A NOTE ON OPERATORS EXTENDING PARTIAL ULTRAMETRICS

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ABSTRACT. We consider the question of simultaneous extension of partial ultrametrics, i.e. continuous ultrametrics defined on nonempty closed subsets of a compact zero-dimenional metrizable space. The main result states that there exists a continuous extension operator that preserves the maximum operation. This extension can also be chosen so that it preserves the Assouad dimension.

1. INTRODUCTION

C. Bessaga [6, 7] formulated a general problem of linear extensions of continuous (pseudo)metrics defined on a closed subset of a metrizable space and gave a partial solution of it. A complete solution was first obtained by T. Banakh [3, 4]; see also [5, 17, 21] for related results.

Recently, the authors [19] considered a problem of simultaneous linear extension of metrics with variable domains in a compact space. The obtained result on existence of linear extension operators is in some sense parallel to the corresponding result due to Künzi and Shapiro [10] on simultaneous linear extensions of partial functions. In the present paper we consider a problem of simultaneous extension of partial ultrametrics, i.e. ultrametrics defined on the nonempty closed subsets of a zero-dimensional compact metrizable space.

Recall that a metric ϱ on a set X is called an *ultrametric* (or *non-Archimedean metric*) if $\varrho(x,y) \le \max\{\varrho(x,z), \varrho(z,y)\}$ for all $x, y, z \in X$. It is well-known (see, e.g. [9]) that a metrizable space X admits an ultrametric compatible with its topology if and only if $\dim X = 0$. Obviously, in the case of ultrametrics, one cannot speak about linear extension operators, because the set of all ultrametrics is not, in general, closed with respect to linear operations (the sum of two ultrametrics need not be an ultrametric). However, the set of all ultrametrics is closed under the operation of pointwise maximum.

Identifying every ultrametric with its graph, one can topologize the set of all partial ultrametrics with the hyperspace topology. We show that there exists a continuous extension operator of partial ultrametrics that preserves the operation of maximum of two ultrametrics. Besides, the constructed operators preserve the so-called Assouad

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dimension of the ultrametric spaces. The results on bi-Lipschitz embeddings of ultrametric spaces [13, 15] allow us to derive from this that the extended ultrametric space bi-Lipschitzely embeds into $\mathbb{R}^n$ if so does the initial ultrametric space.

In this note we focus on the compact case only, as the general case requires completely different approach.

2. Preliminaries

2.1. **Space of partial ultrametrics.** By $\exp X$ we denote the *hyperspace* of X, i. e. the set of all nonempty compact subsets of X endowed with the Vietoris topology. A base of this topology consists of the sets of the form

$$\langle U_1,\dots,U_k\rangle = \{A\in \exp X \mid A\subset \cup_{i=1}^k U_i,\ A\cap U_i\neq\emptyset \text{ for all } i\},$$

where $U_1,\dots,U_k$ run over the family of open subsets in X.

If d is a compatible metric on X, then the Vietoris topology is generated by the Hausdorff metric d_H,

$$d_H(A,B)=\inf\{\varepsilon>0 \mid A\subset O_\varepsilon(B),\ B\subset O_\varepsilon(A)\}.$$

Given a nonempty compact subset A of X, we denote by $\mathcal{UM}(A)$ the set of continuous ultrametrics on A. Set

$$\mathcal{UM}=\cup\{\mathcal{UM}(A)\mid A\in\exp X\}.$$

Identifying every ultrametric $d\in\mathcal{UM}$ with its graph, which is a compact subset of $X\times X\times\mathbb{R}$, we consider the set $\mathcal{UM}$ as a subset of $\exp(X\times X\times\mathbb{R})$ and endow $\mathcal{UM}$ with the subspace topology. Note that such a topologization of functional spaces traces back to Kuratowski [11, 12] and is extensively used in the topological theory of differential equations (see, e.g. [8]).

