem). The presentation of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ for Galois defect extensions $(L|K,v)$ is also studied by Thatte in [29, 30].

Galois defect extensions of degree p of valued fields of characteristic $p>0$ (valued fields of **equal positive characteristic**) have first been classified by the second author in [15]. In [20] the classification was extended to the case of Galois defect extensions of degree p of valued fields of characteristic 0 with residue fields of characteristic $p>0$ (valued fields of **mixed characteristic**), as follows. Take a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree p. For every σ in its Galois group $\mathrm{Gal}\,(L|K)$, with $\sigma \neq \mathrm{id}$, we set

$$\Sigma_\sigma \;:=\; \left\{ v\left(\frac{\sigma b - b}{b}\right) \,\middle|\, b \in L^\times,\, \sigma b \neq b \right\}\,. \tag{5}$$

This set is a final segment of $vL = vK$ and independent of the choice of σ; we denote it by $\Sigma_\mathcal{E}$. The $\mathcal{O}_L$-ideal $I_\mathcal{E} := (a \in L \mid va \in \Sigma_\mathcal{E})$ is the unique **ramification ideal** of $\mathcal{E}$. Detailed information on $\Sigma_\mathcal{E}$ and $I_\mathcal{E}$ is given in Sections 2.7 and 2.8.

We say that $\mathcal{E}$ has **independent defect** if

$$\left\{ \begin{array}{l} \Sigma_\mathcal{E} \;=\; \{\alpha \in vL \mid \alpha > H_\mathcal{E}\} \text{ for some proper convex subgroup } H \\ \qquad\quad \text{of } vL \text{ such that } vL/H \text{ has no smallest positive element.} \end{array} \right. \tag{6}$$

If (6) holds, then we will write $H_\mathcal{E}$ for H. If there is no such subgroup H, we will say that $\mathcal{E}$ has **dependent defect**. If $vL = vK$ is archimedean (i.e., order isomorphic to a subgroup of $\mathbb{R}$), then condition (6) just means that $\Sigma_\mathcal{E}$ consists of all positive elements in vL.

In [11], Gabber and Ramero define deeply ramified fields (K,v) by the property

$$\Omega_{\mathcal{O}_{K^{\mathrm{sep}}}|\mathcal{O}_K} \;=\; 0\,, \tag{7}$$

where K^{sep} denotes the separable-algbraic closure of K. For a valuative definition, which they prove to be equivalent to theirs, and a generalization thereof, see e.g. [20]. In [6] we present a simplified version of their proof. The following is a consequence of the more general part 1) of Theorem 1.10 in [20]:

Theorem 1.3. *Every Galois defect extension of prime degree of a deeply ramified field has independent defect.*

Property (7) together with Theorem 1.3 led us to investigate the connection between Kähler differentials and independent defect. In the present paper we apply the results stated at the beginning of the introduction to the following situation:

$$\left\{ \begin{array}{l} (L|K,v) \text{ is a Galois defect extension of prime degree } p = \mathrm{char}\, Kv \\ \text{ and if } \mathrm{char}\, K = 0\,, \text{ then } K \text{ contains all } p\text{-th roots of unity.} \end{array} \right. \tag{8}$$

In this situation we have $vL = vK$, and $L|K$ is an Artin-Schreier extension if $\mathrm{char}\, K = p$, and a Kummer extension if $\mathrm{char}\, K = 0$. We obtain that their defect is independent if and only if $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$; this is the equivalence of assertions a) and d) in Theorem 1.4 below.

Our results for extensions of the form (8) will be complemented in [6] by applying Theorem 1.1 to Galois extensions of prime degree without defect. Using this together with the results of the present paper, it is shown that a valued field is deeply ramified if and only if the Kähler differentials of all Galois extensions of prime degree are zero; moreover, we compute Kähler differentials for all finite Galois extensions.

Another way to characterize independent defect of Galois extensions $(L|K,v)$ of prime degree is through the trace $\mathrm{Tr}_{L|K}(\mathcal{M}_L)$ where $\mathrm{Tr}_{L|K}$ denotes the trace of $L|K$. Note that if $(L|K,v)$ is unibranched, then $\mathrm{Tr}\,(\mathcal{O}_L) \subseteq \mathcal{O}_K$.

The following theorem summarizes various characterizations of independent defect. The convex subgroups H of vL are in one-to-one correspondence with the coarsenings v_H of v on L in such a way that $v_H L = vL/H$. The maximal ideal of the valuation ring of v_H on L is $\mathcal{M}_{v_H} = (a \in L \mid va > H)$. If H is a subgroup of some ordered abelian group Γ, then we call it a **strongly convex subgroup** if it is a proper convex subgroup such that Γ/H has no smallest positive element.

If $\mathrm{char}\, K = 0$ and $\mathrm{char}\, Kv = p > 0$, then we denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp (cf. [20]). If $\mathrm{char}\, K > 0$, then we set $(vK)_{vp} = vK$.

Theorem 1.4. *Assume that the extension $\mathcal{E} = (L|K,v)$ satisfies (8). Then the following assertions are equivalent:*

a) $\mathcal{E}$ has independent defect,

b) the ramification ideal $I_{\mathcal{E}}$ of $\mathcal{E}$ is equal to $\mathcal{M}_{v_H}$ for some strongly convex subgroup H of vL,

c) $I_{\mathcal{E}}^p = I_{\mathcal{E}}$,

d) $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$,

e) $\mathrm{Tr}_{L|K}(\mathcal{M}_L) = \mathcal{M}_{v_H} \cap K$ for some strongly convex subgroup H of vK.

If assertions b) or e) hold, then $H = H_{\mathcal{E}}$, and the valuation ring of v_H is the localization of $\mathcal{O}_L$ with respect to the ramification ideal $I_{\mathcal{E}}$. The value group of the corresponding coarsening of v does not have a smallest positive element.

If vK is archimedean, or more generally, if $(vK)_{vp}$ is archimedean, then $H_{\mathcal{E}}$ can only be equal to $\{0\}$ and if the ramification ideal is a prime ideal, then it can only be equal to $\mathcal{M}_L$.

If $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ is annihilated by $\mathcal{M}_L$, then $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$.

For more equivalent conditions, see part 1) of Proposition 2.10.

The equivalence of assertions a) and b) follows from the definition of independent defect and the fact that $I_{\mathcal{E}} = (a \in L \mid va \in \Sigma_{\mathcal{E}})$. The equivalence of assertions a) and c) will be proved in part 1) of Proposition 2.10. The equivalence of assertions c) and d) follows from Theorem 1.2. The equivalence of assertions a) and e) follows from the next theorem. The last assertion follows from part 2) of Proposition 4.2. The remaining assertions, in particular the equality $H = H_{\mathcal{E}}$, will be proven together with the next theorem in Section 5.2.

Theorem 1.5. *Assume that the extension* $\mathcal{E} = (L|K, v)$ *satisfies (8). Then*

$$\text{(9)} \qquad \mathrm{Tr}_{L|K}(\mathcal{O}_L) = \mathrm{Tr}_{L|K}(\mathcal{M}_L) = (b \in K \mid vb \in (p-1)\Sigma_{\mathcal{E}}) = (I_{\mathcal{E}} \cap K)^{p-1}.$$

The extension $\mathcal{E}$ *has independent defect if and only if for some strongly convex subgroup* H *of* vL,

$$\text{(10)} \qquad \mathrm{Tr}_{L|K}(\mathcal{O}_L) = \mathrm{Tr}_{L|K}(\mathcal{M}_L) = \mathcal{M}_{v_H} \cap K = (b \in K \mid vb > H).$$

In particular, if $H = \{0\}$ *(which is always the case if* vK *is archimedean, or more generally, if* $(vK)_{vp}$ *is archimedean), then this means that*

$$\mathrm{Tr}_{L|K}(\mathcal{M}_L) = \mathcal{M}_K .$$

In the mixed characteristic case, $\mathcal{M}_{v_H}$ *will always contain* p*, so that* $\mathrm{char}\, Kv_H = p$.

In [4] the authors introduce the notion of *deeply ramified extensions* $(L|K, v)$ where (K, v) is a local field and $L|K$ is algebraic. It follows from [4, Proposition 2.9] that under the conditions above, $(L|K, v)$ is a deeply ramified extension if and only if $\mathrm{Tr}_{F|L}(\mathcal{M}_F) = \mathcal{M}_L$ for every finite extension $F|L$ (note that as an algebraic extension of a local field, (L, v) is henselian, and so the extension of v to F is uniquely determined); see also [10, Theorem 1.1]. Theorem 1.5 shows that the latter condition will in general not characterize deeply ramified fields, unless they have rank 1 (as is the case for algebraic extensions of local fields).

The **Dedekind different** of $(L|K, v)$ is $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) := \mathcal{O}_L :_L \mathcal{C}(\mathcal{O}_L|\mathcal{O}_K)$, where $\mathcal{C}(\mathcal{O}_L|\mathcal{O}_K) := (z \in L \mid \mathrm{Tr}(z\mathcal{O}_L) \subset \mathcal{O}_K)$ is the fractional $\mathcal{O}_L$-ideal called the **complementary ideal**. For $b \in \mathcal{O}_L$ and h_b its minimal polynomial over K, the element $h_b'(b) \in \mathcal{O}_L$ is called the **different** of b. We denote the annihilator of an $\mathcal{O}_L$-ideal I by $\mathrm{ann}\, I$.

In Proposition 4.4 of Section 4.3 and in Section 5.3, we will prove:

Theorem 1.6. *Assume that the extension* $\mathcal{E} = (L|K, v)$ *satisfies (8). Then* $I_{\mathcal{E}}^{p-1}$ *is equal to the* $\mathcal{O}_L$*-ideal generated by the differents of all elements of* $\mathcal{O}_L \setminus \mathcal{O}_K$ *and equal to the* $\mathcal{O}_L$*-ideal generated by the elements of* $\mathrm{Tr}_{L|K}(\mathcal{O}_L)$.

We have that $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = I_{\mathcal{E}}^{p-1}$ *if and only if* $vI_{\mathcal{E}}^{p-1}$ *has no infimum in* vL*. If* $vI_{\mathcal{E}}^{p-1}$ *has infimum* α *in* vL*, then* $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = (a \in L \mid va \geq \alpha)$ *and* $I_{\mathcal{E}}^{p-1} = \mathcal{M}_L\, \mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$*. If* (K, v) *has rank* 1*, then* $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \mathrm{ann}\, \Omega_{\mathcal{O}_L|\mathcal{O}_K}$.

If the extension $\mathcal{E}$ *has independent defect, then* $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \mathcal{O}_L = \mathrm{ann}\, \Omega_{\mathcal{O}_L|\mathcal{O}_K}$ *if* $H_{\mathcal{E}} = \{0\}$*, and* $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \mathcal{M}_{v_{H_{\mathcal{E}}}} \subsetneq \mathcal{O}_L = \mathrm{ann}\, \Omega_{\mathcal{O}_L|\mathcal{O}_K}$ *otherwise. Conversely, if* $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$ *is equal to* $\mathcal{O}_L$ *or to* $\mathcal{M}_{v_H}$ *for a nontrivial strongly convex subgroup* H *of* vL*, then* $\mathcal{E}$ *has independent defect.*

The reader may note that in rank higher than 1, $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$ is not necessarily equal to the annihilator of $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$. For the computation of annihilators in rank higher than 1, see Propositions 4.1 and 4.2. For the Dedekind different of a finite unibranched extension $(L|K, v)$ with $vL = vK$, see Theorem 5.4.

Finally, let us give further characterizations of independent defect, and in particular show that it can be characterized by a simple condition in (K, v) (which is an important point in [12]). If z is an element and S a subset of some valued field (L, v), then we set

$$v(z - S) := \{v(z - c) \mid c \in S\}.$$

We will write $\wp(X)$ for the Artin-Schreier polynomial $X^p - X$.

Theorem 1.7. *Assume that $\mathcal{E} = (L|K,v)$ is an Artin-Schreier defect extension generated by $\vartheta \in L$ with $\vartheta^p - \vartheta = a \in K$. Then $v(\vartheta - c) < 0$ and $v(a - \wp(c)) = pv(\vartheta - c)$ for all $c \in K$, and therefore,*

$$v(a - \wp(K)) \;=\; pv(\vartheta - K)\,. \tag{11}$$

Further, the following are equivalent:

a) $\mathcal{E}$ has independent defect,

b) $v(\vartheta - K) = -\{\alpha \in vK \mid \alpha > H\}$ for some strongly convex subgroup H of vK,

c) $v(a - \wp(K)) = -\{\alpha \in pvK \mid \alpha > H\}$ for some strongly convex subgroup H of vK,

d) there is some strongly convex subgroup H of vK such that for all $b \in K$ such that $vb > H$ there is $c \in K$ satisfying $v(a - \wp(c)) \geq -vb$.

If vK is p-divisible, then these properties are also equivalent to

e) $pv(\vartheta - K) = v(\vartheta - K)$,

f) $v(a - \wp(K)) = v(\vartheta - K)$.

If (K,v) has rank 1, then condition d) is equivalent to:

g) for all $b \in \mathcal{O}_K$ there is $c \in K$ satisfying $v(a - \wp(c)) \geq -vb$.

Theorem 1.8. *Assume that $\mathcal{E}$ is a Kummer defect extension of prime degree p. Then it is generated by a 1-unit $\eta \in L$ with $\eta^p = a \in K$. We have $v(\eta - c) < \frac{1}{p-1}vp$ and $v(a - c^p) = pv(\eta - c)$ for all $c \in K$, and therefore,*

$$v(a - K^p) \;=\; pv(\eta - K)\,. \tag{12}$$

Further, the following are equivalent:

a) $\mathcal{E}$ has independent defect,

b) $v(\eta - K) = \frac{1}{p-1}vp - \{\alpha \in vK \mid \alpha > H\}$ for some strongly convex subgroup H of vK,

c) $v(a - K^p) = \frac{p}{p-1}vp - \{\alpha \in pvK \mid \alpha > H\}$ for some strongly convex subgroup H of vK,

d) there is some strongly convex subgroup H of vK such that for all $b \in K$ such that $vb > H$ there is $c \in K$ satisfying $v(a - c^p) \geq \frac{p}{p-1}vp - vb$.

If vK is p-divisible, then these properties are also equivalent to

e) $pv(\eta - K) = vp + v(\eta - K)$,

f) $v(a - K^p) = vp + v(\eta - K)$.

If (K,v) has rank 1, then condition d) is equivalent to:

g) for all $b \in \mathcal{O}_K$ there is $c \in K$ satisfying $v(a - c^p) \geq \frac{p}{p-1}vp - vb$.

Remark 1.9. Assume the situation of this theorem. Since K contains a primitive root ζ_p of unity and $v(1 - \zeta_p) = \frac{1}{p-1}vp$, assertion b) is equivalent to

b') $v(\frac{\eta}{1-\zeta_p} - K) = -\{\alpha \in vK \mid \alpha > H\}$ for some strongly convex subgroup H of vK.

