VALUE MONOIDS OF ZERO-DIMENSIONAL VALUATIONS OF RANK ONE

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Abstract. Classically, Gröbner bases are computed by first prescribing a fixed monomial order. Moss Sweedler suggested an alternative in the mid 1980s and developed a framework to perform such computations by using valuation rings in place of monomial orders. We build on these ideas by providing a class of valuations on $K(x, y)$ that are suitable for this framework. We then perform such computations for ideals in the polynomial ring $K[x, y]$. Interestingly, for these valuations, some ideals have finite Gröbner bases with respect to a valuation that are not Gröbner bases with respect to any monomial order, whereas other ideals only have Gröbner bases that are infinite.

1. Introduction

Throughout this paper, we denote by $\mathbb{Z}$ the set of integers, $\mathbb{N}$ the set of nonnegative integers, $\mathbb{Z}^+$ the set of positive integers, and $\mathbb{Q}$ the set of rational numbers. Given $r \in \mathbb{Q}$ and a subset $S \subset \mathbb{Q}$, we define $rS = \{rs \mid s \in S\}$ and $r + S = \{r + s \mid s \in S\}$. Whenever K is a field, its algebraic closure will be denoted by $\widetilde{K}$. Whenever R is a monoid, written additively, we denote by R^* the nonzero elements of R. (This applies in particular to the additive group of a ring.) Finally, we denote by $\mathbf{x}$ the n-tuple of indeterminates $x_1, \ldots, x_n$.

In this section, we provide a brief account of a generalized theory of Gröbner bases that uses valuations in place of monomial orders. The fundamental idea is that monomial orders are well-orderings on the set of monomials, which leads to a natural reduction process that includes multivariate polynomial division. Here valuations permit a more general reduction process than provided by monomial orders. The development of this theory can be found entirely in Moss Sweedler's unpublished manuscript [Sw], and it is briefly provided here solely for the sake of completeness. In that manuscript, Sweedler develops the theory in terms of valuation rings. Here we present the same results in terms of valuations rather than valuation rings. Proofs are omitted since they can all be found in Sweedler's original manuscript.

Definition 1.1. A **valuation** on a field F is a homomorphism ν from the multiplicative group of nonzero elements of F onto an ordered group G (called the **value group**) such that $\nu(f + g) \leq \max\{\nu(f), \nu(g)\}$ for all $f, g \in F^*$ with $f + g \in F^*$. Suppose K is a subfield of F. A **valuation on F over K** is a valuation on F such that its restriction to K^* is the zero map.

Note that the triangle inequality in the definition above was chosen to be the opposite of many conventions so that our results most closely align with those concerning monomial orders. In addition, the valuations being used in Sweedler's generalized theory of Gröbner bases require the extra conditions described in the following definition.

Definition 1.2. We say that a valuation ν on $K(\mathbf{x})$ over K is **suitable relative to** $K[\mathbf{x}]$ if it satisfies the following three properties:

(i) For all $f \in K[\mathbf{x}]^*$, we have $\nu(f) = 0$ iff $f \in K^*$.
(ii) Let $f, g \in K(\mathbf{x})^*$. If $\nu(f) = \nu(g)$, then $\exists \lambda \in K^*$ such that either $f = \lambda g$ or both $f \neq \lambda g$ and $\nu(f - \lambda g) < \nu(f)$.
(iii) $\nu(K[\mathbf{x}]^*)$ is a well-ordered set.

It should be noted that if ν is a valuation on $K(\mathbf{x})$ over K that is suitable relative to $K[\mathbf{x}]$, then the choice of λ in part (ii) above is unique.

When using monomial orders, it is clear what is meant by one monomial dividing another. The analogue we use in the case of valuations deals with arithmetic in the set $\nu(K[\mathbf{x}]^*)$.

Definition 1.3. Let ν be a valuation on $K(\mathbf{x})$. Given $f, g \in K[\mathbf{x}]^*$, we say that $\nu(g)$ **divides** $\nu(f)$, denoted $\nu(g) \mid \nu(f)$, if there exists $h \in K[\mathbf{x}]^*$ such that $\nu(f) = \nu(gh)$. We say that h is an **approximate quotient** of f by g (relative to ν) if either $f = gh$ or both $f \neq gh$ and $\nu(f - gh) < \nu(f)$.

The following simple proposition follows from the definition above.

Proposition 1.4. *Let ν be a valuation on $K(\mathbf{x})$ over K that is suitable relative to $K[\mathbf{x}]$. Let $f, g \in K[\mathbf{x}]^*$. Then $\nu(g)$ divides $\nu(f)$ if and only if there exists an approximate quotient h of f by g.*

The following is a generalized form of the standard polynomial reduction algorithm that makes use of valuations.

Algorithm 1.5. Let ν be a valuation on $K(\mathbf{x})$ over K that is suitable relative to $K[\mathbf{x}]$. Let G be a nonempty subset of $K[\mathbf{x}]^*$. The following algorithm computes a reduction of a polynomial $f \in K[\mathbf{x}]^*$ over G relative to ν.

- Set $i = 0$ and $f_0 = f$.
- While $f_i \neq 0$ and $\nu(g) \mid \nu(f_i)$ for some $g \in G$ do:
  Choose $g_i \in G$ such that $\nu(g_i) \mid \nu(f_i)$. Let h_i be an approximate quotient of f_i by g_i. Set $f_{i+1} = f_i - g_i h_i$. Increment i by 1.

Definition 1.6. In the algorithm above, we say that f_n is an n-th **reduction** of f over G. We say that f **reduces** to b if b is a reduction of f.

It can be shown that if ν is suitable relative to $K[\mathbf{x}]$, then the reduction of any nonzero element of $K[\mathbf{x}]$ over G terminates after a finite number of steps. This leads us to one of many possible formulations of the definition of a Gröbner basis.

Definition 1.7. Let ν be a valuation on $K(\mathbf{x})$ over K that is suitable relative to $K[\mathbf{x}]$. Let J be a nonzero ideal in $K[\mathbf{x}]$ and let G be a nonempty subset of J^*. We say that G is a **Gröbner basis** for J relative to ν if every nonzero element of J has a first reduction over G.

We can use Gröbner bases in the generalized setting to solve the ideal membership problem in much the same way that we do in the case when working with monomial orders. As in the classical case with monomial orders, it can be shown that a Gröbner basis necessarily generates the given ideal. In addition, we have the following equivalent conditions for a set to be a Gröbner basis.

Proposition 1.8. *Let ν be a valuation on $K(\mathbf{x})$ over K that is suitable relative to $K[\mathbf{x}]$. Let J be a nonzero ideal in $K[\mathbf{x}]$ and let G be a nonempty subset of J^*. The following are equivalent:*

(i) *G is a Gröbner basis for J.*
(ii) *Every element of J reduces to 0 over G.*
(iii) *Given $f \in K[\mathbf{x}]^*$, we have $f \in J$ if and only if f reduces to 0 over G.*

Definition 1.9. An **ideal** J of a commutative monoid M is a nonempty subset of M such that for any $m \in M$ and $j \in J$, we have $j + m \in J$. The smallest ideal containing $m_1, \ldots, m_\ell$ will be denoted $\langle m_1, \ldots, m_\ell \rangle$ and is called the **ideal generated by** $m_1, \ldots, m_\ell$.

Definition 1.10. Let ν be a valuation on $K(\mathbf{x})$ over K that is suitable relative to $K[\mathbf{x}]$. Given $f, g \in K[\mathbf{x}]^*$, we say that $T \subset \nu(K[\mathbf{x}]^*)$ is a **monoid ideal generating set for** f **and** g **with respect to** ν if T generates the ideal $\langle \nu(f) \rangle \cap \langle \nu(g) \rangle$ in $\nu(K[\mathbf{x}]^*)$. It can be shown that for each $t \in T$, there exist $a, b \in K[\mathbf{x}]^*$ such that the following two properties hold:

(i) $\nu(af) = \nu(bg) = t$.
(ii) Either $af = bg$ or both $af \neq bg$ and $\nu(af - bg) < t$.

Note that the pair (a, b) is not uniquely determined by t in general. Indeed, for any $\lambda \in K^*$, the pair $(\lambda a, \lambda b)$ also satisfies properties (i) and (ii) above. By invoking the Axiom of Choice, we can choose a pair $(a, b) \in K[\mathbf{x}]^*$ for each $t \in T$, which yields a map

$$\begin{aligned} T &\rightarrow K[\mathbf{x}] \\ t &\mapsto af - bg. \end{aligned}$$

The image of this map is a **syzygy family for** f **and** g **indexed by** T. We say that $af - bg$ is the element of the family corresponding to t.

This definition shows one of the main differences between the generalized theory using valuations and the classical theory using monomial orders, namely, that each pair of polynomials may have many syzygies. Sweedler cites an example in [Sw] where this family may consist of multiple elements. There he considers valuations on subalgebras of $K(\mathbf{x})$ other than just $K[\mathbf{x}]$. Here we construct an example of a valuation on $K(x, y)$ where syzygy families may have more than one member even when working in $K[x, y]$.

Example 1.11. It can be shown that there is a unique valuation over K of the form

$$\begin{aligned} \nu : K(x,y)^* &\rightarrow \mathbb{Z} \oplus \mathbb{Z} \\ x &\mapsto (2,2) \\ y &\mapsto (3,3) \\ x^3 - y^2 &\mapsto (2,1). \end{aligned}$$

Furthermore, it can be shown that this valuation is suitable relative to $K[x, y]$ and that $\{(2, 1), (2, 2), (3, 3)\}$ is a minimal set of generators for the monoid $\nu(K(x, y)^*)$. Consider $\langle (2, 2) \rangle \cap \langle (3, 3) \rangle$, which can be shown to be $\langle (5, 5), (6, 6), (7, 6) \rangle$, and hence is clearly not principal.

The algorithm below provides a method for constructing a Gröbner basis for a nonzero ideal J with generating set G.

Algorithm 1.12 (Gröbner Basis Construction Algorithm)**.** Let ν be a valuation on $K(\mathbf{x})$ over K that is suitable relative to $K[\mathbf{x}]$, and let $G \subset J^*$ be a generating set for a nonzero ideal J.

(i) Set $i = 0$ and $G_0 = G$.