If $\varrho\in\mathcal{UM}$, then dom $\varrho=A$ means $\varrho\in\mathcal{UM}(A)$. Note that the map dom : $\mathcal{UM}\to\exp X$, being the restriction of the projection onto the first coordinate, is continuous. For $\varrho\in\mathcal{UM}$ let $\|\varrho\|=\max\{\varrho(x,y)\mid x,y\in \text{dom } \varrho\}$.

For every $A\in\exp X$ the set $\mathcal{UM}(A)$ is a continuous $\vee$-semilattice with respect to the operation $\varrho\vee\varrho'=\max\{\varrho,\varrho'\}$. Also, the set $\mathcal{UM}$ is closed under pointwise multiplication by positive numbers.

2.2. **Assouad dimension.** Let $c,s\geq 0$. We say that a metric space (X,ϱ) is (c,s)-*homogeneous* if the inequality $|X_0|\leq c(b/a)^s$ holds for $a>0$, $b>0$ and $X_0\subset X$ provided that $b\geq a$ and that $a\leq\varrho(x,y)\leq b$ holds for every pair of distinct points x and y of X_0.

The space (X,ϱ) is s-*homogeneous* if it is (c,s)-homogeneous for some $c\geq 0$.

The *Assouad dimension*, $\dim_A(X,\varrho)$ of a metric space (X,ϱ) is defined as follows:

$$\dim_A(X,\varrho)=\inf\{s\geq 0\mid (X,\varrho) \text{ is } s-\text{homogeneous}\}$$

(see [1, 2]).

Proposition 2.1. *Let (X,ϱ) be a metric space and $\varrho=\varrho_1\vee\varrho_2$. Then* $\dim_A(X,\varrho)\le\dim_A(X,\varrho_1)+\dim_A(X,\varrho_2)$.

Proof. Let m denotes the max-metric on $(X,\varrho_1)\times(X,\varrho_2)$, i.e. $m((x,y),(x',y'))=\max\{\varrho_1(x,x'),\varrho_2(y,y')\}$. It is proved in [14] (see Theorem 5.A therein) that $\dim_A(X\times X,m)\le\dim_A(X,\varrho_1)+\dim_A(X,\varrho_2)$. Since the diagonal map $\Delta\colon(X,\varrho)\to(X\times X,m)$ is an isometric embedding and the Assouad dimension is monotonic, the result follows. □

Proposition 2.2. *Let (X,ϱ) be a compact metric space and $c>0$. For the truncated metric ϱ', $\varrho'(x,y)=\min\{\varrho'(x,y),c\}$ we have* $\dim_A(X,\varrho)=\dim_A(X,\varrho')$.

Proof. The result follows from the fact that (X,ϱ) and (X,ϱ') are Lipschitz equivalent and Theorem 5.A. (1) in [14]. □

Proposition 2.3. *There exists an ultrametric d on the Cantor set C with the following properties:*

(i) *d takes only binary rational values;*
(ii) $\dim_A(C,d)=0$.

Proof. Identify C with the set $2^{\mathbb{N}}$ and define d by the formula

$$d((x_i),(y_i))=\sup\{2^{-2^j}\mid x_j\ne y_j\},\ (x_i),(y_i)\in 2^{\mathbb{N}},\ (x_i)\ne(y_i).$$

Obviously, d is an ultrametric on C that takes only binary-rational values.

First, we show that $\dim_A(C,d)=0$. Let $s>0$. We are going to show that C is s-homogeneous.

Given a,b, $0<a\le b$, find the minimal natural number n and the maximal natural number m such that $a\le 2^{-2^m}\le 2^{-2^n}\le b$ (without loss of generality we may suppose that such m,n exist). Suppose that $X_0\subset C$ has the property that $a\le d(x,y)\le b$, for every $x,y\in X_0$, $x\ne y$. Then

$$2^{-2^m}\le d(x,y)\le 2^{-2^n} \tag{2.1}$$

for every $x,y\in X_0$, $x\ne y$. Suppose that $x=(x_i)\in X_0$. For arbitrary $y=(y_i)\in X_0$, $x\ne y$, it easily follows from condition (2.1) and the definition of the metric d, that

$$\{i\mid x_i\ne y_i\}\cap\{m,m+1,\dots,n\}\ne\emptyset.$$

Therefore, $|X_0|\le 2^{n-m+1}$.