We note that conditions g) in the two theorems are elementary in the language of valued rings. It has been pointed out to us by Arno Fehm that in fact also conditions d) are elementary, as it can be stated in the language of valued rings that in the setting of Theorem 1.7, the set

$$\{\alpha \in vK \mid v(a-\wp(K)) \,<\, \alpha \,<\, -v(a-\wp(K))\}$$

and in the setting of Theorem 1.8, the set

$$\left\{\alpha \in vK \;\middle|\; v(a-K^p) - \frac{p}{p-1}vp \,<\, \alpha \,<\, -v(a-K^p) + \frac{p}{p-1}vp \right\}$$

is a strongly convex subgroup H of vK.

The following can be deduced from Theorems 1.7 and 1.8; for details, see [21]. Take a deeply ramified field (K,v) of rank 1 (e.g. a perfectoid field) with char $Kv = p > 0$. If it is of equal characteristic, then it satisfies the sentence

$$\forall a,b \in \mathcal{O}_K \,\exists c \in K :\; v(a-\wp(c)) \,\geq\, -vb\,;$$

likewise, if it is of mixed characteristic, then it satisfies the sentence

$$\forall a,b \in \mathcal{O}_K \,\exists c \in K :\; v(a-c^p) \,\geq\, \frac{p}{p-1}vp - vb\,.$$

Hence the same holds for every elementary extension (K^*,v^*) of (K,v). Assume that v^* admits a proper nontrivial coarsening w. Then (K^*,w) satisfies the sentence

$$\forall a \in \mathcal{O}_{(K^*,w)} \,\exists c \in K^* :\; w(a-\wp(c)) \,\geq\, 0$$

or

$$\forall a \in \mathcal{O}_{(K^*,w)} \,\exists c \in K^* :\; w(a-c^p) \,\geq\, \frac{p}{p-1}wp\,,$$

respectively, which shows that every Artin-Schreier extension or Kummer extension of degree p, respectively, either lies in the henselization of (K^*,w) or is tame (see the definition given in [20]); in particular, they are defectless. This is also a consequence of results obtained in [12], by methods different from those used in [21].

In [21] we will give constructions that will show that all the situations mentioned in the above theorems can appear. To conclude this introduction, let us give two examples of interesting extensions with independent defect.

Example 1.10. Choose a prime p and let $\mathbb{F}_p$ denote the field with p elements. Consider the rational function field $\mathbb{F}_p(t)$, equipped with the t-adic valuation v_t, and its perfect hull $K = \mathbb{F}_p(t)(t^{1/p^n} \mid n \in \mathbb{N})$, equipped with the unique extension of v_t. Take a root ϑ of the Artin-Schreier polynomial

$$X^p - X - \frac{1}{t}\,.$$

The extension $(K(\vartheta)|K,v_t)$ was presented by Shreeram Abhyankar in [1]. It became famous since it shows that there are elements algebraic over $\mathbb{F}_p(t)$ with a power series expansion in which the exponents do not have a common denominator. However, this extension is also an important example for an extension with nontrivial defect. The situation remains the same if we replace $\mathbb{F}_p(t)$ by the field $\mathbb{F}_p((t))$ of formal Laurent series (see [16, Example 3.12]).

In both cases, (K, v_t) is a deeply ramified field, as it is perfect of positive characteristic. Hence by Theorem 1.3, the extension $(K(\vartheta)|K, v_t)$ has independent defect. Now Theorem 1.4 shows that $\Omega_{\mathcal{O}_{K(\vartheta)}|\mathcal{O}_K} = 0$.

An analoguous construction in the mixed characteristic case is given in [16, Example 3.20]. Consider $L = \mathbb{Q}(p^{1/p^n} \mid n \in \mathbb{N})$, equipped with the unique extension of the p-adic valuation v_p of $\mathbb{Q}$. Take a root ϑ of the polynomial

$$X^p - X - \frac{1}{p}\,.$$

Then $(L(\vartheta)|L, v_p)$ is a defect extension. The situation remains the same if we replace $\mathbb{Q}$ by $\mathbb{Q}_p$. In both cases, (L, v_p) is known to be a deeply ramified field (cf. also [21]), hence as before, the extension has independent defect and we have that $\Omega_{\mathcal{O}_{L(\vartheta)}|\mathcal{O}_L} = 0$.

2. Preliminaries

2.1. Defect.

A valued field extension $(L|K, v)$ is **unibranched** if the extension of v from K to L is unique. Note that a unibranched extension is automatically algebraic, since every transcendental extension always admits several extensions of the valuation.

If $(L|K, v)$ is a finite unibranched extension, then by the Lemma of Ostrowski [32, Corollary to Theorem 25, Section G, p. 78]),

$$[L : K] \;=\; \tilde{p}^{\nu} \cdot (vL : vK)[Lv : Kv]\,, \tag{13}$$

where ν is a non-negative integer and $\tilde{p}$ the **characteristic exponent** of Kv, that is, $\tilde{p} = \operatorname{char} Kv$ if it is positive and $\tilde{p} = 1$ otherwise. The factor $d(L|K, v) := \tilde{p}^{\nu}$ is the **defect** of the extension $(L|K, v)$. We call $(L|K, v)$ a **defect extension** if $d(L|K, v) > 1$, and a **defectless extension** if $d(L|K, v) = 1$. Nontrivial defect only appears when $\operatorname{char} Kv = p > 0$, in which case $\tilde{p} = p$. A valued field (K, v) is **henselian** if it satisfies Hensel's Lemma, or equivalently, if all of its algebraic extensions are unibranched. A henselian field (K, v) is called a **defectless field** if all of its finite extensions are defectless.

Throughout this paper, when we talk of a **defect extension** $(L|K, v)$ **of prime degree**, we will always tacitly assume that it is a unibranched extension. Then it follows from (13) that $[L : K] = p = \operatorname{char} Kv$ and that $(vL : vK) = 1 = [Lv : Kv]$; the latter means that $(L|K, v)$ is an **immediate extension**, i.e., the canonical embeddings $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lv$ are onto.

In order to reduce arbitrary finite defect extensions to purely inseparable extensions and Galois extensions of degree $p = \operatorname{char} Kv > 0$, we fix an extension of v from K to its algebraic closure $\tilde{K}$. The **absolute ramification field of** (K, v) (with respect to the chosen extension of v), denoted by (K^r, v), is the ramification field of the normal extension $(K^{\rm sep}|K, v)$. If $a \in \tilde{K}$ is such that $(K(a)|K, v)$ is a defect extension, then $(K^r(a)|K^r, v)$ is a defect extension with the same defect (see [20, Proposition 2.12]). On the other hand, $K^{\rm sep}|K^r$ is a p-extension (see [15, Lemma 2.7]), so $K^r(a)|K^r$ is a tower of purely inseparable extensions and Galois extensions of degree p.

2.2. **Final segments.**

We recall basic notions and facts connected with final segments in ordered abelian groups, and how they relate to elements in valued field extensions. For the details and proofs see Section 2.3 of [15] and Section 3 of [22].

Take a totally ordered set $(T,<)$. For a nonempty subset S of T and an element $t\in T$ we will write $t<S$ if $t<s$ for every $s\in S$, and similarly for $>$ in place of $<$. A set $S\subseteq T$ is called an **initial segment** of T if for each $s\in S$ every $t<s$ also lies in S. Similarly, $S\subseteq T$ is called a **final segment** of T if for each $s\in S$ every $t>s$ also lies in S. By definition, final and initial segments are convex. If S is a final segment of T, then $T\setminus S$ is an initial segment, and vice versa.

For a subset M of T we define

$$M^+ \;:=\; \{t\in T \mid t>M\} \quad\text{ and }\quad M^- \;:=\; \{t\in T\mid \exists m\in M\ t\geq m\}\,.$$

Take a subset S in an ordered abelian group Γ, an element $\gamma\in\Gamma$, and $n\in\mathbb{N}$. We set $-S:=\{-s\mid s\in S\}$, $\gamma+S:=\{\gamma+s\mid s\in S\}$, and $nS:=\{ns\mid s\in S\}$. We note that if S is a final segment of Γ, then $-S$ is an initial segment of Γ, and $\gamma+S$ is a final segment of Γ. If Γ is divisible by n and S is a final segment of Γ, then also nS is a final segment of Γ. Further, we note that

$$(\gamma+S)^+ \;=\; \gamma+S^+ \quad\text{ and }\quad (\gamma+S)^- \;=\; \gamma+S^-\,. \tag{14}$$

2.3. **Strongly convex subgroups.**

Lemma 2.1. *Take an ordered abelian group* $(\Gamma,<)$*, a nonempty final segment* Σ *of* $\Gamma^{\geq 0}:=\{\gamma\in\Gamma\mid\gamma\geq 0\}$*, and* $m\in\mathbb{N}$*,* $m\geq 2$*. Assume that* $\Sigma\neq\Gamma^{\geq 0}$*. Then the following assertions are equivalent:*

a) there is a convex subgroup Δ *of* Γ *such that* $\Sigma=\Gamma^{\geq 0}\setminus\Delta$*,*

b) $\Sigma=(m\Sigma)^-$*.*

Proof. a)⇒b): Since Σ is a final segment of $\Gamma^{\geq 0}$, we know that $(m\Sigma)^-\subseteq\Sigma$. Hence in order to prove that $\Sigma=(m\Sigma)^-$, we have to show the reverse inclusion. For this, we will show that for every $\alpha\in\Sigma$ there is $\beta\in\Sigma$ such that $m\beta<\alpha$. The equality $\Sigma=\Gamma^{\geq 0}\setminus\Delta$ implies that $\Sigma=\{\gamma\in\Gamma\mid\gamma>\Delta\}$, hence we have $\alpha+\Delta>0$ in Γ/Δ. As Γ/Δ has no smallest positive element, the positive elements cannot be bounded away from 0 by an element in the divisible hull of Γ/Δ, and so there must be $\beta\in\Gamma$ such that $0<\beta+\Delta<\frac{1}{m}(\alpha+\Delta)$. This implies $m\beta+\Delta<\alpha+\Delta$, whence $m\beta<\alpha$, as desired.

b)⇒a): Since $\Sigma\neq\Gamma^{\geq 0}$ is a nonempty final segment of $\Gamma^{\geq 0}$, we see that $\Gamma^{\geq 0}\setminus\Sigma$ is nonempty and convex. We prove that $\Gamma^{\geq 0}\setminus\Sigma$ is additively closed. Take $\gamma,\gamma'\in\Gamma^{\geq 0}\setminus\Sigma$; we may assume that $\gamma'\leq\gamma$. Hence $0\leq\gamma+\gamma'\leq m\gamma$. Suppose that $m\gamma\in\Sigma$. Then by the statement of b), there is $\beta\in\Sigma$ such that $m\beta\leq m\gamma$, whence $\beta\leq\gamma$. From the convexity of Σ it follows that $\gamma\in\Sigma$, contradiction. Thus $m\gamma\in\Gamma^{\geq 0}\setminus\Sigma$, and by the convexity of $\Gamma^{\geq 0}\setminus\Sigma$, we obtain that $\gamma+\gamma'\in\Gamma^{\geq 0}\setminus\Sigma$. This proves our claim.

Since $\Gamma^{\geq 0}\setminus\Sigma$ is convex, also $\Delta:=\{\gamma,-\gamma\mid\gamma\in\Gamma^{\geq 0}\setminus\Sigma\}$ is convex. Since $\Gamma^{\geq 0}\setminus\Sigma$ is additively closed, it follows by convexity that Δ is additively closed, hence a convex subgroup of Γ.

Take any $\alpha \in \Gamma$ such that $0 < \alpha + \Delta$. It follows that $0 < \alpha \notin \Delta$, hence $\alpha \in \Sigma$. Take $\beta \in \Sigma$ such that $m\beta \leq \alpha$. We have that $\beta + \Delta < m\beta + \Delta$ since otherwise, $(m-1)\beta \in \Delta$ and by convexity, $\beta \in \Delta$, contradiction. Thus $0 < \beta + \Delta < m\beta + \Delta \leq \alpha + \Delta$. This shows that $\alpha + \Delta$ is not the smallest positive element of Γ/Δ. As $\alpha + \Delta$ was arbitrary, we conclude that Γ/Δ has no smallest positive element. □

2.4. **Immediate and unibranched extensions.**

Take a valued field (K, v). A **henselization** of (K, v) is an algebraic extension of (K, v) which admits a valuation preserving embedding in every other henselian extension of (K, v). Henselizations always exist and are unique up to valuation preserving isomorphism over K; therefore we will talk of *the* henselization of (K, v) and denote it by $(K, v)^h = (K^h, v^h)$. The henselization of (K, v) is an immediate separable-algebraic extension.

In what follows, we will need the following result, which is [3, Lemma 2.1]:

Lemma 2.2. *A finite extension $(L|K, v)$ is unibranched if and only if $L|K$ is linearly disjoint from some (equivalently, every) henselization of (K, v).*

Let us consider an immediate but not necessarily algebraic extension $(K(x)|K, v)$. Then by [15, Theorem 2.19] the set $v(x - K) \subseteq vK$ is a final segment of vK; in particular, it has no maximal element. If $g \in K[X]$ and there is $\alpha \in v(x - K)$ such that for all $c \in K$ with $v(x - c) \geq \alpha$ the value $vg(c)$ is constant, then we will say that **the value of g is ultimately fixed over** K. We call $(K(x)|K, v)$ **pure in** x if the value of every $g(X) \in K[X]$ of degree smaller than $[K(x) : K]$ is ultimately fixed over K. Note that we set $[K(x) : K] = \infty$ if x is transcendental over K.

Lemma 2.3. *Every unibranched immediate extension $(K(x)|K, v)$ of prime degree is pure in x.*

Proof. By the Lemma of Ostrowski, every such extension must have a defect equal to its degree, which must be equal to its residue characteristic $p > 0$.

In the case of a henselian field (K, v), the assertion follows from [22, Proposition 6.5]. We have to consider the general case. Since $(K(x)|K, v)$ is unibranched, Lemma 2.2 shows that $K(x)|K$ is linearly disjoint from $K^h|K$. Hence $(K^h(x)|K^h, v)$ is a unibranched extension of degree p, and it is immediate since $K^h(x) = K(x)^h$ and henselizations are immediate extensions. By what we have said above, the extension $(K^h(x)|K^h, v)$ is pure.

Further, it follows from [17, Theorem 2] that there is no element $b \in K^h$ such that $v(x - b) > v(x - K)$. Thus, if there were a polynomial $g \in K[X]$ of degree less than p whose value is not ultimately fixed over K, then its value would also not be ultimately fixed over K^h, contradiction. □

The next lemma follows from [13, Lemma 8] and [22, Lemma 5.2]. Note that if $(K(x)|K, v)$ is an extension such that $v(x - K)$ has no maximal element, then by the proof of [13, Theorem 1], x is limit of a pseudo Cauchy sequence in (K, v) without limit in K, or equivalently, by [22, part a) of Lemma 4.1] its approximation type over (K,v) is immediate. We use the Taylor expansion

$$f(X) = \sum_{i=0}^{n} \partial_i f(c)(X-c)^i \tag{15}$$

where $\partial_i f$ denotes the i-th **Hasse-Schmidt derivative** of f.