(ii) For each pair of distinct elements $g, h \in G$, find a monoid ideal generating set $T^0_{g,h}$ for g, h and a syzygy family $S^0_{g,h}$ for g, h indexed by $T^0_{g,h}$. Define $U_i = \bigcup_{g \neq h \in G} S^0_{g,h}$.
(iii) Determine a set H_i of nonzero final reductions that occur from reducing the elements of U_i over G_i.
(iv) If H_i is empty, stop.
(v) Define $G_{i+1} = G_i \cup H_i$.
(vi) For each pair of distinct elements $g \in G_{i+1}, h \in H_i$, find a monoid ideal generating set $T^{i+1}_{g,h}$ for g, h, and a syzygy family $S^{i+1}_{g,h}$ for g, h indexed by $T^{i+1}_{g,h}$. Define $U = \bigcup_{g \neq h \in G} S^{i+1}_{g,h}$.
(vii) Increment i and go to step (iii).

Sweedler showed that if G is finite and $\nu(J^*)$ is Noetherian (i.e., every ascending chain of monoid ideals stabilizes), then the construction algorithm can be completed so that it terminates with a finite Gröbner basis. However, even if $\nu(J^*)$ isn't Noetherian, the set $\cup_{n=1}^{\infty} G_n$ is still a Gröbner basis.

These algorithms will allow us to compute Gröbner bases using a class of valuations on $K(x, y)$ originally studied by Zariski in [Za]. In Section 2, we develop the background necessary to work with such valuations and we state one of the main results of the paper, which provides an explicit construction of $\nu(K[x, y]^*)$. In Section 3, we build on these ideas to show that each element of $\nu(K[x, y]^*)$ can be decomposed as a unique sum, which leads to a more precise description of $\nu(K[x, y]^*)$. Finally, we use these decompositions to compute Gröbner bases using Sweedler's algorithms in Section 4. In Section 5, we prove many of the supporting results needed to prove the main theorems, and in Section 6, we prove additional technical results concerning certain sequences related to the valuations of interest.

2. Value Groups and Monoids from Power Series

In this section, we examine a class of valuations on $K(x, y)$ described by Zariski in [Za]. The value groups of these valuations were explicitly constructed by MacLane and Schilling in [MaSc]. In this section, we state one of our main results, which is an explicit construction of the image of the restriction of such valuations to $K[x, y]^*$. Since the valuations of interest are constructed using generalized power series, we begin with a review of the relevant concepts.

Definition 2.1. We say that a set $I \subset \mathbb{Q}$ is **Noetherian** if every nonempty subset of I has a largest element. Given a function $z : \mathbb{Q} \to K$, the **support** of z is defined by

$$\mathrm{Supp}(z) = \{i \in \mathbb{Q} \mid z(i) \neq 0\}.$$

The collection of **Noetherian power series**, denoted by $K((t^{\mathbb{Q}}))$, consists of all functions from $\mathbb{Q}$ to K with Noetherian support.

More commonly in the literature, generalized power series are defined as functions with well-ordered support, and we will freely use the analogues of these results for Noetherian power series. We choose the supports of our series to be opposite of the usual definition so that our results more closely fit with the theory of monomial orders and Gröbner bases. For more details, see [MoSw1], [MoSw2], and [Mo].

As demonstrated by Hahn in [Ha], the collection of Noetherian power series forms a field in which addition is defined point-wise and multiplication is defined via convolution; i.e., if $u, v \in K((t^{\mathbb{Q}}))$ and $i \in \mathbb{Q}$, then $(u+v)(i) = u(i)+v(i)$ and $(uv)(i) = \sum_{j+k=i} u(j)v(k)$. Given a Noetherian power series z with nonempty support I, we will often use the notation

$$z = \sum_{i \in I} z_i t^i$$

where $z_i := z(i)$. For simplicity, we often write t in place of t^1.

Example 2.2. Given $u = t^{1/2} + t^{1/4} + t^{1/8} + \cdots$ and $v = 3t + 1$, we have

$$u + v = 3t + (t^{1/2} + t^{1/4} + t^{1/8} + \cdots) + 1$$

and

$$uv = (3t^{3/2} + 3t^{5/4} + 3t^{9/8} + \cdots) + (t^{1/2} + t^{1/4} + t^{1/8} + \cdots).$$

Definition 2.3. Let $z \in K((t^{\mathbb{Q}}))^*$.

(i) The **leading exponent** of z is the rational number given by

$$\mathrm{le}(z) = \max\{i \mid i \in \mathrm{Supp}(z)\}.$$

(ii) The **leading coefficient** of z is the element of K given by

$$\mathrm{lc}(z) = z(\mathrm{le}(z)).$$

(iii) For each $m \in \mathbb{N}$, we define $z^{(m)}$ as follows. First, define $z^{(0)} = 0$. If $\mathrm{Supp}(z)$ has at most m elements, then define $z^{(m)} = z$. If $\mathrm{Supp}(z)$ has more than m elements, then define $e_1, \ldots, e_m$ to be its m largest elements and define

$$z^{(m)} = z(e_1)t^{e_1} + \cdots + z(e_m)t^{e_m}.$$

(iv) The **leading term** of z is defined to be

$$\mathrm{lt}(z) = z^{(1)}.$$

Example 2.4. For the simple series

$$z = 2t^{1/2} + 3t^{1/3} + 4t^{1/4} + 5t^{1/5} + \cdots,$$

we have the following:

$$\begin{aligned} \mathrm{le}(z) &= 1/2, \\ \mathrm{lc}(z) &= 2, \\ z^{(3)} &= 2t^{1/2} + 3t^{1/3} + 4t^{1/4}, \text{ and} \\ \mathrm{lt}(z) &= 2t^{1/2}. \end{aligned}$$

Note that $\mathrm{le} : K((t^{\mathbb{Q}}))^* \to \mathbb{Q}$ is a valuation, and so given $u, v \in K((t^{\mathbb{Q}}))^*$ with $u - v \in K((t^{\mathbb{Q}}))^*$, the following hold:

$$\mathrm{le}(uv) = \mathrm{le}(u) + \mathrm{le}(v), \tag{2.1}$$

$$\mathrm{lc}(uv) = \mathrm{lc}(u)\,\mathrm{lc}(v), \text{ and} \tag{2.2}$$

$$\mathrm{lt}(uv) = \mathrm{lt}(u)\,\mathrm{lt}(v). \tag{2.3}$$

Moreover, we have the following triangle inequality:

$$\mathrm{le}(u - v) \leq \max\{\mathrm{le}(u), \mathrm{le}(v)\}.$$

If, in addition, $\mathrm{lt}(u) \neq \mathrm{lt}(v)$, then

$$\mathrm{le}(u - v) = \max\{\mathrm{le}(u), \mathrm{le}(v)\}. \tag{2.4}$$

Definition 2.5. We say that $z \in K((t^{\mathbb{Q}}))^*$ is **simple** if it can be written in the form

$$z = \sum_{i=1}^{n} z_i t^{e_i} \tag{2.5}$$

where $z_i \in K^*, n \in \mathbb{Z}^+ \cup \{\infty\}, e_i \in \mathbb{Q}$, and $e_i > e_{i+1}$. Whenever we write a series in this form, we implicitly assume that each z_i is nonzero and the exponents are written in descending order.

Definition 2.6. Let $z \in K((t^{\mathbb{Q}}))^*$ be a simple series written in the form (2.5).

(i) We call $\mathbf{e} = (e_1, e_2, \dots)$ the **exponent sequence** of z.

(ii) Write $e_i = n_i/d_i$ in reduced terms where $n_i, d_i \in \mathbb{Z}$. If $n_i = 0$, then choose $d_i = 1$. Define $r_0 = 1$ and for $i \geq 1$, set $r_i = \operatorname{lcm}(d_1, \dots, d_i)$. We call $\mathbf{r} = (r_0, r_1, r_2, \dots)$ the **ramification sequence** of z.

(iii) Denote the sequence obtained from the ramification sequence after removing repetitions by $(r_0^{red}, r_1^{red}, r_2^{red}, \dots)$. Note that the reduced ramification sequence possibly only consists of finitely many terms (which occurs whenever the ramification sequence has only finitely many distinct terms). For each $i \in \mathbb{N}$, denote by $l(i)$ the smallest nonnegative integer (if it exists) such that $r_i^{red} = r_{l(i)}$; i.e.,

$$l(i) = \min\{j \in \mathbb{N} \mid r_j = r_i^{red}\}.$$

(iv) Define $u_0 = 0$ and for $i \geq 1$,

$$u_i = \sum_{j=0}^{i-1} \left(\frac{r_i}{r_j} - \frac{r_i}{r_{j+1}} \right) e_{j+1}.$$

We call $\mathbf{u} = (u_0, u_1, u_2, \dots)$ the **bounding sequence** of z.

(v) For each positive integer i such that $l(i)$ exists, define

$$\rho_i = u_{l(i)-1} + e_{l(i)}.$$

We call $(\rho_1, \rho_2, \rho_3, \dots)$ the **monoid generating sequence** of z. Note that this sequence may either be finite or infinite.

(vi) For each positive integer i such that $l(i-1)$ and $l(i)$ exist, define

$$s_i = r_{l(i)}/r_{l(i-1)} = r_{l(i)}/r_{l(i)-1}.$$

The **partial ramification sequence** of z is defined as $\mathbf{s} = (s_1, s_2, s_3, \dots)$. If the ramification index increases without bound, then the partial ramification index is infinite. Note, however, that the partial ramification sequence is not even defined unless $l(1)$ exists.

Example 2.7. Given the simple series

$$z = t^2 + t^{3/2} + t^{1/2} + t^{1/3} + t^{1/5} + t^{1/7} + t^{1/11} + \cdots,$$

we have the following associated sequences:

$$\begin{aligned} \mathbf{e} &= \left(2, \frac{3}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \cdots\right), \\ \mathbf{r} &= (1, 1, 2, 2, 6, 30, 210, 2310, \dots), \\ (r_0^{red}, r_1^{red}, r_2^{red}, \dots) &= (1, 2, 6, 30, 210, 2310, \dots), \\ (l(0), l(1), l(2), \dots) &= (0, 2, 4, 5, 6, 7, \dots), \\ \mathbf{u} &= \left(0, 0, \frac{3}{2}, \frac{3}{2}, \frac{31}{6}, \frac{799}{30}, \frac{39331}{210}, \frac{4761151}{2310}, \cdots\right), \\ (\rho_1, \rho_2, \rho_3, \dots) &= \left(\frac{3}{2}, \frac{11}{6}, \frac{161}{30}, \frac{5623}{210}, \frac{432851}{2310}, \cdots\right), \text{ and} \\ (s_1, s_2, s_3, \dots) &= (2, 3, 5, 7, 11, \dots). \end{aligned}$$

Since $l(i)$ marks the point where the ramification index increases, we have $r_j = r_{l(i)}$ for $l(i) \leq j < l(i+1)$, and so

$$r_j / r_{j-1} = 1 \quad \text{whenever} \quad l(i) < j < l(i+1). \tag{2.6}$$

In particular, this yields $r_{l(i-1)} = r_{l(i)-1}$, and so

$$u_{l(i-1)} = u_{l(i)-1} \tag{2.7}$$

despite the fact that $e_{l(i-1)}$ and $e_{l(i)-1}$ need not be the same. Under certain assumptions, the monoid generating sequence is guaranteed to be infinite. In particular, we have the following result.

