Chain union closures

Wiesław Kubiś*
Mathematical Institute, Czech Academy of Sciences, Czech Republic
Cardinal Stefan Wyszyński University, Warsaw, Poland

Franz-Viktor Kuhlmann†
Institute of Mathematics, University of Szczecin, Poland

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Abstract

We study spherical completeness of ball spaces and its stability under expansions. We give some criteria for ball spaces that guarantee that spherical completeness is preserved when the ball space is closed under unions of chains. This applies in particular to the spaces of closed ultrametric balls in ultrametric spaces with linearly ordered value sets, or more generally, with countable narrow value sets. Finally, we show that in general, chain union closures of ultrametric spaces with partially ordered value sets do not preserve spherical completeness.

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1 Introduction

In [3, 4, 5, 6, 7, 8], ball spaces are studied in order to provide a general framework for fixed point theorems that in some way or the other work with contractive functions. A *ball space* $(X, \mathscr{B})$ is a nonempty set X together with any nonempty collection of nonempty subsets of X. The completeness property necessary for the proof of fixed point theorems is then encoded as follows. A *chain of balls* (also called a *nest*) in $(X, \mathscr{B})$ is a nonempty subset of $\mathscr{B}$ which is linearly ordered by inclusion. A ball space $(X, \mathscr{B})$ is called *spherically complete* if every chain of balls has a nonempty

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intersection. Further, we say that a ball space $(X, \mathscr{B})$ is *chain union closed* if the union of every chain in $\mathscr{B}$ is a member of $\mathscr{B}$. We define $\mathrm{cu}(\mathscr{B})$ to be the family of all sets of the form $\bigcup \mathscr{C}$, where $\mathscr{C} \subseteq \mathscr{B}$ is a chain (recall that, by default, chains of sets are supposed to be nonempty). More formally,

$$\mathrm{cu}(\mathscr{B}) = \left\{ \bigcup \mathscr{C} \mid \emptyset \neq \mathscr{C} \subseteq \mathscr{B},\ \mathscr{C} \text{ is a chain} \right\}.$$

Hence a ball space $(X, \mathscr{B})$ is chain union closed if and only if $\mathrm{cu}(\mathscr{B}) = \mathscr{B}$. In the present paper, we study the process of obtaining a chain union closed ball space from a given ball space and the question under which conditions the spherical completeness of $(X, \mathscr{B})$ implies the spherical completeness of $(X, \mathrm{cu}(\mathscr{B}))$.

Main inspiration for these definitions and questions is taken from the theory of ultrametric spaces and their ultrametric balls. An *ultrametric* u on a set X is a function from $X \times X$ to a partially ordered set Γ with smallest element $\perp$, such that for all $x, y, z \in X$ and all $\gamma \in \Gamma$,

(U1) $u(x, y) = \perp$ if and only if $x = y$,

(U2) if $u(x, y) \leq \gamma$ and $u(y, z) \leq \gamma$, then $u(x, z) \leq \gamma$,

(U3) $u(x, y) = u(y, x)$ (symmetry).

Condition (U2) is the ultrametric triangle law; if Γ is linearly ordered, it can be replaced by

(UT) $u(x, z) \leq \max\{u(x, y), u(y, z)\}$.

A *closed ultrametric ball* is a set $B_\alpha(x) := \{y \in X \mid u(x, y) \leq \alpha\}$, where $x \in X$ and $\alpha \in \Gamma$. The problem with general ultrametric spaces is that closed balls $B_\alpha(x)$ are not necessarily precise, that is, there may not be any $y \in X$ such that $u(x, y) = \alpha$. Therefore, we prefer to work only with *precise* ultrametric balls, which we can write in the form

$$B(x, y) := \{z \in X \mid u(x, z) \leq u(x, y)\},$$

where $x, y \in X$. We obtain the *ultrametric ball space* $(X, \mathscr{B}_u)$ from (X, d) by taking $\mathscr{B}_u$ to be the set of all such balls $B(x, y)$. Specifically, $\mathscr{B}_u := \{B(x, y) \mid x, y \in X\}$.

More generally, an ultrametric ball is a set

$$B_S(x) := \{y \in X \mid u(x, y) \in S\},$$

where $x \in X$ and S is an initial segment of Γ. We call X together with the collection of all ultrametric balls the *full ultrametric ball space* of (X, d). Every ultrametric ball can be written as the union over a chain of precise balls:

$$B_S(x) = \bigcup \{B(x, y) \mid u(x, y) \in S\}.$$

Hence the full ultrametric ball space is just $(X, \mathrm{cu}(\mathscr{B}_u))$.