Lemma 2.4. *Take an immediate extension $(K(x)|K,v)$ that is pure in x. Let p be the characteristic exponent of Kv. Then for every nonconstant polynomial $f\in K[X]$ of degree smaller than $[K(x):K]$ there is some $\gamma\in v(x-K)$ such that for all $c\in K$ with $v(x-c)\geq\gamma$ and all i with $1\leq i\leq \deg f$,*

the values $v\partial_i f(c)$ are fixed, equal to $v\partial_i f(x)$, and

the values $v\partial_i f(c)+i\cdot v(x-c)$ are pairwise distinct.

2.5. **Artin-Schreier and Kummer extensions.**

Every Galois extension of degree p of a field K of characteristic $p>0$ is an **Artin-Schreier extension**, that is, generated by an **Artin-Schreier generator** ϑ which is the root of an **Artin-Schreier polynomial** X^p-X-b with $b\in K$. For every $c\in K$, also $\vartheta-c$ is an Artin-Schreier generator as its minimal polynomial is $X^p-X-b+c^p-c$. Every Galois extension of prime degree n of a field K of characteristic different from n which contains all n-th roots of unity is a **Kummer extension**, that is, generated by a **Kummer generator** η which satisfies $\eta^n\in K$. For these facts, see [23, Chapter VI, §6].

A **1-unit** in a valued field (K,v) is an element of the form $u=1+b$ with $b\in\mathcal{M}_K$; in other words, u is a unit in $\mathcal{O}_K$ with residue 1. Take a Kummer extension $(L|K,v)$ of degree p of fields of characteristic 0 with any Kummer generator η. Assume that $v\eta\in vK$, so that there is $c_1\in K$ such that $vc_1=-v\eta$, whence $vc_1\eta=0$. Assume further that $c_1\eta v\in Kv$, so that there is $c_2\in K$ such that $c_2v=(c_1\eta v)^{-1}$. Then $vc_2c_1\eta=0$ and $c_2c_1\eta v=1$. Furthermore, $K(c_2c_1\eta)=K(\eta)$ and $(c_2c_1\eta)^p=c_2^pc_1^p\eta^p\in K$. Hence $c_2c_1\eta$ is a Kummer generator of $(L|K,v)$ and a 1-unit. This shows that in particular, if $(L|K,v)$ is an immediate Kummer extension, it always admits a Kummer generator which is a 1-unit.

We note that if η is a 1-unit and if $v(\eta-c)>v\eta=0$, then also c and c^{-1} are 1-units; conversely, if c is a 1-unit, then $v(\eta-c)>0$.

We will need the following facts.

Lemma 2.5. *Take a valued field (K,v), $n\in\mathbb{N}$, and a primitive n-th root of unity $\zeta\in K$. Then*

$$\prod_{i=1}^{n-1}(1-\zeta^i) \;=\; n\,. \tag{16}$$

If in addition n is prime, then

$$v(1-\zeta) \;=\; \frac{vn}{n-1}\,. \tag{17}$$

Proof. Let $f(X)=X^n-1=\prod_{i=1}^n(X-\zeta^i)$. Then $f'(X)=nX^{n-1}$ and $\prod_{i=1}^{n-1}(1-\zeta^i)=f'(1)=n$. This proves equation (16).

If n is prime, then all powers $\zeta^k \neq 1$ are also primitive n-th roots of unity. We may assume that ζ is chosen among the primitive n-th roots of unity such that $v(1-\zeta)$ is maximal. Take any $k \in \mathbb{N}$ such that $\zeta^k \neq 1$. Then

$$1-\zeta^k \;=\; (1-\zeta)(1+\zeta+\ldots+\zeta^{k-1})\,.$$

Since $v\zeta = 0$, we have $v(1+\zeta+\ldots+\zeta^{k-1}) \geq 0$, whence $v(1-\zeta^k) \geq v(1-\zeta)$. By our choice of ζ, this yields $v(1-\zeta^k) = v(1-\zeta)$. Consequently, all factors of the product in (16) have equal value. This proves equation (17). □

2.6. **Ideals and final segments.**

Take a valued field (L,v). The function

$$v:\; I \;\mapsto\; vI \;:=\; \{vb \mid 0 \neq b \in I\} \tag{18}$$

is an order preserving bijection from the set of all proper ideals of $\mathcal{O}_L$ onto the set of all final segments of $vL^{>0}$ (including the final segment $\emptyset$). The set of these final segments is linearly ordered by inclusion, and the function (18) is order preserving: $J \subseteq I$ holds if and only if $vJ \subseteq vI$ holds. The inverse of the above function is the order preserving function

$$\Sigma \;\mapsto\; I_\Sigma \;:=\; (a \in L \mid va \in \Sigma) \;=\; \{a \in L \mid va \in \Sigma\} \cup \{0\}\,. \tag{19}$$

The following facts from general valuation theory are well known:

Lemma 2.6. *The following statements are equivalent:*
a) I_Σ is a prime ideal of $\mathcal{O}_L$,
b) $\Sigma = vL^{\geq 0} \setminus H$ for some convex subgroup H of vL.
If b) holds, then $I_\Sigma = \mathcal{M}_{v_H}$ for the coarsening v_H of v on L whose value group is vL/H.

We note that for each $S \subset vL^{\geq 0}$, we have the following equalities of $\mathcal{O}_L$-ideals:

$$(a \in L \mid va \in S) \;=\; (a \in L \mid va \in S^-) \;=\; I_{S^-}\,. \tag{20}$$

Lemma 2.7. *Take a nonempty final segment Σ of $vL^{\geq 0}$ and $m \in \mathbb{N}$.*
1) We have

$$I_\Sigma^m \;=\; (b \in L \mid vb \in m\Sigma) \;=\; (b \in L \mid vb \in (m\Sigma)^-)\,.$$

2) We have $I_\Sigma = I_\Sigma^m$ if and only if $\Sigma = (m\Sigma)^-$.

Proof. 1): We have

$$\begin{aligned} I_\Sigma^m \;&=\; (a^m \mid a \in I_\Sigma) \;=\; (a^m \mid a \in L \text{ and } va \in \Sigma) \;=\; (b \in L \mid vb \in m\Sigma) \\ &=\; (b \in L \mid vb \in (m\Sigma)^-)\,, \end{aligned}$$

where the fourth equality follows from (20), and the third equality is seen as follows. The inclusion $\subseteq$ is obvious. Assume that $b \in L$ with $vb \in m\Sigma$, i.e., there is $\alpha \in \Sigma$ such that $m\alpha = vb$. Choose $a \in L$ such that $va = \alpha$. Then $va^m = vb$, hence $b \in (a^m \mid a \in L \text{ and } va \in \Sigma)$. This proves the inclusion $\supseteq$.

2): This follows immediately from part 1). □

2.7. **Ramification jumps and ramification ideals.**

Take a valued field (K,v). Assume that $L|K$ is a Galois extension, and let $G = \mathrm{Gal}\,(L|K)$ denote its Galois group. For proper ideals I of $\mathcal{O}_L$ we consider the (upper series of) **higher ramification groups**

$$G_I \;:=\; \left\{\sigma \in G \;\middle|\; \frac{\sigma b - b}{b} \in I \;\text{ for all } b \in L^\times\right\} \tag{21}$$

(see [32, §12]). Note that $G_{\mathcal{M}_L}$ is the ramification group of $(L|K,v)$. For every ideal I of $\mathcal{O}_L$, G_I is a normal subgroup of G ([32, (d) on p. 79]). The function

$$\varphi:\; I \;\mapsto\; G_I \tag{22}$$

preserves $\subseteq$, that is, if $I \subseteq J$, then $G_I \subseteq G_J$. As $\mathcal{O}_L$ is a valuation ring, the set of its ideals is linearly ordered by inclusion. This shows that also the higher ramification groups are linearly ordered by inclusion. Note that in general, φ will neither be injective nor surjective as a function to the set of normal subgroups of G.

Using the function (19), the higher ramification groups can be represented as

$$G_\Sigma \;:=\; G_{I_\Sigma} \;=\; \left\{\sigma \in G \;\middle|\; v\,\frac{\sigma b - b}{b} \in \Sigma \cup \{\infty\} \;\text{ for all } b \in L^\times\right\},$$

where Σ runs through all final segments of $vL^{>0}$.

Like the function (22), also the function $\Sigma \mapsto G_\Sigma$ is in general not injective. We call Σ a **ramification jump** if

$$\Sigma' \subsetneq \Sigma \;\Rightarrow\; G_{\Sigma'} \subsetneq G_\Sigma$$

for all final segments Σ' of $vL^{>0}$. If Σ is a ramification jump, then I_Σ is called a **ramification ideal**. It follows from the definition that an ideal I of $\mathcal{O}_L$ is a ramification ideal if and only if for some subgroup G' of G, I is the smallest ideal such that $G' = G_I$ (cf. [9, §21]).

In this paper we are particularly interested in the case where $\mathcal{E} = (L|K,v)$ is a Galois extension of prime degree p. Then $G = \mathrm{Gal}\,(L|K)$ is a cyclic group of order p and thus has only one proper subgroup, namely $\{\mathrm{id}\}$, and this subgroup is equal to G_Σ for $\Sigma = \emptyset$. If in this case G itself is the ramification group of the extension, then there must be a unique ramification jump $\Sigma_\mathcal{E}$, and we will call $I_\mathcal{E} = I_{\Sigma_\mathcal{E}}$ *the* ramification ideal of $(L|K,v)$. As we will show in the next section, ramification jump and ramification ideal carry important information about the extension $(L|K,v)$.

2.8. **Galois defect extensions of prime degree.**

The following is part of Theorem 3.5 of [20]:

Theorem 2.8. *Take a Galois defect extension $\mathcal{E} = (L|K,v)$ of prime degree with Galois group G. Then G is the ramification group of $\mathcal{E}$. The set Σ_σ does not depend on the choice of the generator σ of G. Writing $\Sigma_\mathcal{E}$ for Σ_σ, we have that the final segment $\Sigma_\mathcal{E}$ of $vK^{>0}$ is the unique ramification jump of the extension $\mathcal{E}$. Further, $I_\mathcal{E} = I_{\Sigma_\mathcal{E}}$ is the unique ramification ideal of the extension $\mathcal{E}$.*

If $(L|K,v)$ is an Artin-Schreier defect extension with any Artin-Schreier generator ϑ, then

$$\Sigma_{\mathcal{E}} \;=\; -v(\vartheta - K)\,. \tag{23}$$

If K contains a primitive root of unity and $(L|K,v)$ is a Kummer extension with Kummer generator η of value 0, then

$$\Sigma_{\mathcal{E}} \;=\; \frac{1}{p-1}vp - v(\eta - K)\,. \tag{24}$$

Corollary 2.9. *Let the assumptions be as in the preceding theorem. Then for every $c \in K$,*

$$v(\vartheta - c) \;<\; 0 \quad \textit{and} \quad v(\eta - c) \;<\; \frac{1}{p-1}vp\,.$$

Proof. Our assertions follow from (23), (24), and the fact that $\Sigma_{\mathcal{E}} \subseteq vK^{>0}$ by Theorem 2.8. □

Proposition 2.10. *Assume that the extension $\mathcal{E} = (L|K,v)$ satisfies (8).*

1) The following assertions are equivalent:

a) the extension $\mathcal{E}$ has independent defect,

b) $\Sigma_{\mathcal{E}} \;=\; (p\Sigma_{\mathcal{E}})^-$,

c) $I_{\mathcal{E}}^p \;=\; I_{\mathcal{E}}$.

d) $(I_{\mathcal{E}} \cap K)^p \;=\; I_{\mathcal{E}} \cap K$.

If vK is p-divisible, then these properties are also equivalent to $\Sigma_{\mathcal{E}} \;=\; p\Sigma_{\mathcal{E}}$.

2) If $\mathcal{E}$ has independent defect, then the corresponding convex subgroup $H_{\mathcal{E}}$ of vL does not contain vp.

Proof. 1): By definition, $\mathcal{E}$ has independent defect if and only if $\Sigma_{\mathcal{E}} = vL^{\geq 0} \setminus H_{\mathcal{E}}$ for some strongly convex subgroup $H_{\mathcal{E}}$ of vL. By Lemma 2.1, this holds if and only if $\Sigma_{\mathcal{E}} = (p\Sigma_{\mathcal{E}})^-$. By part 2) of Lemma 2.7, this in turn is equivalent to $I_{\mathcal{E}}^p = I_{\mathcal{E}}$ since $I_{\mathcal{E}} = I_{\Sigma_{\mathcal{E}}}$. It is also equivalent to $(I_{\mathcal{E}} \cap K)^p \;=\; I_{\mathcal{E}} \cap K$ because $vL = vK$ and therefore, $I_{\mathcal{E}} \cap K = (a \in K \mid va \in \Sigma_{\mathcal{E}})$.

The last assertion of part 1) holds since if vK is p-divisible, then $p\Sigma_{\mathcal{E}}$ is again a final segment, whence $(p\Sigma_{\mathcal{E}})^- = p\Sigma_{\mathcal{E}}$.

2): If $\operatorname{char} K = p$, then $vp = \infty$ and the assertion is trivial. In the mixed characteristic case, where $\mathcal{E}$ is a Kummer extension and admits a Kummer generator η of value 0, we know that $0 = v(\eta - 0) \in v(\eta - K)$, hence also $0 \in -v(\eta - K)$. It follows from (24) that $\frac{1}{p-1}vp \in \Sigma_{\mathcal{E}}$. Therefore, $vp \notin H_{\mathcal{E}}$ since otherwise $\frac{1}{p-1}vp \in H_{\mathcal{E}}$ by convexity; but this is a contradiction. □

3. Generation of immediate unibranched extensions of valuation rings

In this section we will assume that $(L|K,v)$ is an immediate extension, and in various cases determine generators for the valuation ring $\mathcal{O}_L$ as an $\mathcal{O}_K$-algebra.

3.1. **The general case.**

In this section we consider an immediate but not necessarily algebraic extension $(K(x)|K,v)$. Then the set $v(x-K)$ has no maximal element. For this and the definition of "pure extension", see Section 2.4.

For every $\gamma\in vK$ we choose $t_\gamma\in K$ such that $vt_\gamma=-\gamma$. For every $c\in\mathcal{O}_K$ we know that $v(x-c)\in vK$ since the extension is immediate, so we may set $t_c:=t_{v(x-c)}$ and

$$x_c \;:=\; t_c(x-c)\;\in\;\mathcal{O}_{K(x)}^\times\,.$$

We use the Taylor expansion (15).