There exists $N\in\mathbb{N}$ such that for every $p>N$ we have $p\le 2^{p-1}s$. Let $c=2^{N+1}$. If $n=m$, then $|X_0|=2\le c(b/a)^s$.

Suppose now that $n>m$. If $n\le N$, then $|X_0|\le 2^{N+1}\le c(b/a)^s$.

If $n>N$, then

$$\begin{aligned}\log_2(c(b/a)^s)&\ge N+s\log_2(b/a)\ge N+s(2^n-2^m)\ge N+s2^{n-1}\\&\ge n\ge n-m+1\ge\log_2|X_0|\end{aligned}$$

i.e. $|X_0|\le c(b/a)^s$.

Since C is s-homogeneous for every $s>0$, we conclude that $\dim_A C=0$. □

3. Extension of partial ultrametrics

The following is the main result of this note.

Theorem 3.1. *There exists a map* $u\colon \mathcal{UM} \to \mathcal{UM}(X)$ *that satisfies the following properties for every* $\varrho, \varrho' \in \mathcal{UM}$*:*

(1) u *is continuous;*
(2) $\|u(\varrho)\| = \|\varrho\|$*;*
(3) $u(\varrho)$ *is an extension of* ϱ *for every* $\varrho \in \mathcal{UM}$*;*
(4) $u(\varrho \vee \varrho') = u(\varrho) \vee u(\varrho')$ *and* $u(c\varrho) = cu(\varrho)$ *if* $\mathrm{dom}\ \varrho = \mathrm{dom}\ \varrho'$ *and* $c > 0$.
(5) *if* $\varrho \in \mathcal{UM}$ *takes only (binary) rational values then so does* $u(\varrho)$.
(6) $\dim_A(X, u(\varrho)) = \dim_A(\mathrm{dom}\ \varrho, \varrho)$.

Proof. The set $K = \{(x, A) \in X \times \exp X \mid x \in A\}$ is closed in the space $X \times \exp X$. Denote by C the standard Cantor set. The space $X \times \exp X$ is a zero-dimensional compact metrizable space and therefore there exists a continuous map $f\colon X \times \exp X \to C$ such that $f(K) = \{0\}$ and $f|((X \times \exp X) \setminus K)$ is an embedding. Define a multivalued map $F\colon X \times \exp X \to X$ by the formula

$$F(x, A) = \begin{cases} A & \text{if } x \notin A, \\ \{x\} & \text{if } x \in A. \end{cases}$$

We show that the map F is lower semicontinuous, i.e. the set $U^\sharp = \{(x, A) \in X \times \exp X \mid F(x, A) \cap U \neq \emptyset\}$ is open for every open subset U of X. Let $(x_0, A_0) \in U^\sharp$.

Case 1). $x_0 \notin A_0$. Then $F(x_0, A_0) = A_0$ and $A_0 \cap U \neq \emptyset$. There exist disjoint neighborhoods V of x_0 and W of A in X respectively. Then for every $(x, A) \in \langle W, W \cap U\rangle$ we have $x \notin A$ and thus $F(x, A) = A$. Since $A \cap (W \cap U) \neq \emptyset$, we see that $(x, A) \in U^\sharp$.

Case 2). $x_0 \in A_0$. Then $F(x_0, A_0) = \{x_0\}$ and $x_0 \in U$. Obviously, $U \times \langle X, U\rangle$ is a neighborhood of (x_0, A_0) and for every $(x, A) \in U \times \langle X, U\rangle$ we have $F(x, A) \cap U \neq \emptyset$, i.e. $(x, A) \in U^\sharp$.

Since the space X is zero-dimensional, so is $\exp X$ and therefore $X \times \exp X$. We can apply the zero-dimensional Michael Selection Theorem [16] to find a continuous selection of F, i.e. a continuous map $g\colon X \times \exp X \to X$ such that $g(x, A) \in A$, for every $(x, A) \in X \times \exp X$.