Theorem 1.1. *Let $(X, \mathscr{B}_u)$ be the ball space of an ultrametric space (X, d) with linearly ordered value set. Then the following assertions hold:*

1) The ball space $(X, \mathrm{cu}(\mathscr{B}_u))$ is chain union closed.

2) If $(X, \mathscr{B}_u)$ is spherically complete, then so is $(X, \mathrm{cu}(\mathscr{B}_u))$.

We will deduce this theorem from the more general Theorem 3.2 about "tree-like" ball spaces, together with Proposition 2.1 which deals with the crucial notion of "chain union stable ball spaces" that we will introduce in Section 2.

Assume that $\{B_i \mid i \in I\}$ is any collection of balls in $\mathrm{cu}(\mathscr{B}_u)$ such that $B_i \cap B_j \neq \emptyset$ for all $i, j \in I$. Then from the ultrametric triangle law and the assumption that the value set is linearly ordered it follows that $\{B_i \mid i \in I\}$ is in fact a chain. Hence it follows from part 1) of our theorem that $\mathrm{cu}(\mathscr{B}_u)$ is closed under nonempty unions of *arbitrary* collections of balls.

The structure of ultrametric spaces with partially ordered value sets is in general much more complex than in the case of linearly ordered value sets. But we can at least prove the following. Recall that a partially ordered set ("poset") is *narrow* if it contains no infinite sets of pairwise incomparable elements.

Theorem 1.2. *Let $(X, \mathscr{B}_u)$ be the ball space of an ultrametric space with countable narrow value set. Then the assertions of Theorem 1.1 hold.*

This theorem will be deduced from Theorem 3.5 at the end of Section 3, where we study chain union closures for a class of ball spaces that constitutes a slight generalization of the class of ball spaces whose chain intersection closures we have studied in [2, Section 3].

In order to formulate two open problems, we need a definition. Take two ball spaces $(X, \mathscr{B})$ and $(X, \mathscr{B}')$ on the same set X. We call $(X, \mathscr{B}')$ an *expansion* of $(X, \mathscr{B})$ if $\mathscr{B} \subseteq \mathscr{B}'$. In general, we cannot expect the existence of chain union closed expansions which preserve spherical completeness.

Open questions:

1) Does there exist a countable spherically complete ultrametric space with partially ordered value set whose ultrametric ball space does not admit any expansion that is chain union closed and spherically complete?

2) Does there exist a spherically complete ultrametric space with narrow partially ordered value set whose ultrametric ball space does not admit any expansion that is chain union closed and spherically complete?

2 Chain union and chain intersection closures

Let $\mathscr{B}$ be a nonempty family of nonempty sets. Using transfinite recursion, we define $\mathrm{cu}_\alpha(\mathscr{B})$ and $\mathrm{ci}_\alpha(\mathscr{B})$ for each ordinal α, as follows:

$$\begin{aligned} \mathrm{ci}_0(\mathscr{B}) &= \mathscr{B}, \qquad \mathrm{ci}_\alpha(\mathscr{B}) = \mathrm{ci}\left(\bigcup_{\xi<\alpha} \mathrm{ci}_\xi(\mathscr{B})\right) \text{ for } \alpha > 0\,, \\ \mathrm{cu}_0(\mathscr{B}) &= \mathscr{B}, \qquad \mathrm{cu}_\alpha(\mathscr{B}) = \mathrm{cu}\left(\bigcup_{\xi<\alpha} \mathrm{cu}_\xi(\mathscr{B})\right) \text{ for } \alpha > 0\,. \end{aligned}$$

By the definition given in the introduction, $\mathrm{cu}(\mathscr{B}) = \mathrm{cu}_1(\mathscr{B})$, and we set $\mathrm{ci}(\mathscr{B}) := \mathrm{ci}_1(\mathscr{B})$. We observe that

$$(1) \qquad \mathscr{B} \subseteq \mathscr{B}' \Rightarrow \mathrm{ci}_\alpha(\mathscr{B}) \subseteq \mathrm{ci}_\alpha(\mathscr{B}') \text{ and } \mathrm{cu}_\alpha(\mathscr{B}) \subseteq \mathrm{cu}_\alpha(\mathscr{B}') \text{ for all } \alpha\,.$$

We define the *chain union rank* of $\mathscr{B}$, denoted by $\mathrm{cur}(\mathscr{B})$, to be the smallest ordinal α such that $\mathrm{cu}_{\alpha+1}(\mathscr{B}) = \mathrm{cu}_\alpha(\mathscr{B})$. Thus, $\mathrm{cur}(\mathscr{B}) = 0$ if and only if $\mathscr{B}$ is chain union closed, while $\mathrm{cur}(\mathscr{B}) \leq 1$ means that in order to make $\mathscr{B}$ chain union closed, it suffices to extend it by adding all unions of chains. In general, we call $(X, \mathrm{cu}_\alpha(\mathscr{B}))$, with $\alpha = \mathrm{cur}(\mathscr{B})$, the *chain union closure* of $(X, \mathscr{B})$. It could also be described as a ball space $(X, \mathscr{B}')$, where $\mathscr{B}' \supseteq \mathscr{B}$ is minimal such that $\mathscr{B}'$ is stable under unions of chains.