Lemma 3.1. *Assume that the immediate extension $(K(x)|K,v)$ is pure. Then for every $g(x)\in\mathcal{O}_{K(x)}\cap K[x]$ there is $c\in K$ such that $g(x)\in\mathcal{O}_K[x_c]$. If in addition $K(x)|K$ is algebraic, then*

$$\mathcal{O}_{K(x)} \;=\; \bigcup_{c\in K}\mathcal{O}_K[x_c]\,.$$

Proof. From Lemma 2.4 we infer that whenever for $c\in K$ the value $v(x-c)$ is large enough, then the values $v\partial_i g(c)(x-c)^i$, $0\le i\le\deg g$, are pairwise distinct. It follows that

$$vg(x)\;=\;\min_i v\partial_i g(c)(x-c)^i\;=\;\min_i v\partial_i g(c)t_c^{-i}x_c^i\,.$$

Since $g(x)\in\mathcal{O}_{K(x)}$ and $vx_c=0$, we find that $\partial_i g(c)t_c^{-i}\in\mathcal{O}_K$ for all i and therefore, $g(x)\in\mathcal{O}_K[x_c]$.

If $K(x)|K$ is algebraic, then $K(x)=K[x]$ and the last assertion of our lemma follows from what we have already proved. □

Lemma 3.2. *Take $c_1,c_2\in K$. If $v(x-c_1)=v(x-c_2)$, then $\mathcal{O}_K[x_{c_1}]=\mathcal{O}_K[x_{c_2}]$. If $v(x-c_1)<v(x-c_2)$, then $\mathcal{O}_K[x_{c_1}]\subseteq\mathcal{O}_K[x_{c_2}]$, and if in addition x_{c_1} is integral over K, then $\mathcal{O}_K[x_{c_1}]\subsetneq\mathcal{O}_K[x_{c_2}]$.*

Proof. Assume that $v(x-c_1)=v(x-c_2)$. Then $t_{c_1}=t_{c_2}$ and

$$v(c_1-c_2)\;\ge\;\min\{v(x-c_1),v(x-c_2)\}\;=\;-vt_{c_1}\,,$$

whence $t_{c_1}(c_1-c_2)\in\mathcal{O}_K$. It follows that

$$\mathcal{O}_K[x_{c_1}]\;=\;\mathcal{O}_K[x_{c_1}+t_{c_1}(c_1-c_2)]\;=\;\mathcal{O}_K[t_{c_1}(x-c_2)]\;=\;\mathcal{O}_K[t_{c_2}(x-c_2)]\;=\;\mathcal{O}_K[x_{c_2}]\,.$$

Now assume that $v(x-c_1)<v(x-c_2)$. Then $v(c_1-c_2)=\min\{v(x-c_1),v(x-c_2)\}=v(x-c_1)=-vt_{c_1}<-vt_{c_2}$, whence $t_{c_1}/t_{c_2}\in\mathcal{M}_K$ and $t_{c_1}(c_1-c_2)\in\mathcal{O}_K^\times$. This shows that

$$x_{c_1}\;=\;\frac{t_{c_1}}{t_{c_2}}\cdot t_{c_2}(x-c_2)\;+\;t_{c_1}(c_2-c_1)\;\in\;\mathcal{O}_K[x_{c_2}]\,,\tag{25}$$

whence $\mathcal{O}_K[x_{c_1}]\subseteq\mathcal{O}_K[x_{c_2}]$.

Suppose that $x_{c_2}\in\mathcal{O}_K[x_{c_1}]$ and x_{c_1} is integral over K. Then x_{c_2} can be written in the form

$$x_{c_2}\;=\;a_0+a_1x_{c_1}+a_2x_{c_1}^2+\ldots+a_{p-1}x_{c_1}^{n-1}$$

with $n=[K(x):K]$ and coefficients in $\mathcal{O}_K$. Since the powers of x_{c_1} appearing in this expression form a basis of $K(x)|K$, this representation of x_{c_2} is unique even if one allows the coefficients to be in K. However, according to (25) we can write $x_{c_1} = ax_{c_2} + b$, where $a \in \mathcal{M}_K$ and $b \in \mathcal{O}_K^\times$. This yields the unique representation

$$x_{c_2} \;=\; a^{-1}(x_{c_1} - b)\,,$$

which does not have its coefficients in $\mathcal{O}_K$. This contradiction shows that $\mathcal{O}_K[x_{c_1}] \neq \mathcal{O}_K[x_{c_2}]$. □

From Lemmas 3.1 and 3.2, we obtain:

Proposition 3.3. *Assume that the immediate algebraic extension $(K(x)|K,v)$ is pure. Then the rings $\mathcal{O}_K[x_c]$, $c \in \mathcal{O}_K$, form a chain under inclusion whose union is $\mathcal{O}_{K(x)}$.*

3.2. **The case of immediate Artin-Schreier and Kummer extensions.**

An extension of prime degree is a defect extension if and only if it is immediate and unibranched.

Theorem 3.4. *1) Assume that $(L|K,v)$ is an Artin-Schreier defect extension with Artin-Schreier generator ϑ. The rings $\mathcal{O}_K[\vartheta_c]$, where $\vartheta_c = t_{v(\vartheta-c)}(\vartheta - c)$ and c runs through all elements of K, form a chain under inclusion whose union is $\mathcal{O}_L$. The same holds for the rings $\mathcal{O}_K[t_{v\vartheta}\vartheta]$ when ϑ runs through all Artin-Schreier generators of the extension.*

For $c_1, c_2 \in K$,

$$\mathcal{O}_K[\vartheta_{c_1}] \subsetneq \mathcal{O}_K[\vartheta_{c_2}] \;\Leftrightarrow\; v(\vartheta - c_1) < v(\vartheta - c_2)\,. \tag{26}$$

2) Assume that $(L|K,v)$ is a Kummer defect extension of degree p and that η is a Kummer generator which is a 1-unit. The rings $\mathcal{O}_K[\eta_c]$, where $\eta_c = t_{v(\eta-c)}(\eta - c)$ and c runs through all 1-units of K, form a chain under inclusion whose union is $\mathcal{O}_L$. If $c_1, c_2 \in K$ are 1-units, then

$$\mathcal{O}_K[\eta_{c_1}] \subsetneq \mathcal{O}_K[\eta_{c_2}] \;\Leftrightarrow\; v(\eta - c_1) < v(\eta - c_2)\,. \tag{27}$$

Proof. From Lemma 2.3 and Proposition 3.3 we know that $\mathcal{O}_L$ is the union of the chain $(\mathcal{O}_K[\vartheta_c])_{c\in K}$ in the Artin-Schreier extension case, and of the chain $(\mathcal{O}_K[\eta_c])_{c\in K}$ in the Kummer extension case. Further, (26) and (27) follow from Lemma 3.2.

It remains to prove the additional assertion for the Artin-Schreier extension case. By [15, Lemma 2.26], $\vartheta' \in L$ is another Artin-Schreier generator of $L|K$ if and only if $\vartheta' = i\vartheta - c$ for some $i \in \mathbb{F}_p^\times$ and $c \in K$. In particular, $\vartheta - c$ is an Artin-Schreier generator for all $c \in K$. Moreover, if $\vartheta' = i\vartheta - c$, then $v\vartheta' = v(i\vartheta - c) = v(\vartheta - i^{-1}c)$ and therefore, $t_{v\vartheta'} = t_{i^{-1}c}$ and

$$\mathcal{O}_K[t_{v\vartheta'}\vartheta'] \;=\; \mathcal{O}_K[t_{v\vartheta'}i^{-1}\vartheta'] \;=\; \mathcal{O}_K[t_{v(\vartheta-i^{-1}c)}(\vartheta - i^{-1}c)] \;=\; \mathcal{O}_K[\vartheta_{i^{-1}c}]\,.$$

This shows that the ring $\mathcal{O}_K[t_{v\vartheta'}\vartheta']$ is already a member in the chain constructed above. □

Let us consider the **Artin-Schreier case** a bit further, keeping the assumptions of part 1) of Theorem 3.4. We set $t_c := t_{v(\vartheta-c)}$. Take any elements $c_1, c_2 \in K$. It follows from (25) that

$$\vartheta_{c_1} \;=\; \frac{t_{c_1}}{t_{c_2}}\vartheta_{c_2} + t_{c_1}(c_2 - c_1)\,. \tag{28}$$

If $v(\vartheta - c_1) \leq v(\vartheta - c_2)$, then $t_{c_1}/t_{c_2} \in \mathcal{O}_K$ and $t_{c_1}(c_2 - c_1) \in \mathcal{O}_K$, as shown in the proof of Lemma 3.2.

Now we turn to the **Kummer case**, keeping the assumptions of part 2) of Theorem 3.4. We choose a primitive p-th root ζ_p of unity and set

$$\chi \;:=\; (\zeta_p - 1)^{-1}\,.$$

According to equation (17), the element χ satisfies:

$$v\chi \;=\; -\frac{1}{p-1} vp\,. \tag{29}$$

Hence by Corollary 2.9,

$$v\chi(\eta - c) \;<\; 0$$

for all $c \in K$. In the following we will use the abbreviations

$$T_c \;:=\; t_{v\chi(\eta-c)} \quad \text{and} \quad \eta_{<c>} \;:=\; T_c\chi(\eta - c)$$

in order to simplify our formulas; we observe that

$$vT_c \;=\; \frac{1}{p-1} vp - v(\eta - c) \;>\; 0\,. \tag{30}$$

Corollary 3.5. *Assume that $(L|K, v)$ is a Kummer defect extension of degree p. Then for every Kummer generator η which is a 1-unit, the rings $\mathcal{O}_K[\eta_{<c>}]$, where c runs through all 1-units of K, form a chain under inclusion whose union is $\mathcal{O}_L$.*

We have that

$$\mathcal{O}_K[\eta_{<c>}] \;=\; \mathcal{O}_K[\eta_c]\,. \tag{31}$$

Hence if c_1 and c_2 are 1-units, then

$$\mathcal{O}_K[\eta_{<c_1>}] \;\subsetneq\; \mathcal{O}_K[\eta_{<c_2>}] \;\Leftrightarrow\; v(\eta - c_1) < v(\eta - c_2)\,. \tag{32}$$

Proof. It suffices to prove (31). We compute:

$$v\frac{T_c}{\chi^{-1}t_c} \;=\; v\frac{t_{v\chi(\eta-c)}}{t_{v\chi}t_{v(\eta-c)}} \;=\; 0\,.$$

Hence,

$$\mathcal{O}_K[\eta_{<c>}] \;=\; \mathcal{O}_K\left[\frac{T_c}{\chi^{-1}t_c}\,\chi^{-1}t_c\,\chi(\eta - c)\right] \;=\; \mathcal{O}_K[t_c(\eta - c)] \;=\; \mathcal{O}_K[\eta_c]\,.$$

Now the assertion of our corollary follows from part 2) of Theorem 3.4. □

Take any elements $c_1, c_2 \in K$. Similarly as in the Artin-Schreier case, one derives the equation

$$\eta_{<c_1>} \;=\; \frac{T_{c_1}}{T_{c_2}}\eta_{<c_2>} + T_{c_1}\chi(c_2 - c_1)\,. \tag{33}$$

Assume that $v(\eta - c_1) \leq v(\eta - c_2)$. Then $vT_{c_1} \geq vT_{c_2}$ and

$$v(c_2 - c_1) \;\geq\; \min\{v(\eta - c_1), v(\eta - c_2)\} \;=\; v(\eta - c_1) \;=\; -vT_{c_1}\chi\,,$$

whence $T_{c_1}/T_{c_2} \in \mathcal{O}_K$ and $T_{c_1}\chi(c_2 - c_1) \in \mathcal{O}_K$.

3.3. **Values of derivatives of minimal polynomials.**

In the case of Artin-Schreier extensions and Kummer extensions $(K(x)|K,v)$ of prime degree we have sufficient information about the minimal polynomials f of the various generators x we have worked with in the previous sections, or equivalently, about their conjugates, to work out the values $vf'(x)$. In order to do this, we can either compute f' explicitly, or we can use the formula

$$f'(x) \;=\; \prod_{\sigma\in G\setminus\{\mathrm{id}\}} (x-\sigma x)\,, \tag{34}$$

where G is the Galois group of $K(x)|K$.

We keep the notations from the previous sections.

3.3.1. *Artin-Schreier extensions.*

Take an Artin-Schreier polynomial f with ϑ as its root. Then $f(X)=X^p-X-\vartheta^p+\vartheta$ and $f'(X)=-1$, whence

$$f'(\vartheta)=-1\,. \tag{35}$$

This is also obtained from (34) since $\{\vartheta-\sigma\vartheta\mid\sigma\in G\setminus\{\mathrm{id}\}\}=\mathbb{F}_p^\times$ and the product of all elements of $\mathbb{F}_p^\times$ is -1.

For $t\in K^\times$, denote by f_t the minimal polynomial of $t\vartheta$. Then

$$f_t'(t\vartheta) \;=\; \prod_{\sigma\in G\setminus\{\mathrm{id}\}} (t\vartheta-\sigma t\vartheta) \;=\; t^{p-1}f'(\vartheta) \;=\; -t^{p-1}\,. \tag{36}$$

Lemma 3.6. *Take an Artin-Schreier defect extension $\mathcal{E}=(L|K,v)$ of prime degree p with Artin-Schreier generator ϑ. Denote by g_c the minimal polynomial of ϑ_c. Then the $\mathcal{O}_L$-ideal $(g_c'(\vartheta_c)\mid c\in K)$ is equal to*

$$I^{p-1}$$

where

$$I \;=\; (t_c\mid c\in K) \;=\; (a\in L\mid va\in -v(\vartheta-K)) \;=\; I_{\mathcal{E}} \tag{37}$$

is the ramification ideal of the extension $(L|K,v)$.

Proof. Applying (36) to $\vartheta_c=t_c(\vartheta-c)$ and keeping in mind that $\vartheta-c$ is an Artin-Schreier generator, we obtain:

$$g_c'(\vartheta_c) \;=\; -t_c^{p-1}\,. \tag{38}$$

This shows that $(g_c'(\vartheta_c)\mid c\in K)=(t_c\mid c\in K)^{p-1}=I^{p-1}$.

The second equality in (37) holds since $vt_c=-v(\vartheta-c)$ by definition. The third equality follows from equation (23) of Theorem 2.8. □

3.3.2. *Kummer extensions.*
Take $f(X) = X^p - \eta^p$. Then $f'(X) = pX^{p-1}$, whence

$$f'(\eta) \;=\; p\eta^{p-1}\,. \tag{39}$$

This is also obtained from equations (34) and (16) using that $\{\sigma\eta \mid \sigma \in G\} = \{\zeta_p^i\eta \mid i \in \{0,\dots,p-1\}\}$.

Lemma 3.7. *Take a Kummer defect extension $\mathcal{E} = (L|K,v)$ of prime degree p with Kummer generator η which is a 1-unit. Denote by f_c the minimal polynomial of $\eta_{<c>}$ over K. Then the $\mathcal{O}_L$-ideal $(f_c'(\eta_{<c>}) \mid c \in K$ a 1-unit$)$ is*

$$I^{p-1}$$

where

$$I \;=\; (T_c \mid c \in K) \;=\; \left(a \in L \mid va \in \frac{1}{p-1}vp - v(\eta - K)\right) \;=\; I_{\mathcal{E}} \tag{40}$$

is the ramification ideal of the extension $(L|K,v)$.