Lemma 2.8. *If $z \in K((t^{\mathbb{Q}}))$ is a simple series with positive support such that t and z are algebraically independent over K, then the monoid generating sequence of z is infinite.*

Proof. Since the terms of the exponent sequence are positive and decreasing, the ramification sequence must increase without bound. Thus, $l(i)$ exists for each $i \in \mathbb{N}$, and so the monoid generating sequence is infinite. □

We are now in a position to define valuations on $K(x, y)$ based on Noetherian power series.

Definition 2.9. Let $z \in K((t^{\mathbb{Q}}))$ be a Noetherian power series such that t and z are algebraically independent over K. Consider the embedding

$$\begin{aligned} \varphi_z : K(x, y)^* &\rightarrow K((t^{\mathbb{Q}})) \\ x &\mapsto t \\ y &\mapsto z. \end{aligned}$$

Since le is a valuation on $K((t^{\mathbb{Q}}))$, the composite map $\text{le} \circ \varphi_z : K(x, y)^* \rightarrow \mathbb{Q}$ is a valuation on $K(x, y)$.

In general, given a valuation ν on $K(\mathbf{x})^*$, the set

$$V = \{f \in K[\mathbf{x}]^* \mid \nu(f) \leq 0\}$$

is a valuation ring with maximal ideal

$$\mathfrak{m} = \{f \in K[\mathbf{x}]^* \mid \nu(f) < 0\}.$$

The quotient $V/\mathfrak{m}$ is called the **residue class field** and contains an isomorphic copy of K. The transcendence degree of $V/\mathfrak{m}$ over K is called the **dimension** of the valuation. The

rank of the valuation is defined to be the number of isolated subgroups of $\nu(K[\mathbf{x}]^*)$, which is also the Krull dimension of the valuation ring V (see Theorem 15 from [ZaSa] for details). It follows that $\mathrm{le} \circ \varphi_z$ is a zero-dimensional valuation of rank one.

Example 2.10. Let K be a field such that char $K \neq 2$. Given $z = t^{1/2} + t^{1/4} + t^{1/8} + \cdots$, we have

$$\begin{aligned}(\mathrm{le} \circ \varphi_z)(x) &= \mathrm{le}(t) = 1, \\ (\mathrm{le} \circ \varphi_z)(y) &= \mathrm{le}(z) = 1/2, \text{ and} \\ (\mathrm{le} \circ \varphi_z)(y^2 - x) &= \mathrm{le}(z^2 - t) = \mathrm{le}((t + 2t^{3/4} + 2t^{5/8} + \cdots) - t) = 3/4.\end{aligned}$$

MacLane and Schilling proved the following result in [MaSc], which precisely describes the value group of $\mathrm{le} \circ \varphi_z$ in case K has characteristic zero.

Theorem 2.11. *If K is a field of characteristic zero and $z \in K((t^{\mathbb{Q}}))$ is a simple series with exponent sequence $(e_1, e_2, e_3, \dots)$ such that t and z are algebraically independent over K, then*

$$(le \circ \varphi_z)(K(x, y)^*) = \mathbb{Z} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \cdots .$$

One of the primary goals of this paper is to restrict such valuations to the nonzero elements of the polynomial ring $K[x, y]$ and compute the monoid consisting of all the images.

Definition 2.12. Given a simple series $z \in K((t^{\mathbb{Q}}))$ such that t and z are algebraically independent over K, the **value monoid with respect to** z is given by

$$\Lambda = (\mathrm{le} \circ \varphi_z)(K[x, y]^*) = \{\mathrm{le}(f(t, z)) \mid f(x, y) \in K[x, y]^*\}.$$

Theorem 2.13. *Let $z \in K((t^{\mathbb{Q}}))$ be a simple series with positive support such that t and z are algebraically independent over K. If no term of the ramification index of z is divisible by the characteristic of K, then the value monoid with respect to z is*

$$(le \circ \varphi_z)(K[x, y]^*) = \mathbb{N} + \mathbb{N}\rho_1 + \mathbb{N}\rho_2 + \cdots .$$

A more precise version of this theorem will be given in Section 3. In particular, we demonstrate in Theorem 3.20 that each element of the value monoid has a unique representation as the sum of a nonnegative integer and an $\mathbb{N}$-linear combination of terms of the monoid generating sequence.

3. Construction of the Value Monoid

We begin with an explanation of some notation to be used from this point forward. The field of Laurent series in the variable t over the field K is denoted by $K((t))$ and consists of all functions $z : \mathbb{Z} \to K$ with well-ordered support. In contrast, we will consider $K((t^{-1}))$, which consists of all functions $z : \mathbb{Z} \to K$ with Noetherian support. Note that every element of the rational function field $K(t)$ can be written as a series in either t or t^{-1}, and so we can view $K(t)$ as embedded in both $K((t))$ and $K((t^{-1}))$. Again, since we focus on series with Noetherian support, we consider the following inclusions:

$$K(t) \subset K((t^{-1})) \subset K((t^{\mathbb{Q}})) \subset \widetilde{K}((t^{\mathbb{Q}})).$$

We cite various references concerning power series, most of which use the field $K((t))$ rather than $K((t^{-1}))$, though the proofs can be easily translated. In the first half of this section, we describe the algebraic closure of $K((t^{-1}))$ in $K((t^{\mathbb{Q}}))$, whose description depends on the characteristic of K. We then demonstrate that for each $p(x,y) \in K[x,y]^*$, there exists $f(x,y) \in K[x,y]^*$ such that $\mathrm{le}(p(t,z)) \geq \mathrm{le}(f(t,z))$ where $p(x,y)$ and $f(x,y)$ have the same degree in the variable y and the roots of $f(t,y)$ in $K((t^{\mathbb{Q}}))$ have finite support. Using this intermediate result, we give an explicit form for the elements of the value monoid $(\mathrm{le} \circ \varphi_z)(K[x,y]^*)$.

Definition 3.1. An element of $K((t^{\mathbb{Q}}))$ is said to be **Puiseux** if it lies in $K((t^{-1/r}))$ for some positive integer r. The **ramification index** of a Puiseux series $w \in K((t^{\mathbb{Q}}))$ is the smallest positive integer r such that $w \in K((t^{-1/r}))$.

When K is an algebraically closed field of characteristic zero, the following result (known as Puiseux's Theorem or the Newton-Puiseux Theorem) describes the algebraic closure of the field of Laurent series in $K((t^{\mathbb{Q}}))$. (See [AbMo] or [Du] for further details.) Here we provide the analogue where $K((t))$ is replaced by $K((t^{-1}))$.

Theorem 3.2 (Puiseux's Theorem)**.** *Let K be an algebraically closed field of characteristic zero. The algebraic closure of $K((t^{-1}))$ in $K((t^{\mathbb{Q}}))$ precisely consists of all the elements of $K((t^{\mathbb{Q}}))$ that are Puiseux.*

The case when K has positive characteristic is more complex and is considered by Kiran Kedlaya in [Ke]. First, we review a few preliminary definitions.

Definition 3.3. Let K be a field of characteristic $p > 0$ and let $\{c_n\}$ be a sequence of elements of K. We say that $\{c_n\}$ satisfies the **linearized recurrence relation (LRR)** corresponding to $d_0, \ldots, d_k \in K$ (where $d_k \neq 0$) if for all $n \in \mathbb{N}$,

$$d_0 c_n + d_1 c_{n+1}^p + \cdots + d_k c_{n+k}^{p^k} = 0. \tag{3.1}$$

Definition 3.4. Let p be a positive prime. Given $a \in \mathbb{Z}^+$ and $b, c \in \mathbb{N}$, define

$$S_{a,b,c} = \left\{ \frac{1}{a}\left(n + \frac{b_1}{p} + \frac{b_2}{p^2} + \cdots \right) \,\middle|\, n \leq b,\ b_i \in \{0, \ldots, p-1\},\ \sum b_i \leq c \right\}.$$

Definition 3.5. Let K be a field of characteristic $p > 0$. Let $T_c = S_{1,0,c} \cap (0,1)$ and let k be a positive integer. A function $f : T_c \to K$ is **twist-recurrent (of order k)**, if there exist $d_0, \ldots, d_k \in K$ such that (3.1) holds for any sequence of the form

$$c_n = f\left(\frac{b_1}{p} + \cdots + \frac{b_{j-1}}{p^{j-1}} + \frac{1}{p^n}\left(\frac{b_j}{p^j} + \cdots\right)\right)$$

where $n \in \mathbb{N}, j \in \mathbb{Z}^+, b_i \in \{0, \ldots, p-1\}$ and $\sum b_i \leq c$.

Definition 3.6. Let K be a field of characteristic $p > 0$. We say $w \in K((t^{\mathbb{Q}}))$ is **twist-recurrent** if all the following conditions hold:

(a) There exist positive integers a, b, c such that $\mathrm{Supp}(w) \subset S_{a,b,c}$.
(b) For some positive integers a, b, c such that $\mathrm{Supp}(w) \subset S_{a,b,c}$, and for all $m \leq b$, the function

$$\begin{aligned} f_m : T_c &\to K \\ \tau &\mapsto w_{(m+\tau)/a} \end{aligned}$$

is twist-recurrent.
(c) The functions of the form f_m span a finite-dimensional vector space over K.

The following is the analogue of Corollary 9 from [Ke] where $K((t))$ is replaced by $K((t^{-1}))$.

Theorem 3.7. *Let K be a perfect field of positive characteristic. The algebraic closure of $K((t^{-1}))$ in $K((t^{\mathbb{Q}}))$ precisely consists of all twist-recurrent series whose coefficients lie in a finite extension of K.*

We state the following straightforward lemma without proof.

Lemma 3.8. *For each $\ell \in \mathbb{Z}$, we define the function $\eta_\ell : \mathbb{Q} \to \widetilde{K}^*$ in two different ways, depending on the characteristic of K. For each $b \in \mathbb{Z}^+$, we denote by ζ_b a primitive b-th root of unity. Note that $\widetilde{K}$ contains such a primitive root of unity provided that b is not divisible by char K.*

I. *If char $K = 0$, then for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, define*

$$\eta_\ell\left(\frac{a}{b}\right) = \zeta_b^{a\ell}.$$

II. *If char $K = p > 0$, then for $a \in \mathbb{Z}$, $n \in \mathbb{N}$, and $b \in \mathbb{Z}^+$ such that gcd$(p, b) = 1$, define*

$$\eta_\ell\left(\frac{a}{bp^n}\right) = \zeta_b^{a\ell q^n}$$

where q is an integer such that $pq \equiv 1 \mod b$.