We say that a ball space $(X, \mathscr{B})$ is *chain union stable* if for every nonempty family $\mathbb{F}$ consisting of chains in $\mathscr{B}$ such that $\{\bigcup \mathscr{C} \mid \mathscr{C} \in \mathbb{F}\}$ is a chain, there exists a chain $\mathscr{U} \subseteq \mathscr{B}$ satisfying

$$\bigcup \mathscr{U} = \bigcup_{\mathscr{C}\in\mathbb{F}} \bigcup \mathscr{C}\,.$$

In [2], the notions of *chain intersection rank*, denoted by $\mathrm{cir}(\mathscr{B})$, of *chain intersection closure* and of *chain intersection stable* were defined analogously, just with "cu" replaced by "ci".

The proofs of the following observations are straightforward:

Proposition 2.1. *If $(X, \mathscr{B})$ is chain union stable, then $\mathrm{cur}(\mathscr{B}) \leq 1$ and spherical completeness is preserved when passing from $\mathscr{B}$ to $\mathrm{cu}(\mathscr{B})$.*

If $(X, \mathscr{B})$ is a ball space, then also $(X, \mathrm{cpl}\mathscr{B})$ with

$$\mathrm{cpl}\mathscr{B} := \{X \setminus B \mid B \in \mathscr{B}, B \neq X\}$$

is a ball space; we will call it the *complement ball space* of $(X, \mathscr{B})$. Note that $\mathrm{cpl}\mathscr{B} = \mathrm{cpl}(\mathscr{B} \setminus \{X\})$. We collect a number of useful properties of chain unions and chain intersections.

Lemma 2.2. *Take a ball space* $(X, \mathscr{B})$.

1) If $S \subseteq \mathscr{B}$ *is a finite set and* $\mathscr{B} \setminus S \neq \emptyset$, *then*

i) $\mathrm{ci}_\alpha(\mathscr{B}) = \mathrm{ci}_\alpha(\mathscr{B} \setminus S) \cup S$ *and* $\mathrm{cu}_\alpha(\mathscr{B}) = \mathrm{cu}_\alpha(\mathscr{B} \setminus S) \cup S$ *for all* α,
ii) $\mathrm{cir}(\mathscr{B}) \leq \mathrm{cir}(\mathscr{B} \setminus S)$ *and* $\mathrm{cur}(\mathscr{B}) \leq \mathrm{cur}(\mathscr{B} \setminus S)$.

2) If $X \in \mathscr{B}$ *and* $\mathscr{B} \setminus \{X\} \neq \emptyset$, *then* $\mathrm{ci}_\alpha(\mathscr{B}) \setminus \mathrm{ci}_\alpha(\mathscr{B} \setminus \{X\}) = \{X\}$ *for all* α *and* $\mathrm{cir}(\mathscr{B}) = \mathrm{cir}(\mathscr{B} \setminus \{X\})$.

3) For all α, $(X, \mathrm{ci}_\alpha(\mathrm{cpl}\mathscr{B}))$ *is the complement ball space of* $(X, \mathrm{cu}_\alpha(\mathscr{B}))$.

4) We have that

$$\mathrm{cur}(\mathscr{B}) \geq \mathrm{cir}(\mathrm{cpl}\mathscr{B})\,.$$

If $\mathrm{cir}(\mathrm{cpl}\mathscr{B}) = \alpha$ *and* $X \in \mathrm{cu}_\alpha(\mathscr{B})$, *then* $\mathrm{cur}(\mathscr{B}) = \mathrm{cir}(\mathrm{cpl}\mathscr{B})$.

Proof. 1): We treat the case of chain intersections; the case of chain unions is analogous. We have that $\mathrm{ci}(\mathscr{B}) = \mathrm{ci}(\mathscr{B} \setminus S) \cup S$ as all members of S can be removed from any infinite chain without changing the intersection. This implies assertion i) for $\alpha = 1$ in the case of chain intersections.