Proof. We use (34) to compute:

$$f_c'(\eta_{<c>}) \;=\; \prod_{\sigma\in G}(\eta_{<c>} - \sigma\eta_{<c>}) \;=\; \prod_{\sigma\in G} T_c\chi(\eta - \sigma\eta) \;=\; (T_c\chi\eta)^{p-1}\prod_{i=1}^{p-1}(1-\zeta_p^i)\,,$$

whence by equations (16) and (29), and the fact that $v\eta = 0$,

$$f_c'(\eta_{<c>}) \;=\; p(T_c\chi\eta)^{p-1} \;=\; T_c^{p-1}u \tag{41}$$

with u a unit in $\mathcal{O}_L$. This shows that

$$(f_c'(\eta_{<c>}) \mid c \in K \text{ a 1-unit}) = (T_c \mid c \in K)^{p-1} = I^{p-1}\,.$$

The second equality in (40) follows from (30). The assertion that $I = I_{\mathcal{E}}$ in the Kummer case follows from equation (24) of Theorem 2.8. □

4. Kähler differentials and their annihilators for algebraic field extensions

4.1. **Proof of Theorem 1.1.**

Let $(L|K,v)$ be a unibranched algebraic field extension. Let $A \subseteq K$ be a normal domain whose quotient field is K. Assume that $z \in L$ is integral over A and let $f(X)$ be the minimal polynomial of z over K. Then $f(X) \in A[X]$ (see [31, Theorem 4, page 260]). By the Gauss Lemma (see [28, Theorem A]), $A[z] \cong A[X]/(f(X))$. Thus,

$$\Omega_{A[z]|A} \;\cong\; [A[X]/(f(X), f'(X))]dX \;\cong\; [A[z]/(f'(z))]dX \tag{42}$$

by [24, Example 26.J, page 189] and [24, Theorem 58, page 187]. There is a canonical derivation $d_{A[z]|A} : A \to \Omega_{A[z]|A}$ defined by $g(z) \mapsto g'(z)dX$ for $g(X) \in A[X]$.

We will now proced under the assumptions of Theorem 1.1. Before proving the theorem, we derive a formula that we will need in its proof. Let $n = [L : K]$. Then for all $\alpha \in S$, $n = \deg h_\alpha(X)$ where $h_\alpha(X)$ is the minimal polynomial of b_α over K. Suppose that $\alpha < \beta$. Then

$$\left(\frac{a_\beta}{a_\alpha}\right)^n h_\alpha\left(\frac{a_\alpha}{a_\beta}X + c_{\alpha,\beta}\right)$$

is a monic polynomial of degree n in $K[X]$ which has b_β as a root, so it is the minimal polynomial $h_\beta(X)$ of b_β. By the chain rule,

$$\frac{dh_\beta}{d_{X_\beta}}(b_\beta) = \left(\frac{a_\beta}{a_\alpha}\right)^n \frac{dh_\alpha}{dX}(b_\alpha)\left(\frac{a_\alpha}{a_\beta}\right),$$

and so

$$h'_\alpha(b_\alpha) = \left(\frac{a_\alpha}{a_\beta}\right)^{n-1} h'_\beta(b_\beta). \tag{43}$$

Proof of Theorem 1.1:
From the natural A-algebra isomorphisms $A[b_\alpha] \cong A[X_\alpha]/(h_\alpha(X_\alpha))$, where $h_\alpha(X_\alpha)$ is the minimal polynomial of b_α over K, we have that for $\alpha \le \beta$, the natural inclusion of A-algebras $A[b_\alpha] \subset A[b_\beta]$ is induced by sending b_α to $\frac{a_\alpha}{a_\beta}b_\beta + c_{\alpha,\beta}$.

By (42), for $\alpha \in S$ we have a natural isomorphism of $A[b_\alpha]$-modules

$$\Omega_{A[b_\alpha]|A} \cong [A[b_\alpha]/(h'_\alpha(b_\alpha))]\, dX_\alpha\ ,$$

and so

$$\Omega_{A[b_\alpha]|A} \otimes_{A[b_\alpha]} B \cong [B/(h'_\alpha(b_\alpha))]\, dX_\alpha\ .$$

For $\alpha < \beta$, by the universal property of derivations (Proposition on page 182 [24]), there is a unique B-module homomorphism

$$\lambda_{\alpha,\beta} : \Omega_{A[b_\alpha]|A} \otimes_{A[b_\alpha]} B \to \Omega_{A[b_\beta]|A} \otimes_{A[b_\beta]} B$$

such that there is a commutative diagram of B-module homomorphisms

$$\begin{array}{ccc} \Omega_{A[b_\alpha]|A} \otimes_{A[b_\alpha]} B & \overset{\lambda_{\alpha,\beta}}{\to} & \Omega_{A[b_\beta]|A} \otimes_{A[b_\beta]} B \\ \uparrow & & \uparrow \\ A[b_\alpha] \otimes_{A[b_\alpha]} B & \to & A[b_\beta] \otimes_{A[b_\beta]} B \end{array}$$

where the vertical arrows are the respective maps

$$d_{A[b_\alpha]/A} \otimes 1 : A[b_\alpha] \otimes_{A[b_\alpha]} B \to \Omega_{A[b_\alpha]|A} \otimes_{A[b_\alpha]} B$$

and

$$d_{A[b_\beta]/A} \otimes 1 : A[b_\beta] \otimes_{A[b_\beta]} B \to \Omega_{A[b_\beta]|A} \otimes_{A[b_\beta]} B.$$

Suppose that $z \in A[b_\alpha]$. Then there exists $g(X_\alpha) \in A[X_\alpha]$ such that $z = g(b_\alpha)$, hence $d_{A[b_\alpha]/A}(z) = \frac{dg}{dX_\alpha}(b_\alpha)dX_\alpha$. We have that $z = \left(g(\frac{a_\alpha}{a_\beta}X_\beta + c_{\alpha,\beta})\right)(b_\beta)$ so that by the chain rule,

$$d_{A[b_\beta]/A}(z) \;=\; \left(\frac{d}{dX_\beta}g\left(\frac{a_\alpha}{a_\beta}X_\beta + c_{\alpha,\beta}\right)\right)(b_\beta)dX_\beta \;=\; \frac{dg}{dX_\alpha}(b_\alpha)\frac{a_\alpha}{a_\beta}dX_\beta.$$

Thus $\lambda_{\alpha,\beta}$ is the B-module homomorphism defined by mapping dX_α to $\frac{a_\alpha}{a_\beta}dX_\beta$.

In order to compute the direct limit of the directed system of B-module homomorphisms $\lambda_{\alpha,\beta} : \Omega_{A[b_\alpha]|A} \otimes_{A[b_\alpha]} B \to \Omega_{A[b_\beta]|A}$ for $\alpha < \beta$, we will introduce an equivalent directed system, which is a little simpler. For $\alpha \in S$, let M_α be the

B-module $M_\alpha = B/(h'_\alpha(b_\alpha))$. The M_α are a family of B-modules. We have isomorphisms of B-modules $\tau_\alpha : \Omega_{A[b_\alpha]|A} \otimes_{A[b_\alpha]} B \to M_\alpha$ defined by mapping dX_α to 1.

For all $\alpha, \beta \in S$ with $\alpha < \beta$, we have commutative diagrams of B-module homomorphisms

$$\begin{array}{ccc} \Omega_{A[b_\alpha]|A} \otimes_{A[b_\alpha]} B & \stackrel{\lambda_{\alpha,\beta}}{\to} & \Omega_{A[b_\beta]|A} \otimes_{A[b_\beta]} B \\ \downarrow \tau_\alpha & & \downarrow \tau_\beta \\ M_\alpha & \stackrel{\tau_\beta\lambda_{\alpha,\beta}\tau_\alpha^{-1}}{\to} & M_\beta. \end{array}$$

and we see that $\tau_\beta\lambda_{\alpha,\beta}\tau_\alpha^{-1}$ is just multiplication by $\frac{a_\alpha}{a_\beta}$. Thus the directed systems

$$\Omega_{A[b_\alpha]|A} \otimes_{A[b_\alpha]} B \stackrel{\lambda_{\alpha,\beta}}{\to} \Omega_{A[b_\beta]|A} \otimes_{A[b_\beta]} B \text{ for } \alpha < \beta \quad \text{and} \quad M_\alpha \stackrel{\frac{a_\alpha}{a_\beta}}{\to} M_\beta \text{ for } \alpha < \beta$$

are equivalent.

We have isomorphisms of B-modules

$$\Omega_{B|A} \cong \lim_{\to} \left[\Omega_{A[b_\alpha]|A} \otimes_{A[b_\alpha]} B\right] \cong \lim_{\to} M_\alpha$$

by [8, Corollary 16.7, page 394], where the direct limits are over $\alpha \in S$.

Write $M_\alpha = R_\alpha/T_\alpha$ where R_α is the directed system of B-modules $R_\alpha = B$ for $\alpha \in S$, with B-module homomorphisms $\frac{a_\alpha}{a_\beta} : R_\alpha \to R_\beta$ for $\alpha < \beta$, and T_α is the directed system of B-modules $T_\alpha = h'_\alpha(b_\alpha)R_\alpha$ for $\alpha \in S$, with B-module homomorphisms $\frac{a_\alpha}{a_\beta} : T_\alpha \to T_\beta$ for $\alpha \le \beta$. We have short exact sequences of B-modules

$$0 \to T_\alpha \to R_\alpha \to M_\alpha \to 0$$

which are compatible with multiplication by $\frac{a_\alpha}{a_\beta}$ for $\alpha < \beta$. Thus

$$\lim_{\to} M_\gamma \cong \lim_{\to} R_\gamma / \lim_{\to} T_\gamma$$

by [27, Theorem 2.18].

We now compute $\lim_{\to} R_\gamma$. For $\alpha \in S$, we have B-module homomorphisms $R_\alpha \stackrel{a_\alpha}{\to} B$ which give commutative diagrams

$$\begin{array}{ccc} R_\alpha & \stackrel{\frac{a_\alpha}{a_\beta}}{\to} & R_\beta \\ a_\alpha \searrow & & \swarrow a_\beta \\ & B & \end{array}$$

for $\alpha < \beta$. By the universal property of direct limits ([5, Proposition 11.1] or [2, Exercise 18, page 33]) there exists a unique B-module homomorphism $\Psi : \lim_{\to} R_\gamma \to B$ giving commutative diagrams of B-module homomorphisms

$$\begin{array}{ccc} R_\alpha & & \\ \frac{a_\alpha}{a_\beta} \downarrow & \searrow \phi_\alpha & \\ R_\beta & \stackrel{\phi_\beta}{\to} & \lim_{\to} R_\gamma \\ a_\beta \downarrow & \swarrow \Psi & \\ B & & \end{array}$$

for $\alpha<\beta$. Here $\phi_\alpha : R_\alpha \to \lim\limits_{\to} R_\gamma$ are the canonical maps of the direct product.

As we will now show, it follows from some standard properties of direct limits ([5, Proposition 11.3] or [2, Exercise 15, page 33]) and some diagram chasing that as B-ideals,

$$\lim_{\to} R_\gamma \;\cong\; (a_\alpha \mid \alpha\in S) \;=\; U\,.$$

Suppose that $u\in \lim\limits_{\to} R_\gamma$. Then there exist $\alpha\in S$ and $b\in R_\alpha=B$ such that $u=\phi_\alpha(b)$. Thus $\Psi(u)=a_\alpha b\in U$. Given $\alpha\in S$, $\Psi(\phi_\alpha(1))=a_\alpha$. Thus the image of Ψ is U. Suppose that $0\neq u\in \lim\limits_{\to} R_\gamma$. Then there exist $\alpha\in S$ and $0\neq b\in R_\alpha$ such that $\phi_\alpha(b)=u$ so $\Psi(u)=a_\alpha b\neq 0$. Thus Ψ is an isomorphism and so

$$\lim_{\to} R_\gamma \cong U.$$

Similarly,

$$\lim_{\to} T_\gamma \cong (a_\alpha h'_\alpha(b_\alpha)\mid \alpha\in S).$$

By (43), we have that for $\alpha,\beta<\gamma$,

$$a_\alpha h'_\beta(b_\beta)=\frac{a_\alpha}{a_\gamma}\left(\frac{a_\beta}{a_\gamma}\right)^{n-1} a_\gamma h'_\gamma(b_\gamma).$$

Thus $(a_\alpha h'_\alpha(b_\alpha)\mid \alpha\in S)=UV$ and so $\Omega_{B|A}\cong U/UV$. □

If we choose any $\gamma\in S$, then we will still have

$$\bigcup_{\gamma\le\alpha\in S} A[b_\alpha] \;=\; B$$

since $(A[b_\alpha])_{\alpha\in S}$ is an increasing chain. Hence we can always assume that S has a minimal element.

From now on we consider the case of $A=\mathcal{O}_K$ and $B=\mathcal{O}_L$. Assume that $\alpha\le\beta$. Then $a_\beta\mid a_\alpha$, so that $\frac{a_\alpha}{a_\beta}\,,\ \left(\frac{a_\alpha}{a_\beta}\right)^{n-1}\in\mathcal{O}_K$. From (1) and (43) it thus follows that $b_\alpha\in(b_\beta)$ and $h'_\alpha(b_\alpha)\in(h'_\beta(b_\beta))$. This shows that $\mathcal{O}_L$-ideals $(b_\alpha)_{\alpha\in S}$ and $(h'_\alpha(a_\alpha))_{\alpha\in S}$ form increasing chains.

Now let γ be the minimal element of S. Using (43) with $\beta=\gamma$, we obtain:

$$V \;=\; (h'_\alpha(b_\alpha)\mid\alpha\in S) \;=\; \left(\left(\frac{a_\alpha}{a_\gamma}\right)^{n-1} h'_\gamma(b_\gamma)\mid \alpha\in S\right) \;=\; \frac{h'_\gamma(b_\gamma)}{a_\gamma^{n-1}}U^{n-1}\,. \tag{44}$$

This proves (3).

4.2. **The annihilator.**

We take an arbitrary valued field (L,v) with valuation ring $\mathcal{O}_L$, an $\mathcal{O}_L$-ideal U, an element $b\in\mathcal{O}_L$, and $n\ge 2$. We set $V:=bU^{n-1}$. We will now compute the annihilator of the $\mathcal{O}_L$-ideal U/UV. We note that V is contained in $\operatorname{ann} U/UV$, which is the largest $\mathcal{O}_L$-ideal V' such that $UV'=UV$.

We use the order preserving bijection (18). We have $vI^n=nvI$; recall that for every final segment S of vL, nS is the n-fold sum of S, which is equal to the smallest final segment containing $n\alpha$ for all $\alpha\in S$. Thus, $vV=vb+(n-1)vU$ and

$vUV = vb + nvU$. Note that for each $\alpha \in vL$ there are two associated **principal** final segments:

$$\alpha^- := \{\alpha\}^- = \{\beta \in vL \mid \beta \geq \alpha\} \text{ and } \alpha^+ := \{\alpha\}^+ = \{\beta \in vL \mid \beta > \alpha\}.$$

In order to find V', we have to find the maximal final segment Σ of $vL^{\geq 0}$ such that $vU + \Sigma = vUV$.