Let d be an ultrametric on C generating its topology. We may suppose, by Proposition 2.3, that d takes only binary rational values and $\dim_A(C, d) = 0$.

Define the map $u\colon \mathcal{UM} \to \mathcal{UM}(X)$ by the formula

$$\begin{aligned} u(\varrho)(x, y) = \max\{&\varrho(g(x, \mathrm{dom}\ \varrho), g(y, \mathrm{dom}\ \varrho)),\\ &\min\{d(f(x, \mathrm{dom}\ \varrho), f(y, \mathrm{dom}\ \varrho)), \|\varrho\|\}\} \end{aligned} \tag{3.1}$$

for all $x, y \in X$.

It is easy to see that $u(\varrho)$ is a continuous ultrametric on X for every ultrametric $\varrho \in \mathcal{UM}$.

We are going to verify properties (1)–(6).

(1) Show that the map u is continuous. Let (ϱ_n) be a sequence in $\mathcal{UM}$ that converges to $\varrho \in \mathcal{UM}$. Then, obviously, dom $\varrho_n \to$ dom ϱ. As we have remarked, there exists a continuous ultrametric $\tilde{\varrho}$ on X that extends ϱ over $X \times X$ (take, e.g., $\tilde{\varrho} = u(\varrho)$). Let $\tilde{\varrho}_n = \tilde{\varrho}|(\text{dom } \varrho_n \times \text{dom } \varrho_n)$. Arguing like in the proof of Lemma 3 in [10], we can show that $\tilde{\varrho}_n \to \varrho$ in $\mathcal{UM}$.

Fix $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that for every $n \geq N$ we have

a) $\|\varrho_n - \tilde{\varrho}_n\| < \varepsilon/2$;
b) $|d(f(x, \text{dom } \varrho), f(y, \text{dom } \varrho)) - d(f(x, \text{dom } \varrho_n), f(y, \text{dom } \varrho_n))| < \varepsilon$ for every $x, y \in X \times X$ (this is a consequence of uniform continuity of the maps f and d and compactness of X);
c) $|\varrho(g(x, \text{dom } \varrho), g(y, \text{dom } \varrho)) - \tilde{\varrho}(g(x, \text{dom } \varrho_n), g(y, \text{dom } \varrho_n))| < \varepsilon/2$ for every $x, y \in X \times X$.

Then for every $n \geq N$ we have

$$\begin{aligned} &|\varrho(g(x, \text{dom } \varrho), g(y, \text{dom } \varrho)) - \varrho_n(g(x, \text{dom } \varrho_n), g(y, \text{dom } \varrho_n))| \\ \leq &|\varrho(g(x, \text{dom } \varrho), g(y, \text{dom } \varrho)) - \tilde{\varrho}(g(x, \text{dom } \varrho_n), g(y, \text{dom } \varrho_n))| \\ + &|\tilde{\varrho}(g(x, \text{dom } \varrho_n), g(y, \text{dom } \varrho_n)) - \varrho_n(g(x, \text{dom } \varrho_n), g(y, \text{dom } \varrho_n))| \\ \leq &\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned} \tag{3.2}$$

for every $x, y \in X \times X$.

Inequalities b) and (3.2) together imply $|u(\varrho)(x, y) - u(\varrho_n)(x, y)| < \varepsilon$ for every $n \geq N$ and every $x, y \in X \times X$. This means that the sequence $(u(\varrho_n))$ converges to $u(\varrho)$ in $\mathcal{UM}$. Because of arbitrarity of (ϱ_n), the map u is continuous.

(2) Let $c = \|\varrho\|$. Then

$$u(\varrho)(x, y) \leq \max\{c, \min\{d(f(x, \text{dom } \varrho), f(y, \text{dom } \varrho)), c\}\} = c$$

for every $x, y \in X$, i.e. $\|u(\varrho)\| \leq c$.