Now we proceed by induction on α. Assume that assertion i) holds for ci and ci_α. Then

$$\begin{aligned}\mathrm{ci}_{\alpha+1}(\mathscr{B}) &= \mathrm{ci}(\mathrm{ci}_\alpha(\mathscr{B})) = \mathrm{ci}(\mathrm{ci}_\alpha(\mathscr{B} \setminus S) \cup S) = \mathrm{ci}((\mathrm{ci}_\alpha(\mathscr{B} \setminus S) \cup S) \setminus S) \cup S \\ &= \mathrm{ci}(\mathrm{ci}_\alpha(\mathscr{B} \setminus S)) \cup S = \mathrm{ci}_{\alpha+1}(\mathscr{B} \setminus S) \cup S\,,\end{aligned}$$

where we have used our assertion for ci_α for the second equality, and our assertion for ci for the third equality. This proves the successor case of the induction. The limit case is straightforward.

In order to prove assertion ii), assume that $\mathrm{cir}(\mathscr{B} \setminus S) = \alpha$, that is, $\mathrm{ci}_{\alpha+1}(\mathscr{B} \setminus S)) = \mathrm{ci}_\alpha(\mathscr{B} \setminus S)$. Then by assertion i),

$$\mathrm{ci}_{\alpha+1}(\mathscr{B}) = \mathrm{ci}_{\alpha+1}(\mathscr{B} \setminus S) \cup S = \mathrm{ci}_\alpha(\mathscr{B} \setminus S) \cup S = \mathrm{ci}_\alpha(\mathscr{B})\,.$$

This proves that $\mathrm{cir}(\mathscr{B}) \leq \alpha = \mathrm{cir}(\mathscr{B} \setminus S)$.

2): Since the only chain that has X as its intersection is $\{X\}$, no ball space $\mathscr{B}$ satisfies $X \in \mathrm{ci}(\mathscr{B} \setminus \{X\})$. By induction, $X \notin \mathrm{ci}_\alpha(\mathscr{B} \setminus \{X\})$ for all α. Hence if $X \in \mathscr{B}$ and $\mathscr{B} \setminus \{X\} \neq \emptyset$, then $X \in \mathrm{ci}_\alpha(\mathscr{B}) \setminus \mathrm{ci}_\alpha(\mathscr{B} \setminus \{X\})$, and it follows from assertion i) of part 1) that $\mathrm{ci}_\alpha(\mathscr{B}) \setminus \mathrm{ci}_\alpha(\mathscr{B} \setminus \{X\}) = \{X\}$.

From assertion ii) of part 1) we know that $\mathrm{cir}(\mathscr{B}) \leq \mathrm{cir}(\mathscr{B} \setminus \{X\})$; we have to show that also "$\geq$" holds. Assume that $\mathrm{ci}_{\alpha+1}(\mathscr{B})) = \mathrm{ci}_\alpha(\mathscr{B})$. Then by what we have just proved,

$$\mathrm{ci}_{\alpha+1}(\mathscr{B} \setminus \{X\}) = \mathrm{ci}_{\alpha+1}(\mathscr{B}) \setminus \{X\} = \mathrm{ci}_\alpha(\mathscr{B}) \setminus \{X\} = \mathrm{ci}_\alpha(\mathscr{B} \setminus \{X\})\,.$$

This proves the desired inequality and thus the second assertion of part 2).

3): The assertion is proven by induction on α using the fact that the complement of the union of a chain $\{B_i\}_{i\in I}$ is the intersection of the chain $\{X \setminus B_i\}_{i\in I}$.

4): Assume that $\mathrm{cur}(\mathscr{B}) = \alpha$, that is, $\mathrm{cu}(\mathrm{cu}_\alpha(\mathscr{B})) = \mathrm{cu}_\alpha(\mathscr{B})$. Pick a chain $\mathscr{C}$ in $\mathrm{ci}_\alpha(\mathrm{cpl}\mathscr{B})$ such that $\bigcap\mathscr{C} \neq \emptyset$. By part 3), $\{X\setminus B \mid B \in \mathscr{C}\}$ is a subset of $\mathrm{cu}_\alpha(\mathscr{B})$, and it is also a chain. By assumption, $B' := \bigcup\{X\setminus B \mid B \in \mathscr{C}\} \in \mathrm{cu}_\alpha(\mathscr{B})$. Since $\bigcap\mathscr{C} \neq \emptyset$, we have that $B' \neq X$. Using part 3) again, $\bigcap\mathscr{C} = X \setminus B' \in \mathrm{ci}_\alpha(\mathrm{cpl}\mathscr{B})$. We have proved that $\mathrm{ci}_\alpha(\mathrm{cpl}\mathscr{B})$ is chain intersection closed, which shows that $\mathrm{cir}(\mathrm{cpl}\mathscr{B}) = \alpha$. Hence our first assertion holds.