The function η_ℓ is a homomorphism from the additive group of rational numbers to the multiplicative group of nonzero elements of $\widetilde{K}$. The kernel of this homomorphism contains $\mathbb{Z}$.

Since the kernel of η_ℓ contains $\mathbb{Z}$, the lemma below follows immediately.

Lemma 3.9. *The function $\psi_\ell : \widetilde{K}((t^{\mathbb{Q}})) \to \widetilde{K}((t^{\mathbb{Q}}))$ given by*

$$\sum w_i t^i \mapsto \sum \eta_\ell(i) w_i t^i$$

is a $K((t^{-1}))$-automorphism of $\widetilde{K}((t^{\mathbb{Q}}))$. The collection $\Psi = \{\psi_\ell \mid \ell \in \mathbb{Z}\}$ forms a cyclic group under composition since $\psi_i \circ \psi_j = \psi_{i+j} = \psi_j \circ \psi_i$.

We now reformulate Theorem 2.4 from [MoSw2] in terms functions from $\Psi = \{\psi_\ell \mid \ell \in \mathbb{Z}\}$. This result holds for arbitrary characteristic since we are assuming that the ramification index of w is not divisible by char K.

Proposition 3.10. *Let w be a nonzero Puiseux series with finite, nonnegative support and ramification index n. If n is not divisible by char K, then the minimal polynomial of w over $K(t)$ (and over $K((t^{-1}))$ as well) is*

$$f(y) = \prod_{\ell=1}^{n} (y - \psi_i(w)) \in K[t, y].$$

A similar argument produces the following result.

Proposition 3.11. *Let w be a nonzero Puiseux series with ramification index n. If n is not divisible by char K, then the minimal polynomial of w over $K((t^{-1}))$ is*

$$f(y) = \prod_{\ell=1}^{n} (y - \psi_i(w)) \in K((t^{-1}))[y].$$

For the remainder of this paper, we make the following assumptions.

- The field K is perfect.
- The series $z \in K((t^{\mathbb{Q}}))$ is simple with positive support.
- The series t and z are algebraically independent over K.
- No term of the ramification sequence of z is divisible by char K.
- We adopt the notation introduced in Definition 2.6.
- The value monoid with respect to z will be denoted Λ.

The following proposition is the main building block for the construction of the value monoid. In essence, it aids us in showing that the value monoid can be described purely in terms of polynomials $f(x, y) \in K[x, y]^*$ such that the roots of $f(t, y)$ in $\widetilde{K}((t^{\mathbb{Q}}))$ have finite support. This proposition will be justified in Section 5 immediately following Proposition 5.14.

Proposition 3.12. *For each $p(x, y) \in K[x, y]^*$, there exists $f(x, y) \in K[x, y]^*$ such that the following hold:*

(i) $\deg_y p(x, y) = \deg_y f(x, y)$,
(ii) $le(p(t, z)) \geq le(f(t, z))$, *and*
(iii) *$f(t, y)$ is a product of minimal polynomials of series of the form $z^{(l(j)-1)}$ over $K(t)$.*

Since part (iii) of this proposition suggests that we need to compute leading exponents of minimal polynomials of series of the form $z^{(l(j)-1)}$, we produce a result that precisely allows us to do that (see Lemma 3.15).

Definition 3.13. Given $u, v \in K((t^{\mathbb{Q}}))$, we say that u and v **agree to (finite) order** $m \in \mathbb{N}$ if $u^{(m)} = v^{(m)}$ but $u^{(m+1)} \neq v^{(m+1)}$.

We next use Proposition 3.10 to strengthen Proposition 4.6 from [Mo] to include both fields of characteristic zero and of positive characteristic. Since z is an infinite, simple series with positive support, we know that the ramification sequence increases without bound and so z is not Puiseux. Therefore, for any Puiseux series w, we have that z and w agree to a finite order.

Proposition 3.14. *Let $w \in K((t^{\mathbb{Q}}))^*$ be a finite Puiseux series whose ramification index r is not divisible by char K. Define m to be the order to which z and w agree, and define $p(y) \in K\widetilde{((t^{-1}))}[y]$ to be the minimal polynomial of w over $K((t^{-1}))$. If none of the conjugates of w in $\widetilde{K((t^{-1}))}$ agree with z to an order greater than m, then*

$$le(p(z)) = \left(\frac{r}{r_m}\right)\Big[u_m + le(z - w)\Big] \geq \left(\frac{r}{r_m}\right)\Big[u_m + e_{m+1}\Big].$$

Lemma 3.15. *If $j \in \mathbb{Z}^+$ and $f(t, y) \in K((t^{-1}))[y]$ is the minimal polynomial of $z^{(l(j)-1)}$ over $K((t^{-1}))$, then*

(i) $f(t, y) \in K[t, y]$,
(ii) $\deg_y(f(t, y)) = r_{l(j)-1}$, *and*
(iii) $le(f(t, z)) = \rho_j$.

Proof. Let $f(t,y) \in K((t^{-1}))[y]$ be the minimal polynomial of $z^{(l(j)-1)}$ over $K((t^{-1}))$. Since the exponent sequence of z consists solely of positive numbers, it follows that the finite series $z^{(l(j)-1)}$ has positive support, and so by Proposition 3.10, we know that $f(t,y) \in K[t,y]$. Moreover, since $z^{(l(j)-1)}$ has ramification index $r_{l(j)-1}$, Proposition 3.10 tell us that $\deg_y f(t,y) = r_{l(j)-1}$. Finally, by Proposition 3.14,

$$\mathrm{le}(f(t,z)) = \left(\frac{r_{l(i)-1}}{r_{l(j)-1}}\right)(u_{l(j)-1} + e_{l(j)}) = u_{l(j)-1} + e_{l(j)} = \rho_j.$$

□

The next lemma, which will be demonstrated in Section 6 immediately following Lemma 6.3, allows us to define the minimal possible value of the image of a polynomial of a given degree under the map $\mathrm{le} \circ \varphi_z$.

Lemma 3.16. *The monoid generating sequence is increasing.*

Let $p(x,y) \in K[x,y]$ be a nonzero polynomial of degree d in the variable y. By Proposition 3.12 and Lemma 3.15, there exists a nonzero polynomial $f(x,y)$ of degree d in y such that $\mathrm{le}(p(t,z)) \geq \mathrm{le}(f(t,z))$ and $\mathrm{le}(f(t,z))$ is a sum of terms of the monoid generating sequence. Since the monoid generating sequence is increasing by Lemma 3.16, there exists a choice of $f(x,y) \in K[x,y]$ such that $\mathrm{le}(f(t,z))$ is as small as possible.

Definition 3.17. For each $d \in \mathbb{N}$,

$$\lambda_d = \min\{\mathrm{le}(f(t,z)) \mid f \in K[x,y]^* \text{ and } \deg_y(f(x,y)) = d\}.$$

The following proposition, which will be proved in Section 6 immediately following Lemma 6.6, shows that each λ_d has a unique representation as a sum of terms of the monoid generating sequence.

Proposition 3.18. *For each $d \in \mathbb{Z}^+$, there exist a unique $J \in \mathbb{Z}^+$ and $d_1, \ldots, d_J \in \mathbb{N}$ where $0 \leq d_j < s_j$ for each $j \in \{1, \ldots, J\}$ such that $d_J \neq 0$,*

$$d = \sum_{j=1}^{J} d_j r_{l(j-1)}$$

and

$$\lambda_d = \sum_{j=1}^{J} d_j \rho_j.$$

The following result, which will be proved at the very end of Section 6, shows how to decompose the value monoid as a disjoint union of cosets of $\mathbb{N}$.

Proposition 3.19. *The value monoid with respect to z is the disjoint union*

$$\Lambda = \bigcup_{d=0}^{\infty} (\lambda_d + \mathbb{N}).$$

Combining Proposition 3.18 and Proposition 3.19, we obtain the following theorem.

Theorem 3.20. *For each $m \in \Lambda$, either $m \in \mathbb{N}$ or there exist a unique $J \in \mathbb{Z}^+$, $n \in \mathbb{N}$ and $d_1, \ldots, d_J \in \mathbb{N}$ where $0 \leq d_j < s_j$ for each $j \in \{1, \ldots, J\}$ such that $d_J \neq 0$ and*

$$m = n + \sum_{j=1}^{J} d_j \rho_j.$$

It would be interesting to determine whether this result can be generalized. In particular, it would be desirable to compute the value monoid after either removing the restriction that the exponent sequence must be positive or permitting some of the terms of the exponent sequence to be divisible by the characteristic of the ground field K.

4. ALGORITHMS

In this section, we develop the algorithms necessary to construct Gröbner bases using valuations. To begin, if we use the characteristic-free version of Proposition 3.12 presented in this paper, we can generalize Corollary 5.2 from [Mo] to form the following result where we include the assumption that no term of the ramification sequence of z is divisible by char K.

Theorem 4.1. *The value monoid Λ is well-ordered.*

Using this theorem, we can conclude that $\mathrm{le} \circ \varphi_z$ is suitable with respect to $K[x, y]$ as described in Definition 1.2, and so we can use $\mathrm{le} \circ \varphi_z$ in the algorithms described in Section 1. Throughout this section we will abuse notation and refer to the composite maps $\mathrm{le} \circ \varphi_z$ and $\mathrm{lc} \circ \varphi_z$ simply as le and lc.

Given a rational number $m \in \mathbb{Q}$, we would like to decide whether $m \in \Lambda$, and in case it is, express it in terms of the generators $1, \rho_1, \rho_2, \ldots$. To accomplish this, we need a few preliminary definitions and results. To begin, we describe one method of building the value monoid in stages.

Definition 4.2. Define $\Omega_0 = \mathbb{N}$, and for each $J \in \mathbb{Z}^+$, define

$$\Omega_J = \{n + \sum_{j=1}^{J} d_j \rho_j \;\mid\; n \in \mathbb{N},\ 0 \leq d_j < s_j\}.$$

The next lemma will be demonstrated in Section 6 immediately following Lemma 6.5. It essentially gives restrictions on the denominators of certain $\mathbb{Z}$-linear combinations of terms of the monoid generating sequence.

Lemma 4.3. *Let $J \in \mathbb{Z}^+$ and $0 \leq d_j < s_j$ for $j \in \{1, \ldots, J\}$. If $d_J \neq 0$, then*

$$\sum_{j=1}^{J} d_j \rho_j \in (1/r_{l(J)})\mathbb{Z} - (1/r_{l(J-1)})\mathbb{Z}.$$

Using this lemma, we can prove the following result, which relates Λ and Ω_J.