For a final segment S of vL, we define its **invariance group** to be

$$\mathcal{G}(S) := \{\alpha \in vL \mid \alpha + S = S\}.$$

It is straightforward to show that this is a convex subgroup of vL, that it is a proper subgroup if $S \neq vL$, and that for any two final segments S_1 and S_2,

$$\mathcal{G}(S_1 + S_2) = \mathcal{G}(S_1) \cup \mathcal{G}(S_2) = \max\{\mathcal{G}(S_1), \mathcal{G}(S_2)\}.$$

Further, for each $\alpha \in vL$, $\mathcal{G}(\alpha^-) = \mathcal{G}(\alpha^+) = \{0\}$. For this and more properties of $\mathcal{G}(S)$, see [18, 19]. It follows that

$$\mathcal{G}(vU) = \mathcal{G}((n-1)vU) = \mathcal{G}(nvU) = \mathcal{G}(vb + nvU) = \mathcal{G}(vb + (n-1)vU) = \mathcal{G}(vV).$$

Let us first consider the case where $\mathcal{G}(vU) = \{0\}$ (which for instance holds in rank 1). Denote by Γ the divisible hull of vL. For any final segment S of vL, let S_Γ be the smallest final segment of Γ containing S.

Assume first that vU^{n-1} has an infimum α_0 in vL (that is, vV has infimum $\alpha = vb + \alpha_0$ in vL. Then $(vU)_\Gamma$ has infimum $\alpha_1 := \alpha_0/(n-1)$ in Γ and vU^{n-1} has infimum $(n-1)\alpha_1 = \alpha_0$ in Γ. Since $n - 1 \geq 1$, we have that $(n-1)\alpha_1^-$ is the largest final segment Σ of Γ such that $\alpha_1^- + \Sigma = n\alpha_1^-$. Further, we have $(n-1)\alpha_1^- + \alpha_1^+ = n\alpha_1^+$, and as $(n-1)\alpha_1 + \alpha_1 = n\alpha_1$ is the infimum of $n\alpha_1^+$ in Γ, it follows that the largest final segment Σ of Γ such that $\alpha_1^+ + \Sigma = n\alpha_1^+$ cannot contain any element smaller than $(n-1)\alpha_1$; hence it is again equal to $(n-1)\alpha_1^- = \alpha_0^-$. Consequently, α_0^- is the largest final segment Σ of Γ such that $(vU)_\Gamma + \Sigma = (vU^n)_\Gamma$. Intersecting all sets with vL, we find that α_0^- (now understood as a final segment in vL) is the largest final segment Σ of vL such that $vU + \Sigma = vU^n$. Hence $vb + \alpha_0^-$ is the largest final segment Σ of vL such that $vU + \Sigma = vUV$.

Now assume that vU^{n-1} (or equivalently, $vV = vb + vU$) has no infimum in vL. Let Σ be the largest final segment of vL such that $vU + \Sigma = vU^n$. Suppose that Σ properly contains vU^{n-1}. Then there is some $\alpha \in \Sigma \setminus vU^{n-1}$, but as vU^{n-1} does not have an infimum, there is also some $\alpha' \in \Sigma \setminus vU^{n-1}$ such that $\alpha < \alpha' < vU^{n-1}$. We have $vU + vU^{n-1} = vU^n$, whence

$$vU^n \subseteq vU + \alpha' \subseteq vU + \alpha \subseteq vU + \Sigma = vU^n,$$

so equality holds everywhere. However, we then have that $\alpha' - \alpha \in \mathcal{G}(vU)$, but as $vU \neq vL$, we have $\mathcal{G}(vU) = \{0\}$, whence $\alpha = \alpha'$. This contradiction shows that $\Sigma = vU^{n-1}$.

Using that $vb + vU^{n-1} = vV$, we summarize what we have shown. If vV has an infimum α in vL, then α^- is the largest final segment Σ of vL such that $vU + \Sigma = vUV$. This is equal to vV if vV contains its infimum, and properly contains vV otherwise. If vV has no infimum in vL, then vV is the largest final segment Σ of vL such that $vU + \Sigma = vUV$.

Now we consider the case of $\mathcal{G}(vU) \neq \{0\}$. Given $\alpha \in vL$ and $S \subseteq vL$, by $\bar{\alpha}$ and $\bar{S}$ we denote the images of α and S in $vL/\mathcal{G}(vU)$. We have $\mathcal{G}(\overline{vU}) = \{0\}$ (cf. [19]).

Hence what we have proved already remains valid for the final segments $\overline{vU}$ and $\overline{vV}$ in $\overline{vL}$. So we have: If $\overline{vV}$ has an infimum $\bar{\alpha}$ in $\overline{vL}$, then $\bar{\alpha}^-$ is the largest final segment $\bar{\Sigma}$ of $\overline{vL}$ such that $\overline{vU}+\bar{\Sigma}=\overline{vUV}$. If $\overline{vV}$ has no infimum in $\overline{vL}$, or if $\overline{vV}$ contains its infimum in $\overline{vL}$, then $\overline{vV}$ is the largest final segment $\bar{\Sigma}$ of $\overline{vL}$ such that $\overline{vU}+\bar{\Sigma}=\overline{vUV}$.

Let Σ be the maximal final segment of vL such that $vUV+\Sigma=vUV$. Given a final segment S of vL, the maximal final segment of vL reducing to $\bar{S}$ modulo $\mathcal{G}(vU)$ is

$$\mathcal{G}(vU)+S\,,$$

which is equal to S if $\mathcal{G}(vU)$ is contained in $\mathcal{G}(S)$. Hence, $\Sigma=\mathcal{G}(vU)+\alpha^-=\mathcal{G}(vV)+\alpha^-$ if there is $\alpha\in vL$ such that $\overline{vV}$ has infimum $\bar{\alpha}$ in $\overline{vL}$. If $\overline{vV}$ contains its infimum $\bar{\alpha}$ in $\overline{vL}$, then $\Sigma=\mathcal{G}(vU)+\alpha^-=\mathcal{G}(vV)+vV=vV$. If $\overline{vV}$ has no infimum in $\overline{vL}$, then $\Sigma=\mathcal{G}(vV)+vV=vV$.

At this point, let us take a moment to discuss a special case. Assume that U is a proper $\mathcal{O}_L$-ideal, $\mathcal{G}(vU)$ is nontrivial, $vb=0$, and $\overline{vU}$ admits 0 as an infimum. Then $\overline{vV}$ has infimum $\bar{0}$ in $\overline{vL}$ and it follows that $\Sigma=\mathcal{G}(vU)+0^-$, which contains negative elements of vL. So the largest $\mathcal{O}_L$-ideal V' such that $UV'=UV$ is $V'=(a\in L\mid va\in\mathcal{G}(vU)+0^-)$, which is a fractional $\mathcal{O}_L$-ideal. In contrast, by definition the annihilator is an ideal contained in $\mathcal{O}_L$, so $\operatorname{ann}U/UV=(a\in\mathcal{O}_L\mid va\in\mathcal{G}(vU)+0^-)=\mathcal{O}_L$. As vU does not contain 0 by our assumption on U, we have $vU=\{\alpha\in vL\mid \alpha>\mathcal{G}(vU)\}$, which means that $U=\mathcal{M}_{\mathcal{G}(vU)}$. All idempotent ideals other than $\mathcal{O}_L$ and $\mathcal{M}_L$ are of this form (for $\mathcal{O}_L$ and $\mathcal{M}_L$ we have infimum 0 with trivial invariance group). For all idempotent ideals U we have $U=V=UV$.

From what we have shown, we obtain:

Proposition 4.1. *We use the notation introduced above. If there is $\alpha\in vL$ such that $\overline{vV}$ has infimum $\bar{\alpha}$ in $\overline{vL}$, then*

$$\operatorname{ann}U/UV \;=\; (a\in\mathcal{O}_L\mid va\in\mathcal{G}(vU)+\alpha^-)\,. \tag{45}$$

This is equal to V if and only if $\overline{vV}$ contains its infimum in $\overline{vL}$.

If $\overline{vV}$ has no infimum in $\overline{vL}$, then $V=\operatorname{ann}U/UV$.

Proposition 4.2. *1) If there are $\alpha\in vL$ and a convex subgroup H of vL such that $V=(a\in\mathcal{O}_L\mid va>\alpha+H\}$, then*

$$\operatorname{ann}U/UV \;=\; (a\in\mathcal{O}_L\mid va\in H+\alpha^-)$$

which properly contains V, and

$$\mathcal{M}_{v_H}\operatorname{ann}U/UV\;\subseteq\;V\,. \tag{46}$$

In all other cases, $V=\operatorname{ann}U/UV$.

2) If $\mathcal{M}_L$ annihilates U/UV, then $\mathcal{O}_L=\operatorname{ann}U/UV$, except if $\mathcal{M}_L$ is principal and either $n=2$ and $U=V=\mathcal{M}_L$, or $U=\mathcal{O}_L$ and $V=\mathcal{M}_L=(b)$. In these latter cases, $\mathcal{M}_L=\operatorname{ann}U/UV$.

Proof. 1): A part of the statement follows from the previous proposition if we take into account that $\mathcal{G}(\{\beta\in vL\mid\beta>\alpha+H\})=H$. It only remains to prove (46) in case $\operatorname{ann}U/UV$ properly contains V. We have $v\mathcal{M}_{v_H}=\{\beta\in vL\mid\beta>H\}$ and

$$\{\beta\in vL\mid\beta\in H+\alpha^-\}\;=\;(\alpha+H)\cup\{\beta\in vL\mid\beta>\alpha+H\}\;=\;(\alpha+H)\cup vV\,.$$

As $\mathcal{M}_{v_H} V \subseteq V$, it suffices to prove that $v\mathcal{M}_{v_H} + (\alpha + H) \subseteq vV$. This holds as $\beta > H$ implies $\beta + \gamma > H$ for all $\gamma \in H$, as H is a convex subgroup, and this in turn implies $\alpha + \beta + \gamma > \alpha + H$.

2): Assume that $\mathcal{M}_L$ annihilates U/UV. If $\operatorname{ann} U/UV = \mathcal{O}_L$, then there is nothing to prove, so suppose that $\operatorname{ann} U/UV = \mathcal{M}_L$.

Assume first that $\mathcal{M}_L^2 = \mathcal{M}_L$. We have $U\mathcal{M}_L = UV = bU^n$. If U is principal, then the same is true for bU^n, but as $U\mathcal{M}_L$ is not principal, this leads to a contradiction. If U is not principal, then $UV = U\mathcal{M}_L = U$, so $\operatorname{ann} U/UV = \mathcal{O}_L$, which again is a contradiction.

Now assume that $\mathcal{M}_L$ is principal. By Proposition 4.1, either $\operatorname{ann} U/UV = V$ or $\operatorname{ann} U/UV$ is of the form (45). Assume that $\operatorname{ann} U/UV = V$. Then $\mathcal{M}_L = V = bU^{n-1}$. If $U = \mathcal{O}_L$, then $V = b\mathcal{O}_L$ and $\operatorname{ann} U/UV = \mathcal{M}_L = (b)$. If $U \neq \mathcal{O}_L$, then $U \subseteq \mathcal{M}_L = bU^{n-1}$ and it follows that $n = 2$ and $U = \mathcal{M}_L = V$.

Now assume that $\operatorname{ann} U/UV = \mathcal{M}_L$ is of the form (45). Then $\mathcal{G}(vU) + \alpha^-$ must contain the smallest value of $v\mathcal{M}_L$. By our earlier discussion, $\mathcal{G}(vU)$ must then be trivial since otherwise, $\mathcal{G}(vU) + \alpha^-$ contains 0 so that $\operatorname{ann} U/UV = \mathcal{O}_L$. It follows that the smallest element of $v\mathcal{M}_L$ is the infimum of vV. But as vL is discretely ordered, this shows that $vV = v\mathcal{M}_L$, whence $V = \mathcal{M}_L = \operatorname{ann} U/UV$. Since $U \subseteq V$, we have either $U = \mathcal{O}_L$ or $U = \mathcal{M}_L$. If $U = \mathcal{O}_L$, then $V = bU^{n-1} = (b)$. If $U = \mathcal{M}_L$, then $\mathcal{M}_L = V = b\mathcal{M}_L^{n-1}$ and as $\mathcal{M}_L$ is principal, we must have $n - 1 = 1$. □

Corollary 4.3. *Assume that $(L|K, v)$ satisfies (8). If $\mathcal{M}_L$ annihilates $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$, then $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ and $\operatorname{ann} \Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{O}_L$.*

Proof. Assume that $\mathcal{M}_L$ annihilates $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$; then since $\Omega_{\mathcal{O}_L|\mathcal{O}_K} \simeq U/UV$, $\mathcal{M}_L$ also annihilates U/UV. Under the assumption of the corollary, we have that $U = I_{\mathcal{E}}$ and $V = I_{\mathcal{E}}^{p-1}$ are not principal, hence the exceptional cases of part 2) of the preceding proposition cannot appear. Consequently, $\mathcal{O}_L = \operatorname{ann} U/UV$, which implies that U/UV and thus also $\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ are zero. □

4.3. The ideal V of differents.

Under the assumptions of Theorem 1.1, we will now show that V is the B-ideal generated by the differents of the generators b_α of B over A. Take any generator $b \in \mathcal{O}_L$ of $L|K$ and let h be its minimal polynomial over K. Denote the different of b by $\delta(b) := h'(b)$. For $i \geq 1$ we have

$$b^i - \sigma b^i \;=\; b^i - (\sigma b)^i \;=\; b^i - (b + (\sigma b - b))^i \;=\; \sum_{j=0}^{i-1} \binom{i}{j} b^j (\sigma b - b)^{i-j}\,.$$

Since the extension is unibranched, we have $v\sigma b = vb$, whence $v(b - \sigma b) \geq vb \geq 0$. Consequently,

$$v(b^i - \sigma b^i) \;\geq\; v(\sigma b - b) \;=\; v(b - \sigma b)\,.$$

Every $b \in \mathcal{O}_K[b_\alpha] \setminus \mathcal{O}_K$ is of the form

$$b \;=\; \sum_{i=0}^{n-1} c_i b_\alpha^i$$

with $c_i \in \mathcal{O}_K$. Therefore,

$$\begin{aligned} v\delta(b) &= v\prod_{\mathrm{id}\neq\sigma\in G}(b-\sigma b) = \sum_{\mathrm{id}\neq\sigma\in G} v\left(\sum_{i=0}^{n-1} c_i b_\alpha^i - \sigma\sum_{i=0}^{n-1} c_i (b_\alpha)^i\right) \\ &= \sum_{\mathrm{id}\neq\sigma\in G} v\sum_{i=1}^{n-1} c_i(b_\alpha^i - \sigma b_\alpha^i)\,. \end{aligned}$$

For $1 \leq i \leq n-1$, we have

$$vc_i(b_\alpha^i - \sigma b_\alpha^i) \;\geq\; v(b_\alpha^i - \sigma b_\alpha^i) \;\geq\; v(b_\alpha - \sigma b_\alpha)\,,$$

showing that

$$v\sum_{i=1}^{n-1} c_i(b_\alpha^i - \sigma b_\alpha^i) \;\geq\; v(b_\alpha - \sigma b_\alpha)\,.$$