Now assume that $\mathrm{cir}(\mathrm{cpl}\mathscr{B}) = \alpha$ and that $X \in \mathrm{cu}_\alpha(\mathscr{B})$. By what we have proved before, it suffices to show that $\mathrm{cur}(\mathscr{B}) \leq \mathrm{cir}(\mathrm{cpl}\mathscr{B})$. Pick a chain $\mathscr{C}$ in $\mathrm{cu}_\alpha(\mathscr{B})$; we wish to show that $\bigcup\mathscr{C} \in \mathrm{cu}_\alpha(\mathscr{B})$. As $X \in \mathrm{cu}_\alpha(\mathscr{B})$, we may assume that $\bigcup\mathscr{C} \neq X$, so that $B' := \bigcap\{X \setminus B \mid B \in \mathscr{C}\} \neq \emptyset$. By part 3), $\{X \setminus B \mid B \in \mathscr{C}\}$ is a subset of $\mathrm{ci}_\alpha(\mathrm{cpl}\mathscr{B})$, and it is also a chain. Since $\mathrm{cir}(\mathrm{cpl}\mathscr{B}) = \alpha$, we find that $B' \in \mathrm{ci}_\alpha(\mathrm{cpl}\mathscr{B})$. Using part 3) again, $\bigcup\mathscr{C} = X \setminus B' \in \mathrm{cu}_\alpha(\mathscr{B})$. This shows that $\mathrm{cur}(\mathscr{B}) \leq \mathrm{cir}(\mathrm{cpl}\mathscr{B})$, as desired. □

Note that it can happen that

$$\mathrm{cir}(\mathrm{cpl}\mathscr{B}) \;=\; \mathrm{cur}(\mathscr{B}) \;<\; \mathrm{cur}(\mathscr{B} \setminus \{X\})\,.$$

For example, take $X = \mathbb{N}$ and $\mathscr{B}$ to be the collection of all initial segments of $\mathbb{N}$. Then $(X, \mathscr{B})$ is chain union closed, while $\mathrm{cur}(\mathscr{B} \setminus \{X\}) = 1$. Further, we see that $\mathrm{cpl}(\mathscr{B} \setminus \{X\})$ is the collection of all final segments of $\mathbb{N}$. It is chain intersection closed, i.e., $\mathrm{cir}(\mathrm{cpl}(\mathscr{B} \setminus \{X\})) = 0$.

We will now demonstrate by an example that both the chain union rank and the chain intersection rank of a ball space can be equal to any ordinal α. Since the ball space $(X, \mathscr{P}(X) \setminus \{\emptyset\})$ for nonempty X is both chain union and chain intersection closed, we have to show this only for the case of $\alpha \geq 1$.

Example 2.3. Take an ordinal $\alpha \geq 1$ and set $X := \aleph_\alpha$. For β any ordinal, define $\mathscr{B}_\beta$ to be the collection of all nonempty subsets of $\aleph_\alpha$ of cardinality smaller than or equal to $\aleph_\beta$. Set $\mathscr{B} := \mathscr{B}_0$. We note that $X \notin \mathscr{B}$ since $\alpha \geq 1$.

By (possibly transfinite) induction on $\beta \geq 1$ we show that $\mathrm{cu}_\beta(\mathscr{B}) = \mathscr{B}_\beta$. This holds for $\beta = 0$ by definition. We assume that it holds for all $\gamma < \beta$. If $\beta = \gamma + 1$ is a successor ordinal, then we observe that every subset of $\aleph_\alpha$ of cardinality $\aleph_\beta$ can be written as the union over a chain of subsets of cardinality $\aleph_\gamma$; on the other hand, the union over such a chain cannot be of higher cardinality than $\aleph_{\gamma+1} = \aleph_\beta$. Therefore,

$$\mathrm{cu}_\beta(\mathscr{B}) \;=\; \mathrm{cu}(\mathrm{cu}_\gamma(\mathscr{B})) \;=\; \mathrm{cu}(\mathscr{B}_\gamma) \;=\; \mathscr{B}_\beta\,.$$

Now we assume that β is a limit ordinal. By definition and our induction hypothesis,

$$\mathrm{cu}_\beta(\mathscr{B}) \;=\; \mathrm{cu}\left(\bigcup_{\gamma<\beta} \mathrm{cu}_\gamma(\mathscr{B})\right) \;=\; \mathrm{cu}\left(\bigcup_{\gamma<\beta} \mathscr{B}_\gamma\right)$$

Once again, every subset of $\aleph_\alpha$ of cardinality $\aleph_\beta$ can be written as the union over a chain of subsets of cardinality smaller than $\aleph_\beta$, that is, subsets in $\bigcup_{\gamma<\beta}\mathscr{B}_\gamma$, but the union of such chains cannot have cardinality higher than $\aleph_\beta$. This proves that $\mathrm{cu}_\beta(\mathscr{B}) = \mathscr{B}_\beta$ also in the limit case.