Lemma 4.4. *For all $J \in \mathbb{Z}^+$,*

$$\Omega_J = \Lambda \cap \mathbb{Z} \cdot \{1, \rho_1, \rho_2, \ldots, \rho_J\}.$$

Proof. We begin by demonstrating that $\Omega_J \subset \Lambda \cap \mathbb{Z} \cdot \{1, \rho_1, \rho_2, \ldots, \rho_J\}$. Given $m \in \Omega_J$, there exists $n \in \mathbb{N}$ and $d_1, \ldots, d_J \in \mathbb{N}$ where $0 \le d_j < s_j$ for $j \in \{1, \ldots, J\}$ such that

$$m = n + \sum_{j=1}^{J} d_j \rho_j$$

and so $m \in \mathbb{Z} \cdot \{1, \rho_1, \rho_2, \ldots, \rho_J\}$. Moreover, if we define $f_j(t, y)$ to be the minimal polynomial $z^{l(j)-1}$ over $K((t^{-1}))$, then by Lemma 3.15, we have $f_j(x, y) \in K[x, y]$ and

$$\operatorname{le}\left(x^n \prod_{j=1}^{J} f_j(x, y)^{d_j}\right) = n + \sum_{j=1}^{J} d_j \rho_j.$$

Thus, $m \in \Lambda$ and so we have shown that $\Omega_j \subset \Lambda \cap \mathbb{Z} \cdot \{1, \rho_1, \rho_2, \ldots, \rho_J\}$.

We now demonstrate the reverse containment. Given $m \in \Lambda \cap \mathbb{Z} \cdot \{1, \rho_1, \rho_2, \ldots, \rho_J\}$, we have $m \in (1/r_{l(J)})\mathbb{Z}$. Since $m \in \Lambda$, we know by Theorem 3.20 that m has a unique representation $m = n + \sum_{j=1}^{k} d_j \rho_j$ where $0 \le d_j < s_j$, in which case $m \in \Omega_k$. Suppose, toward contradiction that $J < k$. By Lemma 4.3,

$$m \in (1/r_{l(k)})\mathbb{Z} - (1/r_{l(k-1)})\mathbb{Z}.$$

Since $m \notin (1/r_{l(k-1)})\mathbb{Z}$ and $J \le k - 1$, it follows that $m \notin (1/r_{l(J)})\mathbb{Z}$, a contradiction. Thus, $J \ge k$, and so $m \in \Omega_k \subset \Omega_J$. □

The corollary below follows immediately from this lemma.

Corollary 4.5. *Each Ω_J is closed under addition.*

We are now in a position to describe how to determine whether a positive rational number is an element of the value monoid Λ. Given a positive rational number m, write m as a/b where a and b are relatively prime positive integers. If $m \in \mathbb{N}$, then $m \in \Lambda$, and so we only need to consider the case where $b > 1$.

Our goal is to decide using modular arithmetic whether $m \in \Lambda$. First, find the smallest index J such that $b \mid r_{l(J)}$. It can be shown that the set of all $\mathbb{Z}$-linear combinations of $1, \rho_1, \ldots, \rho_J$ is precisely the set $(1/r_{l(J)})\mathbb{Z}$; thus, $m \in \mathbb{Z} \cdot \{1, \rho_1, \ldots, \rho_J\}$. By Lemma 4.4, we know that $m \in \Lambda$ if and only $m \in \Omega_j$, in which case there exist integers $d_1, \ldots, d_J$ where $0 \le d_j < s_j$ for $j \in \{1, \ldots, J\}$ such that

$$m = n + \sum_{j=1}^{J} d_j \rho_j.$$

Since the validity of this last statement can be tested by solving for $d_1, \ldots, d_J$, we have the following algorithm.

Algorithm 4.6. Let m be a positive rational number. The following algorithm determines whether $m \in \Lambda$. If $m \in \Lambda$, then the algorithm produces a decomposition of m as a $\mathbb{Z}$-linear combination of $1, \rho_1, \ldots, \rho_J$. For each j, write $\rho_j = c_j / r_{l(j)}$.

(1) Write m as a/b where a, b are relatively prime, positive integers.
(2) Define J to be the smallest index such that $b \mid r_{l(J)}$.
(3) Define $m_J = m$.
(4) Solve the congruence $c_J d_J \equiv m r_{l(J)} \mod s_J$ for d_J where $0 \le d_J < s_J$. If there are no solutions, then $m \notin \Lambda$.

(5) For $j = J-1, J-2, \ldots, 1$, define $m_j = m_{j+1} - d_{j+1}\rho_{j+1}$ and solve the congruence $c_j d_j \equiv m r_{l(j)} \mod s_j$ for d_j where $0 \le d_j < s_j$. If any of the congruences fail to yield a solution, then $m \notin \Lambda$.
(6) Define $n = m_1 - d_1\rho_1$. If n is negative, then $m \notin \Lambda$. Otherwise, if $n \in \mathbb{N}$, we have $m \in \Lambda$ and
$$m = n + \sum_{j=1}^{J} d_j \rho_j.$$

Now that we have a test for whether a rational number is in the value monoid, we need to construct preimages of elements of the value monoid under the valuation. The following algorithm accomplishes this task.

Algorithm 4.7. Let $m \in \Lambda$. This algorithm constructs $p(x,y) \in K[x,y]^*$ such that $\mathrm{le}(p(x,y)) = m$.

(1) Using Algorithm 4.6, write $m = n + \sum_{j=1}^{J} d_j\rho_j$.
(2) For each $j \in \{1, \ldots, J\}$, use Proposition 3.10 to compute $p_j(x,y)$, where $p_j(t,y)$ the minimal polynomial of $z^{l(j)-1}$ over $K(t)$.
(3) Define $p(x,y) = x^n \prod_{j=1}^{J} p_j(x,y)^{d_j}$. By Lemma 3.15 (iii), we have $\mathrm{le}(p(x,y)) = m$.

The following algorithm describes how to perform division in $K[x,y]$ relative to le.

Algorithm 4.8. Let $f, g \in K[x,y]^*$. This algorithm constructs $h \in K[x,y]^*$ such that $\mathrm{le}(f - gh) < \mathrm{le}(f)$ provided that such an h exists.

(1) Compute $m = \mathrm{le}(f) - \mathrm{le}(g)$.
(2) Use Algorithm 4.6 to determine whether $m \in \Lambda$, thus determining whether h exists.
(2) Using Algorithm 4.7, find $p(x,y) \in K[x,y]$ such that $\mathrm{le}(p) = m$.
(3) Define $h(x,y) = (\mathrm{lc}(f)/\mathrm{lc}(gp))p(x,y)$. Then $\mathrm{lc}(f) = \mathrm{lc}(gh)$, and since $\mathrm{le}(f) = \mathrm{le}(gh)$, it follows that $\mathrm{le}(f - gh) < \mathrm{le}(f)$.

We state the following simple lemma without proof.

Lemma 4.9. *Let M be a monoid such that $\mathbb{Z} \subset M \subset \mathbb{Q}$, and let q be an element of the quotient group of M (i.e., the set of differences of elements of M). Then for sufficiently large $n \in \mathbb{N}$, we have $q + n \in M$.*

Using this lemma, we demonstrate a proposition that will be useful in the construction of generators of intersections of ideals in Λ of the form $\langle \mathrm{le}(f)\rangle$.

Proposition 4.10. *Given $f, g \in K[x,y]^*$ such that $le(f), le(g) \in \Omega_J$, there exists a finite subset of Ω_J that generates $\langle le(f)\rangle \cap \langle le(g)\rangle$.*

Proof. The proof is simple when $J = 0$, and so we consider the case when $J \in \mathbb{Z}^+$. In this proof, whenever the upper limit of summation is less than the lower limit, we declare the empty sum to be zero. For each positive integer J, define
$$\Omega'_J = \{\sum_{j=1}^{J} d_j\rho_j \mid 0 \le d_j < s_j\}.$$
By Lemma 4.9, for each $\omega \in \Omega'_J$, there exists a smallest $n_\omega \in \mathbb{N}$ such that $\omega - \mathrm{le}(f) + n_\omega$ and $\omega - \mathrm{le}(g) + n_\omega$ are both elements of Λ; that is, $\omega + n_\omega \in \langle \mathrm{le}(f)\rangle \cap \langle \mathrm{le}(g)\rangle$. Define $\Gamma_J \subset \Omega_J$ to be the finite collection
$$\Gamma_J = \{\omega + n_\omega \mid \omega \in \Omega'_J\}.$$

We will show that Γ_J generates $\langle \mathrm{le}(f)\rangle \cap \langle \mathrm{le}(g)\rangle$.

Given $m \in \langle \mathrm{le}(f)\rangle \cap \langle \mathrm{le}(g)\rangle$, we can write $m = \mathrm{le}(f) + m_f$ and $m = \mathrm{le}(g) + m_g$ where $m_f, m_g \in \Lambda$. By Theorem 3.20, there exists $k \in \mathbb{Z}^+$ and $n_f, n_g, d_1, \ldots, d_k, d'_1, \ldots, d'_k \in \mathbb{N}$ where $0 \le d_j, d'_j < s_j$ for each $j \in \{1, \ldots, k\}$ such that

$$\begin{aligned} m_f &= n_f + \sum_{j=1}^{k} d_j \rho_j, \\ m_g &= n_g + \sum_{j=1}^{k} d'_j \rho_j. \end{aligned}$$

Then

$$m = \mathrm{le}(f) + n_f + \alpha_f + \sum_{j=J+1}^{k} d_j \rho_j, \tag{4.1}$$

$$m = \mathrm{le}(g) + n_g + \alpha_f + \sum_{j=J+1}^{k} d'_j \rho_j, \tag{4.2}$$

where $\alpha_f = \sum_{j=1}^{J} d_j \rho_j$ and $\alpha_g = \sum_{j=1}^{J} d'_j \rho_j$. Since $\mathrm{le}(f), \mathrm{le}(g), \alpha_f, \alpha_g \in \Omega_J$, we know by Corollary 4.5 that $\mathrm{le}(f) + n_f + \alpha_f$ and $\mathrm{le}(g) + n_g + \alpha_g$ are both elements of Ω_J. Thus, it follows from the uniqueness of representation promised by Theorem 3.20 that $d_j = d'_j$ for $j \in \{J+1, \ldots, k\}$. Thus,

$$\mathrm{le}(f) + n_f + \alpha_f = \mathrm{le}(g) + n_g + \alpha_g. \tag{4.3}$$

If we define $m' \in \Omega_J$ to be the quantity expressed in (4.3), then by Theorem 3.20, we can write

$$m' = \mathrm{le}(f) + n_f + \alpha_f = n + \sum_{j=1}^{J} \delta_j \rho_j, \tag{4.4}$$

where $n \in \mathbb{N}$ and $0 \le \delta_j < s_j$. Define $\omega = \sum_{j=1}^{J} \delta_j \rho_j \in \Omega'_J$, and let n_ω be the smallest nonnegative integer such that $\omega + n_\omega \in \langle \mathrm{le}(f)\rangle \cap \langle \mathrm{le}(g)\rangle$ (whose existence is guaranteed by Lemma 4.9). Since $m' = \omega + n \in \langle \mathrm{le}(f)\rangle \cap \langle \mathrm{le}(g)\rangle$, it follows that $n \ge n_\omega$. Combining (4.1) and (4.4), we have

$$\begin{aligned} m &= m' + \sum_{j=J+1}^{k} d_j \rho_j \\ &= (\omega + n_\omega) + (n - n_\omega) + \sum_{j=J+1}^{k} d_j \rho_j. \end{aligned}$$

Since $n \ge n_\omega$, it follows that $n - n_\omega \in \mathbb{N}$, and so $(n - n_\omega) + \sum_{j=J+1}^{k} d_j \rho_j \in \Lambda$. Moreover, $\omega + n_\omega \in \Gamma_J$ and so Γ_J generates $\langle \mathrm{le}(f)\rangle \cap \langle \mathrm{le}(g)\rangle$. □

The following algorithm uses the lemma above to produce a syzygy family.