(3) If $x, y \in \text{dom } \varrho$, then $f(x) = f(y)$, $g(x, \text{dom } \varrho) = x$, $g(y, \text{dom } \varrho) = y$ and hence $u(\varrho)(x, y) = \varrho(x, y)$, i.e. u is an extension operator.

(4) If $\varrho = \varrho_1 \vee \varrho_2$, then dom $\varrho_1 =$ dom ϱ_2 and so

$$
\begin{aligned}
(u(\varrho_1)\vee u(\varrho_2))(x,y) = \max\big\{ & \max\{\varrho_1(g(x,\mathrm{dom}\ \varrho_1), g(y,\mathrm{dom}\ \varrho_1)),\\
& \min\{d(f(x,\mathrm{dom}\ \varrho_1), f(y,\mathrm{dom}\ \varrho_1)), \|\varrho_1\|\}\},\\
& \{\max\{\varrho_2(g(x,\mathrm{dom}\ \varrho_2), g(y,\mathrm{dom}\ \varrho_2)), \min\{d(f(x,\mathrm{dom}\ \varrho_2), f(y,\mathrm{dom}\ \varrho_2)), \|\varrho_2\|\}\}\big\}\\
= \max\big\{ & \varrho_1(g(x,\mathrm{dom}\ \varrho_1), g(y,\mathrm{dom}\ \varrho_1)), \varrho_2(g(x,\mathrm{dom}\ \varrho_2), g(y,\mathrm{dom}\ \varrho_2)),\\
& \min\{d(f(x,\mathrm{dom}\ \varrho_1), f(y,\mathrm{dom}\ \varrho_1)), \|\varrho_1\|\},\\
& \min\{d(f(x,\mathrm{dom}\ \varrho_2), f(y,\mathrm{dom}\ \varrho_2)), \|\varrho_2\|\}\big\}\\
= \max\{ & (\varrho_1 \vee \varrho_2)(g(x,\mathrm{dom}\ \varrho_1), g(y,\mathrm{dom}\ \varrho_1)), \min\{d(f(x,\mathrm{dom}\ \varrho_1), f(y,\mathrm{dom}\ \varrho_1)),\\
& \max\{\|\varrho_1\|, \|\varrho_2\|\}\}\}\\
= \max\big\{ & (\varrho_1 \vee \varrho_2)(g(x,\mathrm{dom}\ \varrho_1), (g(y,\mathrm{dom}\ \varrho_1)), \min\{d(f(x,\mathrm{dom}\ \varrho_1), f(y,\mathrm{dom}\ \varrho_1)),\\
& \|\varrho_1 \vee \varrho_2\|\}\big\}\\
= & u(\varrho_1 \vee \varrho_2)(x,y),
\end{aligned}
$$

i.e. u is a homomorphism of $\vee$-semilattices. It easily follows from (3.1) that $u(c\varrho) = cu(\varrho)$.

(5) Follows from formula (3.1).

(6) It follows from Propositions 2.1, 2.2, formula (3.1), and the choice of (C,d) that

$$\dim_A(X, u(\varrho)) \le \dim_A(\mathrm{dom}\ \varrho, \varrho) + \dim_A(C,d) = \dim_A(\mathrm{dom}\ \varrho, \varrho).$$

□

Corollary 3.2. *The operator u from Theorem 3.1 has the following property: if $\varrho \in \mathcal{UM}$ and the space* (dom ϱ, ϱ) *can be bi-Lipschitzely embedded into the Euclidean space $\mathbb{R}^n$, for some n, then $(X, u(\varrho))$ can also be bi-Lipschitzely embedded into $\mathbb{R}^n$.*

Proof. It is proved in [13] that if an ultrametric space can be bi-Lipschitz embedded in $\mathbb{R}^n$, then its Assouad dimension is less than n. Since $\dim_A(X, u(\varrho)) = \dim_A(\mathrm{dom}\ \varrho, \varrho) < n$, it follows from [15, Theorem 3.8] that $(X, u(\varrho))$ can also be bi-Lipschitzely embedded into $\mathbb{R}^n$. □

4. Remarks and open questions

4.1. **Generalized ultrametric spaces.** One can consider an extension problem also for generalized ultrametric spaces.