Since the subsets of $\aleph_\alpha$ have cardinality at most $\aleph_\alpha$, it follows that $\mathrm{cu}_\alpha(\mathscr{B}) = \mathscr{B}_\alpha = \mathscr{P}(\aleph_\alpha)\setminus\{\emptyset\} = \mathscr{P}(X)\setminus\{\emptyset\}$. Therefore, $\mathrm{cu}_\alpha(\mathscr{B})$ is chain union closed. On the other hand, the above arguments show that $\mathrm{cu}_\beta(\mathscr{B})$ is not chain union closed for any $\beta<\alpha$. Hence, $\mathrm{cur}(\mathscr{B}) = \alpha$.

Finally, we show that $\mathrm{cir}(\mathrm{cpl}\mathscr{B}) = \alpha$. By part 4) of the preceding lemma, $\beta := \mathrm{cir}(\mathrm{cpl}\mathscr{B}) \le \mathrm{cur}(\mathscr{B}) = \alpha$. Applying part 1)i) of the same lemma with $S = \{X\}$, we obtain that $\mathrm{cu}_\beta(\mathscr{B}\cup\{X\}) = \mathscr{B}_\beta(\mathscr{B})\cup\{X\}$. We have that

$$\beta \;=\; \mathrm{cir}(\mathrm{cpl}\mathscr{B}) \;=\; \mathrm{cir}(\mathrm{cpl}(\mathscr{B}\cup\{X\})) \;=\; \mathrm{cur}(\mathscr{B}\cup\{X\})\,,$$

where the last equality follows from part 4) of the previous lemma. This implies that $\mathscr{B}_\beta(\mathscr{B})\cup\{X\} = \mathscr{B}_\beta\cup\{X\}$ is chain union closed. But as we have seen above, this can only be if $\mathscr{B}_\beta\cup\{X\} = \mathscr{B}_\alpha$. Since X is not the only subset of X of cardinality $\aleph_\alpha$, this can only be the case if $\beta=\alpha$, showing that $\mathrm{cir}(\mathrm{cpl}\mathscr{B}) = \alpha$.

3 Chain union closed ball spaces

In view of Proposition 2.1 we wish to find conditions for a ball space $(X,\mathscr{B})$ to be chain union stable. Let us describe a general approach. Take a chain $\mathscr{D}$ in $\mathrm{cu}(\mathscr{B})$. For each $D\in\mathscr{D}$, choose a chain $\mathscr{C}_D\subseteq\mathscr{B}$ such that $D=\bigcup\mathscr{C}_D$. Pick some $z\in\bigcup\mathscr{D}$ and set

$$\mathscr{D}_z \;:=\; \{D\in\mathscr{D}\mid z\in D\}\,.$$

Then $(\mathscr{D}_z,\subseteq)$ is cofinal in $(\mathscr{D},\subseteq)$ and therefore, $\bigcup\mathscr{D}=\bigcup\mathscr{D}_z$. For every $D\in\mathscr{D}_z$, set

$$\mathscr{C}_{D,z} \;:=\; \{B\in\mathscr{C}_D\mid z\in B\}\,.$$

Then $(\mathscr{D}_{D,z},\subseteq)$ is cofinal in $(\mathscr{C}_D,\subseteq)$ and therefore, $\bigcup\mathscr{C}_{D',z}=\bigcup\mathscr{C}_{D'}$. Altogether, we find that for

$$\mathscr{C}_z \;:=\; \{B\in\mathscr{C}_{D,z}\mid D\in\mathscr{D}_z\} \;\subseteq\; \mathscr{B}$$

we have that $\bigcup\mathscr{D}=\bigcup\mathscr{C}_z$. Therefore, we wish to search for conditions that imply that $\mathscr{C}_z$ is a chain (as in Theorem 3.2 below), or more generally, that $\mathscr{C}_z$ contains a chain $\mathscr{C}$ such that $\bigcup\mathscr{C}_z=\bigcup\mathscr{C}$ (as in Theorem 3.5 below).

We will start with a condition that covers the case of classical ultrametric spaces. A ball space $(X,\mathscr{B})$ will be called *tree-like* if for every $B_1,B_2\in\mathscr{B}$ the following implication holds.

(I) $$B_1\cap B_2\neq\emptyset \implies B_1\subseteq B_2 \text{ or } B_2\subseteq B_1.$$

See [2] for some remarks on tree-like ball spaces.