Algorithm 4.11. Let $f, g \in K[x,y]^*$. This algorithm will produce $m_1, \ldots, m_\ell \in \Lambda$ such that $\langle \mathrm{le}(f) \rangle \cap \langle \mathrm{le}(g) \rangle = \langle m_1, \ldots, m_\ell \rangle$. In addition, for each $j \in \{1, \ldots, \ell\}$, we will produce $a_j, b_j \in K[x,y]^*$ such that either $a_j f = b_j g$ or both $a_j f \neq b_j g$ and $\mathrm{le}(a_j f - b_j g) < m_j$.

(1) Using Algorithm 4.6, write $\mathrm{le}(f) = n + \sum_{j=1}^{J} d_j \rho_j$ and $\mathrm{le}(g) = n' + \sum_{j=1}^{J} d'_j \rho_j$ where $n, n' \in \mathbb{N}$ and $0 \leq d_j, d'_j < s_j$.

(2) Let $\omega_1, \ldots, \omega_t$ be all the elements of $\{\sum_{j=1}^{J} d_j \rho_j \mid 0 \leq d_j < s_j\}$. For each t, find the smallest $n_t \in \mathbb{N}$ such that $\omega_t - \mathrm{le}(f) + n_t$ and $\omega_t - \mathrm{le}(g) + n_t$ are both in Λ. (Note that the existence of such an integer n_t is guaranteed by Lemma 4.9.) To accomplish this, begin with $n_t = 0$ and keep incrementing n_t until $\omega_t - \mathrm{le}(f) + n_t$ and $\omega_t - \mathrm{le}(g) + n_t$ are both in Λ (which can be tested using Algorithm 4.6).

(3) For each t, define $m_t = n_t + \omega_t$. By the proof of Proposition 4.10, $\{m_1, \ldots, m_\ell\}$ generates $\langle \mathrm{le}(f) \rangle \cap \langle \mathrm{le}(g) \rangle$.

Bernd Sturmfels posed the following question at the Third Annual Colloquiumfest at the University of Saskatchewan in 2002: Given a generalized Gröbner basis with respect to a valuation, does it necessarily follow that there exists a monomial order such that G is a Gröbner basis with respect to the monomial order? The following example demonstrates that the answer is negative.

Example 4.12. Let K be a field that is not of characteristic two. Define $f_1 = y^2 - x$ and $f_2 = xy$. Then one can check that the set $G = \{f_1, f_2\}$ is a Gröbner basis for the ideal $I = \langle f_1, f_2 \rangle$ with respect to the valuation induced by $z = t^{1/2} + t^{1/4} + t^{1/8} + t^{1/16} + \cdots$ using Algorithm 1.12.

We now demonstrate that G is not a Gröbner basis with respect to any monomial order. Suppose, for contradiction, that G is a Gröbner basis with respect to some monomial order '$<$'. Note that $x^2, y^3 \in I$ since $x^2 = yf_2 - xf_1$ and $y^3 = yf_1 + f_2$. We consider two cases, depending on whether $x > y^2$ or $x < y^2$. If $x < y^2$, then $\mathrm{lt}(f_1) = y^2$ and $\mathrm{lt}(f_2) = xy$. However, $x^2 \in I$, and so if G were a Gröbner basis with respect to '$<$', then either $y^2 \mid x^2$ or $xy \mid x^2$, a contradiction. Now suppose $x > y^2$, in which case $\mathrm{lt}(f_1) = -x$ and $\mathrm{lt}(f_2) = xy$. However, $y^3 \in I$, and so if G were a Gröbner basis with respect to '$<$', then either $-x \mid y^3$ or $xy \mid y^3$, a contradiction.

Lastly, we note by example that some ideals do not have finite Gröbner bases with respect to a given valuation.

Example 4.13. Consider the ideal $\langle x, y \rangle$ of $K[x,y]$, and let G be a Gröbner basis with respect to $z \in K((t^{\mathbb{Q}}))$. For each ρ_i, choose $p_i(x,y) \in K[x,y]$ such that $\mathrm{le}(p_i) = \rho_i$. Since G is a Gröbner basis, there exists $g_i \in G$ such that $\mathrm{le}(g_i) \mid \mathrm{le}(p_i)$. That is, for some $h_i \in K[x,y]$, we have $\mathrm{le}(g_i h_i) = \rho_i$. By Lemma 4.3, we have $\rho_i \notin (1/r_{l(i-1)})\mathbb{Z}$. Since $G \cap K = \emptyset$, it follows that $\mathrm{le}(g_i) > 0$, and so $\mathrm{le}(h_i) < \rho_i$. Suppose, for contradiction, $\mathrm{le}(g_i) \neq \rho_i$. Then $\mathrm{le}(g_i) < \rho_i$, and so by Theorem 3.20 and Lemma 3.16, $\mathrm{le}(g_i) = n + \sum_{j=1}^{i-1} d_j \rho_j$ and $\mathrm{le}(h_i) = n' + \sum_{j=1}^{i-1} d'_j \rho_j$. Thus, $\rho_i = \mathrm{le}(g_i h_i) \in (1/r_{l(i-1)})\mathbb{Z}$, a contradiction. Therefore, $\mathrm{le}(g_i) = \rho_i$, and since the monoid generating sequence is infinite, it follows that G is infinite.

5. Approximating Algebraic Series with Finite Puiseux Series

In this section, we provide justification for the results that lead to Proposition 3.12. After defining some new terminology, we present a proposition that states that a twist-recurrent series remains twist-recurrent under the action of the functions described in Lemma 3.9.

Throughout this section, we adopt the notation ψ_ℓ for the functions described in Lemma 3.9.

Definition 5.1. Given $E \subset K((t^{\mathbb{Q}}))$, a function $\psi : E \to K((t^{\mathbb{Q}}))$ is called *support-preserving* if the following two conditions hold:

(i) For all $w \in E$, we have $\operatorname{Supp}(w) = \operatorname{Supp}(\psi w)$.
(ii) For all $w_1, w_2 \in E$ and $q \in \mathbb{Q}$,

$$w_1(q) = w_2(q) \Rightarrow [\psi(w_1)](q) = [\psi(w_2)](q).$$

Note that each function ψ_ℓ is support-preserving.

Proposition 5.2. *Let K be a field of characteristic $p > 0$. If w is a twist-recurrent series, then $\psi_\ell(w)$ is also a twist-recurrent series.*

Proof. Suppose $w = \sum w_i t^i$ is twist-recurrent. We will show that $\psi_\ell(\sum w_i t^i)$ is twist-recurrent. By Definition 3.6, w is supported on a set of the form $S_{a,b,c}$. Since ψ_ℓ is support-preserving, $\psi_\ell(w)$ is also supported on the set $S_{a,b,c}$. Thus, we have shown that $\psi_\ell(w)$ satisfies part (a) of Definition 3.6.

Next, we proceed to show that $\psi_\ell(w)$ satisfies part (b) of Definition 3.6. Let $S_{a,b,c}$ be a set on which $\psi_\ell(w)$ is supported. Since ψ_ℓ is support-preserving, w must be supported on the set $S_{a,b,c}$ as well. Let $m \le b$. To show that $\psi_\ell(w)$ satisfies part (b) of Definition 3.6, we must demonstrate that the function

$$\begin{array}{rcl} g_m : T_c & \to & \widetilde{K} \\ \tau & \mapsto & \psi_\ell(w)_{(m+\tau)/a} \end{array}$$

is twist-recurrent. That is, we must show that given any $n \in \mathbb{N}, j \in \mathbb{Z}^+, b_i \in \{0, \ldots, p-1\}$, and $\sum b_i \le c$, the sequence

$$c'_n = g_m(\tau_n) = \eta_\ell\left(\frac{m+\tau_n}{a}\right) w_{(m+\tau_n)/a}$$

satisfies a LRR of the form (3.1) where τ_n is given by

$$\tau_n = \frac{b_1}{p} + \cdots + \frac{b_{j-1}}{p^{j-1}} + \frac{1}{p^n}\left(\frac{b_j}{p^j} + \cdots\right). \tag{5.1}$$

Since w is twist-recurrent, the function

$$\begin{array}{rcl} f_m : T_c & \to & K \\ \tau & \mapsto & w_{(m+\tau)/a} \end{array}$$

is twist-recurrent, and so the sequence $c_n = w_{(m+\tau_n)/a}$ satisfies an LRR of the form

$$d_0 c_n + d_1 c_{n+1}^p + \cdots + d_k c_{n+k}^{p^k} = 0.$$

Note that for all n,

$$c'_n = \eta_\ell\left(\frac{m+\tau_n}{a}\right) c_n. \tag{5.2}$$

Now, there exist nonnegative integers $\alpha, \beta, \gamma, r, s, u$ where $\gcd(\alpha, p) = \gcd(\beta, p) = \gcd(\gamma, p) = 1$, $r \le j-1$ and $s \ge j$ such that

$$\begin{aligned} m + b_1 p^{-1} + \cdots + b_{j-1} p^{-(j-1)} &= \alpha p^{-r} \\ b_j p^{-j} + b_{j+1} p^{-(j+1)} + \cdots &= \beta p^{-s} \\ a &= \gamma p^u. \end{aligned}$$