Let $(\Gamma, \le)$ be a partially ordered set with smallest element, denoted by 0. Let X be a non-empty set and $d : X \times X \to \Gamma$ be a mapping, satisfying the following conditions:

1. $d(x,y) = 0$ if and only if $x = y$
2. $d(x,y) = d(y,x)$
3. If $d(x,y) \le \gamma$ and $d(x,z) \le \gamma$, then $d(y,z) \le \gamma$.

The pair (X, d, Γ) is then called a (generalized) *ultrametric space* [20]. We leave to the reader the precise formulation of the problem of extension for partial generalized ultrametrics.

4.2. **Extension of partial metrics with n-dimensional Nagata property.** A metric space (X, d) satisfies the *n-dimensional Nagata property* if for every $r > 0$, every $x \in X$, and every collection of elements $y_1, \dots, y_{n+2}$ of the set

$$\{y \in X \mid \text{ there exists } z \in X \text{ with } d(x,z) < r,\ d(y,z) < 2r\}$$

there exist i, j, $i \neq j$ such that $d(y_i, y_j) < 2r$. It is proved in [1] that a separable metric space X admits a compatible metric with n-dimensional Nagata property if and only if $\dim X \leq n$. For every $A \in \exp X$ denote by $\mathrm{P}_n\mathcal{M}(A)$ the set of all compatible metrics on A with n-dimensional Nagata property. Set $\mathrm{P}_n\mathcal{M} = \cup\{\mathrm{P}_n\mathcal{M}(A) \mid A \in \exp X\}$.

The set of metrics with 0-dimensional Nagata property is easily seen to coincide with the set of all ultrametrics. Therefore, the set $\mathrm{P}_0\mathcal{M}(A)$ is closed under the operation max. There is no counterpart of this property for the spaces $\mathrm{P}_n\mathcal{M}(A)$ if $n \geq 1$. However, the sets $\mathrm{P}_n\mathcal{M}(A)$ are closed under multiplication by positive real numbers.

Question 4.1. Let X be an n-dimensional compact metrizable space, $n \geq 1$. Is there a continuous (homogeneous) extension operator $\mathrm{P}_n\mathcal{M} \to \mathrm{P}_n\mathcal{M}(X)$?

A version of this question can be formulated about the existence of a continuous extension operator $\mathrm{P}_n\mathcal{M} \to \mathrm{P}_n\mathcal{M}(X)$ that preserves the partial order relation $\leq$ on $\mathrm{P}_n\mathcal{M}$.

4.3. **Non-metrizable case.** If we replace the axiom $\varrho(x,y) = 0 \Leftrightarrow x = y$ by $x = y \Rightarrow \varrho(x,y) = 0$, we obtain the notion of ultrapseudometric. One can similarly formulate the problem of simultaneous extension of partial ultrapseudometrics. Denote by $\mathcal{UPM}(A)$ the set of all continuous ultrapseudometrics defined on a nonempty closed subset A of a compact zero-dimensional Hausdorff space X. Let $\mathcal{UPM} = \cup\{\mathcal{UPM}(A) \mid A \in \exp X\}$. As in [19], one can prove the following result.

Theorem 4.2. *For a compact zero-dimensional Hausdorff space X the following are equivalent:*

(1) *there exists a continuous extension operator $u\colon \mathcal{UPM} \to \mathcal{UPM}(X)$;*
(2) *there exists a continuous map $\Psi\colon (X \times X) \setminus \Delta_X \to \mathcal{UPM}(X)$, $(x,y) \mapsto \Psi_{(x,y)}$, with $\Psi_{(x,y)}(x,y) \neq 0$ for all $(x,y) \in X^2 \setminus \Delta_X$;*
(3) *X is metrizable.*

Proof. Coincides with that of Theorem 6.1 from [19], in which, in turn, implication (2) $\Rightarrow$ (3) is based on a result of Stepanova [18] on extension of partial continuous functions. □

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