The following result is obvious:

Lemma 3.1. *If $\mathscr{C}$ is a subset of a tree-like ball space such that all $B \in \mathscr{C}$ contain a common element d, then $\mathscr{C}$ is a chain.*

As a corollary, we obtain:

Theorem 3.2. *Every tree-like ball space is chain union stable. In particular, every ultrametric space with linearly ordered value set is chain union stable.*

Every ultrametric space with linearly ordered value set is tree-like, since in this case property (I) follows from the ultrametric triangle law. Hence Theorem 1.1 follows from Proposition 2.1 in conjunction with Theorem 3.2.

For our next theorem, we will need two lemmas that reflect important and well known properties of narrow posets.

Lemma 3.3. *Let $(\mathscr{P}, \leq)$ be a narrow poset and $\mathscr{A} \subseteq \mathscr{P}$ infinite. Then there exists a chain $\mathscr{C} \subseteq \mathscr{A}$ such that $|\mathscr{C}| = |\mathscr{A}|$.*

Proof. Given $B, B' \in \mathscr{A}$, colour the pair $\{B, B'\}$ *green* if $B \subseteq B'$ or $B' \subseteq B$. Otherwise, color it *red*. By the Erdős-Dushnik-Miller Theorem, there is $\mathscr{C} \subseteq \mathscr{A}$ of the same cardinality as $\mathscr{A}$ and such that all pairs in $\mathscr{C}$ have the same color. Suppose that this color were red. Then any two distinct $B, B' \in \mathscr{C}$ are incomparable in $(\mathscr{P}, \leq)$. But as $(\mathscr{P}, \leq)$ is narrow, the set $\mathscr{C}$ would have to be finite, a contradiction. Thus the color of every pair of elements in $\mathscr{C}$ is green, that is, $\mathscr{C}$ is a chain. □

Recall that a subset A of a poset $(P, \leq)$ is *directed* if for every $a_0, a_1 \in A$ there is $b \in A$ with $a_0 \leq b$ and $a_1 \leq b$. The following fact can be found in [1]. We presented a proof in [2, Lemma 3.3].

Lemma 3.4. *Every narrow poset is a finite union of directed subsets.*

We will now generalize Theorem 3.2 to a larger class of ball spaces $(X, \mathscr{B})$. For every $z \in X$, we set

$$\mathscr{B}_z := \{B \in \mathscr{B} \mid z \in B\}.$$

Theorem 3.5. *Let $(X, \mathscr{B})$ be a ball space such that for every $z \in X$, the poset $(\mathscr{B}_z, \subseteq)$ is narrow and admits only countable strictly increasing sequences. Then $(X, \mathscr{B})$ is chain union stable.*

Proof. Take a chain $\mathscr{D}$ in $\mathrm{cu}(\mathscr{B})$ and d, $\mathscr{D}_z$, $\mathscr{D}_{D,z}$ and $\mathscr{C}_z$ as in the beginning of this paragraph. As $\mathscr{C}_z \subseteq \mathscr{B}_z$, $(\mathscr{C}_z, \subseteq)$ satisfies the same assumptions as $(\mathscr{B}_z, \subseteq)$. In order to prove our theorem, we wish to show that $\mathscr{C}_z$ contains a chain $\mathscr{C}$ such that $\bigcup \mathscr{C}_z = \bigcup \mathscr{C}$. If $\mathscr{C}_z$ is finite, then this is trivial, thus we may assume it is infinite. Then by Lemma 3.3, $\mathscr{C}_z$ contains a chain of cardinality $|\mathscr{C}_z|$. By the assumption of our theorem, this cardinality must be countable. Therefore there are countably many members D of $\mathscr{D}$ such that the union over their associated chains $\mathscr{C}_D$ is $\mathscr{C}_z$; they form a subchain of $\mathscr{D}$. If there is a largest D among them, then we can choose

$\mathscr{C} = \mathscr{C}_D$ and we are done again. Otherwise, we can enumerate the members of the chain as $\{D_i\}_{i<\omega}$ with $D_i \subsetneq D_j$ whenever $i < j < \omega$. We obtain that

$$\mathscr{C}_z \;=\; \bigcup_{i<\omega} \mathscr{C}_{D_i}\,.$$

As every chain $\mathscr{C}_{D_i}$ is countable, it contains a subchain $\{B_{i,j}\}_{j<\omega}$ which is cofinal in $(\mathscr{C}_{D_i}, \subseteq)$, meaning that for each $B \in \mathscr{C}_{D_i}$ there is B' in the subchain such that $B \subseteq B'$. We obtain that $D_i = \bigcup \mathscr{C}_{D_i} = \bigcup \{B_{i,j}\}_{j<\omega}$.