From this, we see by (5.1) that

$$\frac{m+\tau_n}{a} = \frac{\alpha p^{-r} + p^{-n}\beta p^{-s}}{\gamma p^u} = \frac{\alpha p^{n+s-r} + \beta}{\gamma p^{u+s+n}}.$$

If we denote the multiplicative inverse of p modulo γ by q and let ζ denote a primitive γ-th root of unity in $\widetilde{K}$, then since $s > r$, by using the notation of Lemma 3.8, we have

$$\eta_\ell\left(\frac{m+\tau_n}{a}\right) = \zeta^{(\alpha p^{n+s-r}+\beta)\ell q^{u+s+n}} = \zeta^{\alpha \ell q^{r+u}} \cdot \zeta^{\beta \ell q^{u+s+n}}.$$

If we define

$$d_i' = \zeta^{-\alpha \ell q^{r+u} p^i} d_i,$$

then by (5.2),

$$\begin{aligned} \sum_{i=0}^k d_i' c_{n+i}'^{p^i} &= \sum_{i=0}^k \zeta^{-\alpha\ell q^{r+u}p^i} d_i \left[\eta_\ell\left(\frac{m+\tau_{n+i}}{a}\right) c_{n+i}\right]^{p^i} \\ &= \sum_{i=0}^k \zeta^{-\alpha\ell q^{r+u}p^i} d_i \zeta^{\alpha\ell q^{r+u}p^i} \cdot \zeta^{\beta\ell q^{u+s+n+i}p^i} c_{n+i}^{p^i} \\ &= \sum_{i=0}^k \zeta^{\beta\ell q^{u+s+n+i}p^i} d_i c_{n+i}^{p^i} \\ &= \sum_{i=0}^k \zeta^{\beta\ell q^{u+s+n}} d_i c_{n+i}^{p^i} \\ &= \zeta^{\beta\ell q^{u+s+n}} \sum_{i=0}^k d_i c_{n+i}^{p^i} \\ &= 0, \end{aligned}$$

and so $\psi_\ell(w)$ satisfies part (b) of Definition 3.6.

Lastly, we need to show that the functions of the form g_m span a finite-dimensional vector space. Since $w = \sum w_i t^i$ is twist-recurrent, the functions of the form f_m span a finite-dimensional vector space. Let $f_{m_1}, \ldots, f_{m_E}$ be a finite collection of functions that span this space. For any $\tau \in T_c \subset \{hp^{-n} \mid h \in \mathbb{Z}, n \in \mathbb{N}\}$, we have

$$\eta_\ell\left(\frac{m+\tau}{a}\right) \in \{\zeta^0, \zeta^1, \ldots, \zeta^{\gamma-1}\}.$$

For $\ell \in \{0, \ldots, \gamma - 1\}$, define $f_{m,\ell} : T_c \to K$ by

$$f_{m,\ell}(\tau) = \begin{cases} w_{(m+\tau)/a} & \text{if } \eta((m+\tau)/a) = \zeta^\ell; \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $f_m = \sum_{\ell=0}^{\gamma-1} f_{m,\ell}$, and so

$$g_m = \sum_{\ell=0}^{\gamma-1} \zeta^\ell f_{m,\ell}.$$

Thus, the vector space spanned by all functions of the form g_m is a subspace of the finite-dimensional vector space spanned by the set $\{f_{m_e,\ell} \mid 1 \le e \le E,\ 0 \le \ell \le \gamma - 1\}$. □

Proposition 5.3. *Each ψ_ℓ sends $\widetilde{K((t^{-1}))}$ to itself.*

Proof. Let $w \in \widetilde{K((t^{-1}))}$. If K has characteristic zero, then by Theorem 3.2, w is a Puiseux series of ramification index r with coefficients in $\widetilde{K}$. Then there exists a finite extension L of K such that $w \in L((t^{-1/r})) \subset \widetilde{K((t^{-1}))}$ (see [Du] for details). By the definition of ψ_ℓ, there exists a primitive r-th root of unity ζ_r such that $\psi_\ell(w) \in L(\zeta_r)((t^{-1/r})) \subset \widetilde{K((t^{-1}))}$.

If K has positive characteristic, then the result follows from Theorem 3.7 and Proposition 5.2. □

Given $w \in \widetilde{K((t^{-1}))}$, denote by $\mathcal{O}(w)$ the orbit of w under the action by $\Psi = \{\psi_\ell \mid \ell \in \mathbb{Z}\}$. The next result provides us with the cardinality of the orbit of w.

Proposition 5.4. *Let w be a nonzero element of $\widetilde{K((t^{-1}))}$.*

(i) *If char $K = 0$, then $|\mathcal{O}(w)|$ is the ramification index of w.*

(ii) *If char $K = p > 0$, then $|\mathcal{O}(w)|$ is the smallest positive integer n such that*

$$\mathrm{Supp}(w) \subset \{k \in \mathbb{Q} \mid \exists e \in \mathbb{N} \text{ such that } knp^e \in \mathbb{Z}\}.$$

Proof. (i) Since char $K = 0$, by Puiseux's Theorem, w can be expressed as

$$w = \sum_{i=1}^{N} w_i t^{m_i/n},$$

where $N \in \mathbb{Z}^+ \cup \{\infty\}$, $w_i \in \widetilde{k}^*$, n is the ramification index of w and $m_i > m_{i+1}$. By Lemma 3.9, we have that for all j,

$$\psi_j(w) = \sum_{i=1}^{N} w_i (\zeta_n^j t^{1/n})^{m_i}.$$

We will show that $\psi_j(w) = \psi_{j'}(w)$ if and only if $j \equiv j' \mod n$, from which we can conclude $|\mathcal{O}(w)| = n$.

Suppose $\psi_j(w) = \psi_{j'}(w)$. Then $w_i(\zeta_n^j t^{1/n})^{m_i} = w_i(\zeta_n^{j'} t^{1/n})^{m_i}$ for each i. Therefore, $\zeta_n^{jm_i} = \zeta_n^{j'm_i}$ for each i, and so $jm_i \equiv j'm_i \mod n$. Since n is the ramification index of w, the integers $n, m_1, m_2, \ldots$ do not all have a common factor, and so $j \equiv j' \mod n$. Conversely, if $j \equiv j' \mod n$, then $w_i(\zeta_n^j t^{1/n})^{m_i} = w_i(\zeta_n^{j'} t^{1/n})^{m_i}$ for each i, and so $\psi_j(w) = \psi_{j'}(w)$.

(ii) Let char $K = p > 0$, and write w as

$$w = \sum_{i \in I} w_i t^i,$$

where I is a nonempty, Noetherian subset of $\mathbb{Q}$. Define n to be the smallest positive integer n such that

$$\mathrm{Supp}(w) \subset \{k \in \mathbb{Q} \mid \exists e \in \mathbb{N} \text{ such that } knp^e \in \mathbb{Z}\}. \tag{5.3}$$

The existence of such an integer n is promised by Kedlaya's description of $\widetilde{K((t^{-1}))}$. (More specifically, Kedlaya demonstrates in Corollary 9 from [Ke] that any finite extension E of $K((t^{-1}))$ can be expressed as a tower of Artin-Schreier extensions over $M((t^{-1/m}))$, where m is a positive integer and M is the integral closure of K in E.)

We will show that $\psi_j(w) = \psi_{j'}(w)$ if and only if $j \equiv j' \mod n$, from which we can conclude $|\mathcal{O}(w)| = n$.

Suppose $\psi_j(w) = \psi_{j'}(w)$. Using the notation of Lemma 3.9, we see that $\eta_j(i) = \eta_{j'}(i)$ for all $i \in I$. Since n is the smallest positive integer that satisfies (5.3), there exists $s \in \mathbb{Z}^+$ and $i_1, \ldots, i_s \in \mathrm{Supp}(w)$ of the form $i_r = a_r/(np^{m_r})$ where $r \in \{1, \ldots, s\}$, $a_r \in \mathbb{Z}^*, m_r \in \mathbb{N}$, $p \nmid a_r$ such that $n, a_1, \ldots, a_s$ do not all share a common factor. By the definition of n, it follows that $\gcd(n, p) = 1$. Since $\eta_j(i_r) = \eta_{j'}(i_r)$ for each r, we have $\zeta_n^{a_r q^{m_r} j} = \zeta_n^{a_r q^{m_r} j'}$ where $q \in \{1, \ldots, n-1\}$ is chosen so that $pq \equiv 1 \mod n$. Thus, $a_r q^{m_r} j \equiv a_r q^{m_r} j' \mod n$, and so $n \mid a_r q^{m_r}(j - j')$. Since $\gcd(n, q) = 1$, it follows that $n \mid a_r(j - j')$. Since $n, a_1, \ldots, a_s$ do not all have a common factor, it follows that $n \mid j - j'$, and so $j \equiv j' \mod n$.

Conversely, suppose $j \equiv j' \mod n$. Each $i \in I$ can be written as $a/(np^e)$ where $a \in \mathbb{Z}$, $n \in \mathbb{Z}^+$, and $e \in \mathbb{N}$. Then $\eta_j(i) = \zeta_n^{ajq^e}$ and $\eta_{j'}(i) = \zeta_n^{aj'q^e}$ where $q \in \{1, \ldots, n-1\}$ is chosen so that $pq \equiv 1 \mod n$. Since $j \equiv j' \mod n$, it follows that $\eta_j(i) = \eta_{j'}(i)$, and so $\psi_j(w) = \psi_{j'}(w)$. ☐

Using this proposition, we can more precisely describe the orbit of any $w \in \widetilde{K((t^{-1}))}$ as below.

Corollary 5.5. *Given $w \in \widetilde{K((t^{-1}))}$, denote by $\mathcal{O}(w)$ the orbit of w under the action by $\Psi = \{\psi_\ell \mid \ell \in \mathbb{Z}\}$. If $|\mathcal{O}| = n$, then*

$$\mathcal{O}(w) = \{\psi_1(w), \ldots, \psi_n(w)\}.$$

Proof. In the proofs of both parts (i) and (ii) of Proposition 5.4, it was shown that $\psi_j(w) = \psi_{j'}(w)$ if and only if $j \equiv j' \mod n$, and so $\mathcal{O}(w) = \{\psi_1(w), \ldots, \psi_n(w)\}$. ☐

Definition 5.6. Let $w = \sum_{i \in I} w_i t^i \in K((t^{\mathbb{Q}}))$.

(i) If K has characteristic zero, then define $\mathcal{P}(w) = w$.