Hence,

$$v\delta(b) \;\geq \sum_{\mathrm{id}\neq\sigma\in G} v(b_\alpha - \sigma b_\alpha) \;=\; vh'_\alpha(b_\alpha)\,.$$

We use this to conclude:

Proposition 4.4. *Under the assumptions of Theorem 1.1, we have:*

$$V \;=\; (\delta(b) \mid b \in \mathcal{O}_L \setminus \mathcal{O}_K)\,. \tag{47}$$

Proof. Since $\mathcal{O}_L$ is the union of the chain of rings $\mathcal{O}_K[b_\alpha]$, $\alpha \in S$, we have

$$(\delta(b) \mid b \in \mathcal{O}_L \setminus \mathcal{O}_K) \;=\; \bigcup_{\alpha\in S}(\delta(b) \mid b \in \mathcal{O}_K[b_\alpha] \setminus \mathcal{O}_K) \;=\; \bigcup_{\alpha\in S}(h'_\alpha(b_\alpha)) \;=\; V\,.$$

□

4.4. **The case of Artin-Schreier defect extensions.**

Theorem 4.5. *Let $(L|K,v)$ be an Artin-Schreier defect extension with ramification ideal $I_\mathcal{E}$. Then there is an $\mathcal{O}_L$-module isomorphism*

$$\Omega_{\mathcal{O}_L/\mathcal{O}_K} \cong I_\mathcal{E}/I_\mathcal{E}^p.$$

Proof. Let ϑ be an Artin-Schreier generator of $L|K$. By Theorem 3.4, the $\mathcal{O}_K$-algebras $\mathcal{O}_K[\vartheta_c]$, with $c \in K$, form a chain of subrings of L whose union is $\mathcal{O}_L$. For $c \in K$, $\vartheta_c = t_c(\vartheta - c)$, where $t_c \in \mathcal{O}_K$ is such that $v\vartheta_c = 0$. For $v(\vartheta - c_1) < v(\vartheta - c_2)$, by (28) we have that

$$\vartheta_{c_1} = \frac{t_{c_1}}{t_{c_2}}\vartheta_{c_2} + t_{c_1}(c_2 - c_1)$$

with $\frac{t_{c_1}}{t_{c_2}}$, $t_{c_1}(c_2 - c_1) \in \mathcal{O}_K$. By equation (38), the minimal polynomial g_c of ϑ_c satisfies $g'_c(\vartheta_c) = -t_c^{p-1}$. We will apply Theorem 1.1 together with equation (3). Let $A = \mathcal{O}_K$, $B = \mathcal{O}_L$ and $S = K$ ordered by $\alpha < \beta$ if $v(\vartheta - \alpha) < v(\vartheta - \beta)$. Let $b_\alpha = \vartheta_\alpha$, $a_\alpha = t_\alpha$, $c_{\alpha,\beta} = t_\alpha(\beta - \alpha)$ and $h_\alpha = f_\alpha$ so that $h'_\alpha(b_\alpha) = f'_\alpha(\vartheta_\alpha) = -t_\alpha^{p-1}$. Hence for every $\alpha \in S$, $h'_\alpha(b_\alpha)/a_\alpha^{p-1} = -1$. The theorem now follows from equation (3) together with Lemma 3.6, where U is the $\mathcal{O}_L$-ideal $U = (t_\alpha \mid \alpha \in S) = I_\mathcal{E}$. □

4.5. **The case of Kummer defect extensions of degree p.**

Theorem 4.6. *Assume that* $\operatorname{char} K = 0$ *and let* $(L|K, v)$ *be a Kummer defect extension of degree* p *with ramification ideal* $I_{\mathcal{E}}$. *Then there is an* $\mathcal{O}_L$*-module isomorphism*

$$\Omega_{\mathcal{O}_L|\mathcal{O}_K} \cong I_{\mathcal{E}}/I_{\mathcal{E}}^p \,.$$

Proof. Let η be a Kummer generator of $L|K$ which is a 1-unit. We use the notation from Section 3.2. By Corollary 3.5, the $\mathcal{O}_K$-algebras $\mathcal{O}_K[\eta_{<c>}]$ with c a 1-unit in K form a chain of subrings of L whose union is $\mathcal{O}_L$. For $v(\eta - c_1) < v(\eta - c_2)$, by (33) we have that

$$\eta_{<c_1>} = \frac{T_{c_1}}{T_{c_2}}\eta_{<c_2>} + T_{c_1}\chi(c_2 - c_1)$$

with $\frac{T_{c_1}}{T_{c_2}}$, $T_{c_1}\chi(c_2 - c_1) \in \mathcal{O}_K$. By equation (41), the minimal polynomial f_c' of $\eta_{<c>}$ satisfies $f_c'(\eta_{<c>}) = T_c^{p-1}u$ with u a unit in $\mathcal{O}_L$. We will apply Theorem 1.1 together with equation (3). Let $A = \mathcal{O}_K, B = \mathcal{O}_L$ and $S = K$ ordered by $\alpha < \beta$ if $v(\eta - \alpha) < v(\eta - \beta)$. Let $b_\alpha = \eta_{<\alpha>}$, $a_\alpha = T_\alpha$, $c_{\alpha,\beta} = T_\alpha\chi(\beta - \alpha)$ and $h_\alpha = f_\alpha$ so that $h_\alpha'(b_\alpha) = f_\alpha'(\eta_{<\alpha>}) = T_c^{p-1}u$. Hence for every $\alpha \in S$, $h_\alpha'(b_\alpha)/a_\alpha^{p-1} = u$. The theorem now follows from equation (3) together with Lemma 3.7, where U is the $\mathcal{O}_L$-ideal $U = (T_\alpha \mid \alpha \in S) = I_{\mathcal{E}}$. □

5. The trace of valuation rings and Dedekind differents

5.1. Defect extensions of prime degree.

In this section we will consider the trace on an extension $\mathcal{E} = (L|K, v)$ that satisfies (8). If $L|K$ is an Artin-Schreier extension, then we write $L = K(\vartheta)$ where ϑ is an Artin-Schreier generator. If $\operatorname{char} K = 0$ and $L|K$ is a Kummer extension, then we write $L = K(\eta)$ where η is a Kummer generator, that is, $\eta^p \in K$; as explained at the beginning of Section 2.8, we can assume that η is a 1-unit.

The proof of the following fact can be found in [14, Section 6.3].

Lemma 5.1. *Take a separable field extension* $K(a)|K$ *of degree* n *and let* $f(X) \in K[X]$ *be the minimal polynomial of* a *over* K. *Then*

$$\operatorname{Tr}_{K(a)|K}\left(\frac{a^m}{f'(a)}\right) = \begin{cases} 0 & \text{if } 1 \le m \le n-2 \\ 1 & \text{if } m = n-1\,. \end{cases} \tag{48}$$

□

For arbitrary $b, c \in K$, we note:

$$b(a-c)^{p-1} \in \mathcal{M}_{K(a)} \iff vb > -(p-1)v(a-c)\,. \tag{49}$$

First we consider Artin-Schreier extensions. By Lemma 5.1 and equation (35),

$$\operatorname{Tr}_{K(\vartheta)|K}\left((\vartheta - c)^i\right) = \begin{cases} 0 & \text{if } 1 \le i \le p-2 \\ -1 & \text{if } i = p-1 \end{cases} \tag{50}$$

for arbitrary $c \in K$ since $\vartheta - c$ is also an Artin-Schreier generator. In particular,

$$\operatorname{Tr}_{K(\vartheta)|K}\left(b(\vartheta - c)^{p-1}\right) = -b\,.$$

By (49) it follows that

$$\begin{aligned}(51)\quad \mathrm{Tr}_{K(\vartheta)|K}\left(\mathcal{M}_{K(\vartheta)}\right) &\supseteq \{b\in K \mid vb > -(p-1)v(\vartheta-c) \text{ for some } c\in K\}\\ &= \{b\in K \mid vb \geq -(p-1)v(\vartheta-c) \text{ for some } c\in K\}\\ &= (b\in K \mid vb \in -(p-1)v(\vartheta-K)\}\\ &= (b\in K \mid vb\in (p-1)\Sigma_{\mathcal{E}}\}\ ,\end{aligned}$$

where the first equality follows from the fact that $-(p-1)v(\vartheta-c)$ has no smallest element, and the last equality follows from equation (23) of Theorem 2.8.

Now we consider Kummer extensions. Since $(\eta^i)^p\in K$, we have that

$$\mathrm{Tr}_{K(\eta)|K}(\eta^i) \;=\; 0$$

for $1\leq i\leq p-1$. For $c\in K$ and $0\leq j\leq p-1$, we compute:

$$(\eta-c)^j \;=\; \sum_{i=1}^{j}\binom{j}{i}\eta^i(-c)^{j-i} \;+\; (-c)^j\ .$$

Thus for every $b\in K$,

$$(52)\qquad \mathrm{Tr}_{K(\eta)|K}(b(\eta-c)^j) \;=\; pb(-c)^j\ .$$

By (49), $b(\eta-c)^{p-1}\in\mathcal{M}_{K(\eta)}$ holds if and only if $vb>-(p-1)v(\eta-c)$; the latter remains true if we make $v(\eta-c)$ even larger. Since η is a 1-unit, there is $c\in K$ such that $v(\eta-c)>0$, which implies that $vc=0$. Hence we may choose $c\in K$ with $vb>-(p-1)v(\eta-c)$ *and* $vc=0$. Applying (52) with $j=p-1$, we find that $\mathrm{Tr}_{K(\eta)|K}(b(-c)^{-(p-1)}(\eta-c)^{p-1})=pb$. We obtain:

$$\begin{aligned}(53)\quad \mathrm{Tr}_{K(\eta)|K}\left(\mathcal{M}_{K(\eta)}\right) &\supseteq \{pb \mid b\in K,\, vb>-(p-1)v(\eta-c) \text{ for some } c\in K\}\\ &= \{b\in K \mid vb\geq vp-(p-1)v(\eta-c) \text{ for some } c\in K\}\\ &= \left(b\in K \mid vb\in (p-1)\left(\frac{1}{p-1}vp-v(\eta-K)\right)\right)\\ &= (b\in K \mid vb\in(p-1)\Sigma_{\mathcal{E}})\ ,\end{aligned}$$

where the first equality follows from the fact that $-(p-1)v(\vartheta-c)$ has no smallest element, and the last equality follows from equation (24) of Theorem 2.8.

In order to prove the opposite inclusions in (51) and (53), we have to find out enough information about the elements $g(a)\in K(a)$ that lie in $\mathcal{M}_{K(a)}$. Using the Taylor expansion, we write

$$g(a) \;=\; \sum_{i=0}^{p-1}\partial_i g(c)(a-c)^i\ .$$

By Lemma 2.4 there is $c\in K$ such that among the values $v\partial_i g(c)(a-c)^i$, $0\leq i\leq p-1$, there is precisely one of minimal value, and the same holds for all $c'\in K$ with $v(a-c')\geq v(a-c)$. In particular, we may assume that $v(a-c)>va$. For all such c, we have:

$$vg(a) \;=\; \min_{0\leq i\leq p-1} v\partial_i g(c)(a-c)^i\ .$$

Hence for $g(a)$ to lie in $\mathcal{M}_{K(a)}$ it is necessary that $v\partial_i g(c)(a-c)^i>0$, or equivalently,

$$(54)\qquad v\partial_i g(c) \;>\; -iv(a-c)$$

for $0 \leq i \leq p-1$ and $c \in K$ as above.

In the Artin-Schreier extension case, for $g(\vartheta) \in \mathcal{M}_{K(\vartheta)}$ and $c \in K$ as above, using (50) we find:

$$\mathrm{Tr}_{K(\vartheta)|K}(g(\vartheta)) \;=\; \sum_{i=0}^{p-1} \mathrm{Tr}_{K(\vartheta)|K}(\partial_i g(c)(\vartheta - c)^i) \;=\; -\partial_{p-1} g(c)\,.$$

Since $\partial_{p-1} g(c) > -(p-1)v(\vartheta - c)$ by (54), this proves the desired equality in (51).

In the Kummer extension case, for $g(\eta) \in \mathcal{M}_{K(\eta)}$ and $c \in K$ as above, using (52) we find:

$$\mathrm{Tr}_{K(\eta)|K}(g(\eta)) \;=\; \sum_{j=0}^{p-1} \mathrm{Tr}_{K(\eta)|K}(\partial_j g(c)(\eta - c)^j) \;=\; p\sum_{j=0}^{p-1} \partial_j g(c)(-c)^j\,.$$

As we assume that $v(\eta - c) > 0$, we have that $vc = 0$ and

$$-iv(\eta - c) \geq -(p-1)v(\eta - c) \quad \text{ for } \; 0 \leq i \leq p-1\,.$$

Hence by (54), $v\sum_{i=0}^{p-1} \partial_i g(c)(-c)^i \geq -(p-1)v(\eta - c)$. This proves the desired equality in (53).

Remark 5.2. In the case of a Kummer extension, it follows from (53) that $p\mathcal{O}_K \subseteq \mathrm{Tr}_{K(\eta)|K}\left(\mathcal{M}_{K(\eta)}\right)$ since $vp = vp - (p-1)v\eta$ as η is a unit.

5.2. **Proof of Theorem 1.5.**

We have just proved the second equality of (9) in the previous section; the third equality follows from part 1) of Lemma 2.7. The first equality is seen as follows. Take any $a \in \mathcal{O}_L$. As the defect extension $(L|K,v)$ is immediate, there is some $b \in \mathcal{O}_K$ such that $a - b \in \mathcal{M}_L$. Then $\mathrm{Tr}_{L|K}(a) = \mathrm{Tr}_{L|K}(b) + \mathrm{Tr}_{L|K}(a-b)$ with $\mathrm{Tr}_{L|K}(a-b) \in \mathrm{Tr}_{L|K}(\mathcal{M}_L)$. If $\mathrm{char}\, L = p$, then $\mathrm{Tr}_{L|K}(b) = 0$. If $\mathrm{char}\, L = 0$, then $\mathrm{Tr}_{L|K}(b) = pb \in p\mathcal{O}_K$, which is contained in $\mathrm{Tr}_{L|K}(\mathcal{M}_L)$ as stated in Remark 5.2.

Assume that the extension $\mathcal{E}$ has independent defect. Then by part 1) of Proposition 2.10, $I_{\mathcal{E}}^p = I_{\mathcal{E}}$ as well as $(I_{\mathcal{E}} \cap K)^p = I_{\mathcal{E}} \cap K$. This implies that $I_{\mathcal{E}}^{p-1} = I_{\mathcal{E}}$ and $(I_{\mathcal{E}} \cap K)^{p-1} = I_{\mathcal{E}} \cap K$. On the other hand, we know from the equivalence of statements a) and b) in Theorem 1.4 that $I_{\mathcal{E}} = \mathcal{M}_{v_H}$, where H is a strongly convex subgroup of vL, and that H is equal to the convex subgroup $H_{\mathcal{E}}$ appearing in the definition of independent defect. Now the second equation of (10) follows from (9) as $(I_{\mathcal{E}} \cap K)^{p-1} = I_{\mathcal{E}} \cap K = \mathcal{M}_{v_H} \cap K$. The third equation of (10) holds since $\mathcal{M}_{v_H} = (a \in L \mid va > H)$.