Since $(\mathscr{C}_z, \subseteq)$ is narrow, it follows from Lemma 3.4 that $\mathscr{C}_z = \mathscr{W}_1 \cup \cdots \cup \mathscr{W}_k$ where each $(\mathscr{W}_i, \subseteq)$ is a directed poset. We claim that there is $\ell \in \{1, \ldots, k\}$ such that $\bigcup \mathscr{C}_z = \bigcup \mathscr{W}_\ell$.

Fix $i < \omega$. For each $j < \omega$ there is $\ell \in \{1, \ldots, k\}$ such that $B_{i,j} \in \mathscr{W}_\ell$, so there is $\ell_i \in \{1, \ldots, k\}$ such that $B_{i,j} \in \mathscr{W}_{\ell_i}$ for infinitely many j. Denote by J_i the set of all such indices j. As J_i is infinite, it is cofinal in ω, so

$$D_i \;=\; \bigcup_{j<\omega} B_{i,j} \;=\; \bigcup_{j\in J_i} B_{i,j}\,.$$

Further, there is some $\ell < k$ such that $\ell_i = \ell$ for infinitely many i. Denote by I the set of all such indices i. As I is infinite, it is cofinal in ω, so

$$\bigcup \mathscr{C}_z \;=\; \bigcup_{i<\omega} D_i \;=\; \bigcup_{i\in I} D_i \;=\; \bigcup_{i\in I}\bigcup_{j\in J_i} B_{i,j} \;\subseteq\; \bigcup \mathscr{W}_\ell\,.$$

Since $\mathscr{W}_\ell \subseteq \mathscr{C}_z$, equality must hold, and we have proved our claim.

Finally, since $(\mathscr{W}_\ell, \subseteq)$ is a directed poset and admits only countable strictly increasing sequences, it contains a chain $\mathscr{C}$ that is cofinal in $(\mathscr{W}_\ell, \subseteq)$, meaning that for each $B \in \mathscr{W}_\ell$ there is $B' \in \mathscr{C}$ satisfying $B \subseteq B'$. Hence

$$\bigcup \mathscr{C}_z \;=\; \bigcup \mathscr{W}_\ell \;=\; \bigcup \mathscr{C}\,.$$

This completes our proof. □

Lemma 3.6. *Let $(X, \mathscr{B})$ be the ball space of an ultrametric space with countable narrow value set Γ. Then for each $z \in X$, $(\mathscr{B}_z, \subseteq)$ is narrow and admits only countable strictly increasing sequences.*

Proof. Take $\{B(x_i, y_i)\}_{i<\omega} \subseteq \mathscr{B}_z$. Then there are $k < \ell < \omega$ such that $u(x_k, y_k) \leq u(x_\ell, y_\ell)$ or $u(x_\ell, y_\ell) \leq u(x_k, y_k)$. Since the intersection of $B(x_k, y_k)$ and $B(x_\ell, y_\ell)$ is nonempty as they have the element z in common, it follows that $B(x_k, y_k) \subseteq B(x_\ell, y_\ell)$ or $B(x_\ell, y_\ell) \subseteq B(x_k, y_k)$. This proves that $(\mathscr{B}_z, \subseteq)$ is narrow.

Take a ball $B(x, y)$ and $x', y' \in B(x, y)$, and set $\gamma := u(x, y)$. Then $u(x, x') \leq \gamma$ and $u(x, y') \leq \gamma$, hence by (U2), $u(x', y') \leq \gamma$. Assume that $u(x', y') = \gamma$, and take any $a \in B(x, y)$, i.e., $u(x, a) \leq \gamma$. The latter together with $u(x, x') \leq \gamma$ implies that

$u(x',a) \leq \gamma = u(x',y')$, that is, $a \in B(x',y')$. Consequently, $u(x',y') = \gamma$ implies $B(x,y) \subseteq B(x',y')$. Therefore, $B(x',y') \subsetneq B(x,y) \Rightarrow u(x',y') < u(x,y)$. Hence if Γ is countable, then $(\mathscr{B}, \subseteq)$, and thus also $(\mathscr{B}_z, \subseteq)$, admits only countable strictly increasing sequences. □

Note that more generally, the assertions of this lemma also hold when a ball space $(X, \mathscr{B})$ admits an ultradiameter with values in a countable narrow poset, in the sense of [2]. This is shown in the proof of [2, Theorem 3.4].

Take an ultrametric space (X, d) with countable narrow value set. Then the previous lemma shows that the assumptions of Theorem 3.5 are satisfied, which proves Theorem 1.2.

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