Now assume that (10) holds for some strongly convex subgroup H of $vL = vK$. Set $\Sigma := vL \setminus H = vK \setminus H$. From (9) we now obtain:

$$(b \in K \mid vb \in (p-1)\Sigma_{\mathcal{E}}) \;=\; (b \in K \mid vb \in \Sigma)\,.$$

On the one hand, we have

$$(b \in K \mid vb \in (p-1)\Sigma_{\mathcal{E}}) \;=\; (b \in K \mid vb \in ((p-1)\Sigma_{\mathcal{E}})^-)\,,$$

and on the other,

$$(b \in K \mid vb \in \Sigma) \;=\; (b \in K \mid vb \in ((p-1)\Sigma)^-)$$

by Lemma 2.1. This implies that $((p-1)\Sigma_{\mathcal{E}})^- = ((p-1)\Sigma)^-$. Suppose that $\Sigma_{\mathcal{E}} \neq \Sigma$. If there is $\alpha \in \Sigma_{\mathcal{E}} \setminus \Sigma$, then $\alpha < \Sigma$, whence $(p-1)\alpha < (p-1)\Sigma$ and therefore, $((p-1)\Sigma_{\mathcal{E}})^- \supsetneq ((p-1)\Sigma)^-$, contradiction. Symmetrically, we obtain a contradiction if there is $\alpha \in \Sigma \setminus \Sigma_{\mathcal{E}}$. This shows that $\Sigma = \Sigma_{\mathcal{E}}$, and we thus obtain that $\mathcal{E}$ has independent defect and that $H = H_{\mathcal{E}}$.

The last statement of the theorem holds since $vp \notin H$ by part 2) of Proposition 2.10. If $(vK)_{vp}$ is archimedean, this forces $H = \{0\}$. □

5.3. **Dedekind differents.**

In this section, we compute the Dedekind differents $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$ in the case where $vL = vK$. We will employ the following auxiliary results.

Lemma 5.3. *Take any $\mathcal{O}_L$-fractional ideal I.*

1) If vI has an infimum α in vL, then $\mathcal{O}_L :_L I = (a \in L \mid va \geq -\alpha)$. If vI has no infimum in vL, then $\mathcal{O}_L :_L I = (a \in L \mid va > -vI)$, which is not a principal ideal, and $v(\mathcal{O}_L :_L I)$ has no infimum.

2) If vI has an infimum α in vL, then $\mathcal{O}_L :_L (\mathcal{O}_L :_L I) = (a \in L \mid va \geq \alpha)$. If vI has no infimum in vL, then $\mathcal{O}_L :_L (\mathcal{O}_L :_L I) = I$.

Proof. 1): We compute:

$$\begin{aligned} \mathcal{O}_L :_L I &= (a \in L \mid aI \subseteq \mathcal{O}_L) = (a \in L \mid va + \beta \geq 0 \text{ for all } \beta \in vI) \\ &= (a \in L \mid va \geq -vI) . \end{aligned}$$

Since α is an infimum of vI if and only if $-\alpha$ is a supremum of $-vI$, this yields our assertions.

2): This follows by applying part 1) twice. □

Denote by $\mathcal{T}(\mathcal{O}_L|\mathcal{O}_K)$ the $\mathcal{O}_L$-ideal generated by $\mathrm{Tr}_{L|K}(\mathcal{O}_L)$. We use the previous lemma to show:

Theorem 5.4. *Assume that $(L|K,v)$ is a finite unibranched extension with $vL = vK$. Then we have:*

1) $\mathcal{C}(\mathcal{O}_L|\mathcal{O}_K) = \mathcal{O}_L :_L \mathcal{T}(\mathcal{O}_L|\mathcal{O}_K)$ and $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \mathcal{O}_L :_L (\mathcal{O}_L :_L \mathcal{T}(\mathcal{O}_L|\mathcal{O}_K))$.

2) If $v\mathrm{Tr}_{L|K}(\mathcal{O}_L)$ has an infimum α in vK, then $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = (a \in L \mid va \geq \alpha)$ and if $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) \neq \mathcal{T}(\mathcal{O}_L|\mathcal{O}_K)$, then

$$\mathcal{M}_L\,\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \mathcal{T}(\mathcal{O}_L|\mathcal{O}_K) . \tag{55}$$

If $v\mathrm{Tr}_{L|K}(\mathcal{O}_L)$ has no infimum in vK, then $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \mathcal{T}(\mathcal{O}_L|\mathcal{O}_K)$, which is not principal.

Proof. 1): Take any $z \in L$. Since $vL = vK$, there is $z_1 \in K$ such that $vz_1 = vz$, so that $z = z_1 z_2$ with z_2 a unit in $\mathcal{O}_L$. Then

$$\mathrm{Tr}_{L|K}(z\mathcal{O}_L) = z_1 \mathrm{Tr}_{L|K}(z_2\mathcal{O}_L) = z_1 \mathrm{Tr}_{L|K}(\mathcal{O}_L) .$$

Observe that $z_1 \mathrm{Tr}_{L|K}(\mathcal{O}_L) \subseteq \mathcal{O}_K$ if and only if $z\mathcal{T}(\mathcal{O}_L|\mathcal{O}_K) \subseteq \mathcal{O}_L$. This proves the assertions of part 1).

2): A part of the assertions follow from part 1) together with part 2) of Lemma 5.3; it only remains to prove (55) in case $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$ properly contains $\mathcal{T}(\mathcal{O}_L|\mathcal{O}_K)$. In this case, $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = (a \in L \mid va \geq \alpha)$ and $\mathcal{T}(\mathcal{O}_L|\mathcal{O}_K) = (a \in L \mid va > \alpha) = v\mathcal{M}_L + (a \in L \mid va \geq \alpha)$, which proves (55). □

Proof of Theorem 1.6:

Lemmas 3.6 and 3.7 together with Proposition 4.4 show that $I_{\mathcal{E}}^{p-1}$ is equal to the $\mathcal{O}_L$-ideal generated by the differents of all elements of $\mathcal{O}_L \setminus \mathcal{O}_K$. From Theorem 1.5 we know that $\mathrm{Tr}_{\,L|K}(\mathcal{O}_L) = (I_{\mathcal{E}} \cap K)^{p-1}$, so $\mathcal{T}(\mathcal{O}_L|\mathcal{O}_K) = I_{\mathcal{E}}^{p-1}$, whence $v\mathrm{Tr}_{\,L|K}(\mathcal{O}_L) = v\mathcal{T}(\mathcal{O}_L|\mathcal{O}_K) = vI_{\mathcal{E}}^{p-1}$. If this has no infimum in $vK = vL$, then from Theorem 5.4 we obtain that $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \mathcal{T}(\mathcal{O}_L|\mathcal{O}_K) = I_{\mathcal{E}}^{p-1}$. As the extension $(L|K, v)$ is immediate, $vI_{\mathcal{E}}^{p-1}$ has no smallest element. Hence if $vI_{\mathcal{E}}^{p-1}$ has an infimum α, it does not contain it, and again by Theorem 5.4, $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = (a \in L \mid va \geq \alpha) \supsetneq I_{\mathcal{E}}^{p-1}$ and $\mathcal{M}_L\,\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \mathcal{T}(\mathcal{O}_L|\mathcal{O}_K) = I_{\mathcal{E}}^{p-1}$.

Assume that (K, v) has rank 1. Then the only proper convex subgroup of vL is $\{0\}$. Thus Proposition 4.1 shows that $\mathrm{ann}\,\Omega_{\mathcal{O}_L|\mathcal{O}_K} = I_{\mathcal{E}}^{p-1}$ if $vI_{\mathcal{E}}^{p-1}$ has no infimum in vL, and $\mathrm{ann}\,\Omega_{\mathcal{O}_L|\mathcal{O}_K} = (a \in L \mid va \geq \alpha)$ if $vI_{\mathcal{E}}^{p-1}$ has infimum α. From what we have already shown, it follows that $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \mathrm{ann}\,\Omega_{\mathcal{O}_L|\mathcal{O}_K}$.

Assume that the extension $\mathcal{E}$ has independent defect; then $\Omega_{\mathcal{O}_L|\mathcal{O}_K} = 0$ by Theorem 1.4 and therefore, $\mathrm{ann}\,\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{O}_L$. If $H_{\mathcal{E}} = \{0\}$, then $\mathrm{Tr}_{\,L|K}(\mathcal{O}_L) = \mathrm{Tr}_{\,L|K}(\mathcal{M}_L) = \mathcal{M}_K$ by Theorem 1.5, hence $I_{\mathcal{E}}^{p-1} = \mathcal{T}(\mathcal{O}_L|\mathcal{O}_K) = \mathcal{M}_L$. Since $v\mathcal{M}_K$ has infimum 0, it follows from part 2) of Theorem 5.4 that $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \mathcal{O}_L$. Again, Proposition 4.1 shows that $\mathrm{ann}\,\Omega_{\mathcal{O}_L|\mathcal{O}_K} = \mathcal{O}_L$.

On the other hand, if $H_{\mathcal{E}} \neq \{0\}$, then again by Theorem 1.4, $\mathrm{Tr}_{\,L|K}(\mathcal{O}_L) = \mathrm{Tr}_{\,L|K}(\mathcal{M}_L) = \mathcal{M}_{v_{H_{\mathcal{E}}}} \cap K$, and as $v\mathcal{M}_{v_{H_{\mathcal{E}}}} \cap K$ has no infimum in vK, we have $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K) = \mathcal{T}(\mathcal{O}_L|\mathcal{O}_K) = \mathcal{M}_{v_{H_{\mathcal{E}}}} \subsetneq \mathcal{O}_L = \mathrm{ann}\,\Omega_{\mathcal{O}_L|\mathcal{O}_K}$ in this case.

Finally, assume that $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$ is equal to $\mathcal{O}_L$ or to $\mathcal{M}_{v_H}$ for a strongly convex nontrivial subgroup H of vL. If the former holds, then $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$ is not equal to $I_{\mathcal{E}}^{p-1}$, so from what we have already proved it follows that 0 is the infimum of $vI_{\mathcal{E}}^{p-1}$. This implies that $I_{\mathcal{E}} = \mathcal{M}_L = \mathcal{M}_{v_H}$ where $H = \{0\}$. Since $v\mathcal{M}_L = vI_{\mathcal{E}}$ has no smallest element, it follows that $\{0\}$ is a strongly convex subgroup of vL.

If the latter holds, then $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$ is not principal and therefore by what we have shown, $\mathcal{D}(\mathcal{O}_L|\mathcal{O}_K)$ can only be equal to $I_{\mathcal{E}}^{p-1}$. This implies that $I_{\mathcal{E}} = \mathcal{M}_{v_H}$ for some strongly convex subgroup of vL. In both cases, it follows from from the equivalence a)⇔b) of Theorem 1.4 that $\mathcal{E}$ has independent defect. □

6. Proof of Theorems 1.7 and 1.8

Take a defect extension $\mathcal{E} = (L|K, v)$ of degree p. Then by the fundamental inequality, the extension is unibranched and immediate, and in particular, $vL = vK$.

We first prove Theorem 1.7, so we assume that $\mathcal{E}$ is an Artin-Schreier extension generated by $\vartheta \in L$ with $\vartheta^p - \vartheta = a \in K$. By Corollary 2.9, $v(\vartheta - c) < 0$ for all $c \in K$. It follows that $v(\vartheta^p - c^p) = v(\vartheta - c)^p = pv(\vartheta - c) < v(\vartheta - c)$ and thus,

$$(56)\quad v(a - \wp(c)) \;=\; v(\vartheta^p - \vartheta - c^p + c) \;=\; \min\{v(\vartheta^p - c^p), v(\vartheta - c)\} \;=\; pv(\vartheta - c)\,.$$

This proves equation (11).

The equivalence a)⇔b) follows from the definition of independent defect together with equation (23) of Theorem 2.8 which says that $\Sigma_{\mathcal{E}} = -v(\vartheta - K)$. The equivalences b)⇔c) and e)⇔f) follow from equation (11). The equivalence c)⇔d) is trivial.

Now assume that vK is p-divisible. Then the equivalence a)⇔e) follows from part 1) of Proposition 2.10 together with the equation $\Sigma_{\mathcal{E}} = -v(\vartheta - K)$.

Finally, assume that (K, v) has rank 1. Then the only proper convex subgroup of vK is $H = \{0\}$, so that condition d) is equivalent to condition g).

Now we prove Theorem 1.8, so we assume that $\mathcal{E}$ is a Kummer extension. As explained at the beginning of Section 2.8, it is generated by a 1-unit $\eta \in L$ with $\eta^p = a \in K$. By Corollary 2.9, $v(\eta - c) < \frac{1}{p-1} vp$ for all $c \in K$.

The following is part of Lemma 2.18 of [20]:

Lemma 6.1. *Take $\eta \in \tilde{K}$ such that $\eta^p \in K$ and $v\eta = 0$. Then for $c \in K$ such that $v(\eta - c) > 0$, $v(\eta - c) < \frac{1}{p-1} vp$ holds if and only if $v(\eta^p - c^p) < \frac{p}{p-1} vp$, and if this is the case, then $v(\eta^p - c^p) = pv(\eta - c)$.*

In order to prove equation (12), we have to discuss the remaining case of $v(\eta - c) \leq 0$. If $v(\eta - c) = 0$, then $vc \geq 0$ and c is not a 1-unit. Hence c^p is not a 1-unit while $a = \eta^p$ is, whence $v(a - c^p) = 0 = pv(\eta - c)$. If $v(\eta - c) < 0$, then $vc < 0$ and $v(\eta - c) = vc$. It follows that $vc^p < 0 = v\eta^p$, so that $v(\eta^p - c^p) = vc^p = pvc = pv(\eta - c)$. This completes the proof of equation (12).

The equivalence a)⇔b) follows from the definition of independent defect together with equation (24) of Theorem 2.8, which says that $\Sigma_{\mathcal{E}} = \frac{1}{p-1} vp - v(\vartheta - K)$. The equivalences b)⇔c) and e)⇔f) follow from equation (12) and the fact that $\alpha < H$ if and only if $p\alpha < H$ since H is a convex subgroup of vK. The equivalence c)⇔d) is trivial.

Now assume that vK is p-divisible. Then by part 1) of Proposition 2.10 together with the equation $\Sigma_{\mathcal{E}} = \frac{1}{p-1} vp - v(\vartheta - K)$, condition a) is equivalent to the equation

$$\frac{p}{p-1} vp - pv(\eta - K) \;=\; \frac{1}{p-1} vp - v(\eta - K)\,.$$

Multiplying both sides with -1 and then adding $\frac{p}{p-1} vp$ to both sides, we find that this equation is equivalent to condition e).

Finally, if (K, v) has rank 1, then it follows as in the proof of Theorem 1.7 that condition d) is equivalent to condition g).

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