(ii) If K has positive characteristic p, then define

$$I_p = \{a/b \in I \mid a \in \mathbb{Z}, b \in \mathbb{Z}^+ \text{ and } p \nmid b\}.$$

If I_p is empty, define $\mathcal{P}(w) = 0$; otherwise, define

$$\mathcal{P}(w) = \sum_{i \in I_p} w_i t^i.$$

Note that if w is algebraic over $K((t^{-1}))$, then $\mathcal{P}(w)$ is a Puiseux series whose ramification index is not divisible by char K.

Example 5.7. If K has characteristic 2 and

$$w = t^{1/2} + t^{1/3} + t^{1/5} + t^{/1/7} + t^{1/11} + t^{1/13} + t^{1/17} + \cdots,$$

then

$$\mathcal{P}(w) = t^{1/3} + t^{1/5} + t^{/1/7} + t^{1/11} + t^{1/13} + t^{1/17} + \cdots .$$

If $w = t^{1/2} + t^{1/4} + t^{1/8} + t^{1/16} + \cdots$, then $\mathcal{P}(w) = 0$.

A simple proof provides the following corollary of Proposition 5.4.

Corollary 5.8. *Let $w \in \widetilde{K((t^{-1}))}$. The ramification index of $\mathcal{P}(w)$ divides $|\mathcal{O}(w)|$.*

The following simple lemma follows directly from Definitions 5.1 and 5.6.

Lemma 5.9. *Let E be a field contained in $K((t^{\mathbb{Q}}))$ and let ψ be a support-preserving function on E. Then $\psi(\mathcal{P}(w)) = \mathcal{P}(\psi(w))$.*

The next few results show how to place bounds on leading exponents in special cases.

Lemma 5.10. *For all $w \in K((t^{\mathbb{Q}}))$,*

$$le(z - w) \geq le(z - \mathcal{P}(w)).$$

Proof. If $\mathcal{P}(w) = w$, then the result follows trivially. Otherwise, K is a field of positive characteristic p and $w - \mathcal{P}(w) \neq 0$. Note that each element of the support of $w - \mathcal{P}(w)$ can be written as a/b where $a \in \mathbb{Z}$, $b \in \mathbb{Z}^+$ such that $p \nmid a$ and $p \mid b$. Since $\mathcal{P}(w)$ is a Puiseux series whose ramification index is not divisible by p and no term of the ramification sequence of z is divisible by p, it follows that the supports of $z - \mathcal{P}(w)$ and $w - \mathcal{P}(w)$ are disjoint. Thus, $\operatorname{lt}(z - \mathcal{P}(w)) \neq \operatorname{lt}(w - \mathcal{P}(w))$, and so by the triangle inequality (2.4),

$$\begin{aligned} \operatorname{le}(z - w) &= \operatorname{le}((z - \mathcal{P}(w)) - (w - \mathcal{P}(w))) \\ &= \max\{\operatorname{le}(z - \mathcal{P}(w)), \operatorname{le}(w - \mathcal{P}(w))\} \\ &\geq \operatorname{le}(z - \mathcal{P}(w)). \end{aligned}$$

□

Definition 5.11. Let $w \in K((t^{\mathbb{Q}}))$ such that $\mathcal{P}(w)$ agrees with z to finite order m. If $m = 0$, then define $\mathcal{S}(w) = 0$; otherwise, let L be the largest index such that $r_L = r_m$ and define

$$\mathcal{S}(w) = z^{(L)}.$$

Note that any Puiseux series agrees with z to a finite order. Therefore, for any $w \in \widetilde{K((t^{-1}))}$, the series $\mathcal{P}(w)$ agrees with z to a finite order, and hence, $\mathcal{S}(w)$ is well-defined.

Lemma 5.12. *Let $w \in \widetilde{K((t^{-1}))}$ be a Puiseux series whose ramification index is not divisible by char K, and let E be an algebraic extension of $K((t^{-1}))$ containing w. Let Ψ be a collection of support-preserving $K((t^{-1}))$-automorphisms of E such that for any $\psi \in \Psi$,*

$$le(z - \psi(w)) \geq le(z - w).$$

Then for all $\psi \in \Psi$,

$$le(z - \psi(w)) \geq le(z - \psi(\mathcal{S}(w))).$$

Proof. For this proof, if we write z as

$$z = \sum_{j=1}^{\infty} z_j t^{e_j},$$

where each z_j is nonzero, then by the assumption that ψ is support-preserving we can write

$$\psi(z) = \sum_{j=1}^{\infty} z_j{}' t^{e_j},$$

where each $z_j{}'$ is nonzero.

Let m denote the order to which z and w agree. Note that since $\mathrm{le}(z - \psi(w)) \geq \mathrm{le}(z - w)$ by assumption, it necessarily follows that z and $\psi(w)$ agree to an order of at most m. First, we consider the case when $m = 0$. In this case, $\mathcal{S}(w) = 0$ and so

$$\mathrm{le}(z - \psi(\mathcal{S}(w))) = \mathrm{le}(z).$$

Moreover, if $\mathrm{lt}(\psi(w)) = \mathrm{lt}(z)$, then z and $\psi(w)$ agree to an order greater than zero, a contradiction. Thus, $\mathrm{lt}(\psi(w)) \neq \mathrm{lt}(z)$, and so by the triangle inequality (2.4),

$$\mathrm{le}(z - \psi(w)) = \max\{\mathrm{le}(z), \mathrm{le}(\psi(w))\} \geq \mathrm{le}(z) = \mathrm{le}(z - \psi(\mathcal{S}(w))).$$

For the remainder of the argument, we consider the case $m > 0$. Define

$$z' = z - z^{(m)}, \tag{5.4}$$

$$w' = w - z^{(m)}. \tag{5.5}$$

Claim 1: $\mathrm{le}(z - \psi(w)) \geq \mathrm{le}(z - \psi(z^{(m)}))$.

Proof of Claim. Using (5.4) and (5.5), we see that

$$z - \psi(w) = \left[z^{(m)} - \psi(z^{(m)})\right] + z' - \psi(w'), \tag{5.6}$$

$$z - \psi(z^{(m)}) = \left[z^{(m)} - \psi(z^{(m)})\right] + z'. \tag{5.7}$$

We divide the proof of this claim into two cases, depending on whether $z^{(m)} = \psi(z^{(m)})$ or $z^{(m)} \neq \psi(z^{(m)})$.

First, we consider the case where

$$z^{(m)} = \psi(z^{(m)}), \tag{5.8}$$

in which case

$$z - \psi(w) = z' - \psi(w'), \tag{5.9}$$

$$z - \psi(z^{(m)}) = z'. \tag{5.10}$$

Since ψ is support-preserving,

$$\psi(\mathrm{lt}(w')) = \mathrm{lt}(\psi(w')). \tag{5.11}$$

Moreover, since z and w agree to order m, we have

$$z^{(m+1)} = z^{(m)} + \mathrm{lt}(z'), \tag{5.12}$$

$$w^{(m+1)} = z^{(m)} + \mathrm{lt}(w'). \tag{5.13}$$

Suppose, for contradiction, that $\text{lt}(z') = \text{lt}(\psi(w'))$. By combining (5.8), (5.11), (5.12), and (5.13), we obtain

$$\begin{aligned}[\psi(w)]^{(m+1)} &= \psi(w^{(m+1)}) \\ &= \psi(z^{(m)}) + \psi(\text{lt}(w')) \\ &= z^{(m)} + \text{lt}(\psi(w')) \\ &= z^{(m)} + \text{lt}(z') \\ &= z^{(m+1)}.\end{aligned}$$

Thus, $\psi(w)$ and z agree to order $m+1$, which contradicts our assumption that $\psi(w)$ and z agree to an order of at most m, and so we may assume $\text{lt}(z') \neq \text{lt}(\psi(w'))$.

Therefore, we can invoke the triangle inequality (2.4) together with (5.9) to obtain

$$\text{le}(z - \psi(w)) = \text{le}(z' - \psi(w')) = \max\{\text{le}(z'), \text{le}(\psi(w'))\} \geq \text{le}(z').$$

Moreover, by (5.10), we have $\text{le}(z - \psi(z^{(m)})) = \text{le}(z')$, and so $\text{le}(z - \psi(w)) \geq \text{le}(z - \psi(z^{(m)}))$.

Secondly, we consider the case where

$$z^{(m)} \neq \psi(z^{(m)}).$$

Note that every element of $\text{Supp}(z^{(m)} - \psi(z^{(m)}))$ is at least as large as e_m, and so $\text{le}(z^{(m)} - \psi(z^{(m)})) \geq e_m$. Since z and w agree to order m, we can see by (5.4) and (5.5) that any element of either $\text{Supp}(z')$ or $\text{Supp}(w')$ must be smaller than e_m. Therefore, $\text{le}(z') < e_m$ and $\text{le}(z' - \psi(w')) < e_m$, and so

$$\begin{aligned}\text{le}(z^{(m)} - \psi(z^{(m)})) &> \text{le}(z' - \psi(w')), \\ \text{le}(z^{(m)} - \psi(z^{(m)})) &> \text{le}(z').\end{aligned}$$

Applying these inequalities to (5.6) and (5.7), we obtain

$$\text{le}(z - \psi(w)) = \text{le}(z^{(m)} - \psi(z^{(m)})$$

and

$$\text{le}(z - \psi(z^{(m)})) = \text{le}(z^{(m)} - \psi(z^{(m)})),$$

in which case

$$\text{le}(z - \psi(w)) = \text{le}(z - \psi(z^{(m)})),$$

and so the claim is justified.

Since our next goal is to demonstrate that $\text{le}(z - \psi(z^{(m)})) \geq \text{le}(z - \mathcal{S}(w))$, we focus on the definition of $\mathcal{S}(w)$. Since w is a Puiseux series whose ramification index is not divisible by char K, it follows that $\mathcal{P}(w) = w$. Let L be the largest integer such that $r_m = r_L$, in which case

$$\mathcal{S}(w) = z^{(L)}. \tag{5.14}$$

Claim 2: $\text{le}(z - \psi(z^{(m)})) \geq \text{le}(z - \psi(z^{(L)}))$.

Proof of Claim. If $z_j = z'_j$ for all $j \in \{1, \ldots, m\}$, then $\text{le}(z - \psi(z^{(m)})) = e_{m+1}$ and $\text{le}(z - \psi(z^{(L)})) \leq e_{m+1}$. Otherwise, let j be the smallest index between 1 and m such that $z_j \neq z'_j$, in which case $\text{le}(z - \psi(z^{(m)})) = e_j$ and $\text{le}(z - \psi(z^{(L)})) = e_j$. Thus, the claim is justified.

Combining Claim 1 and Claim 2, we obtain $\mathrm{le}(z-\psi(w)) \geq \mathrm{le}(z-\psi(z^{(L)}))$. By (5.14), this becomes $\mathrm{le}(z-\psi(w)) \geq \mathrm{le}(z-\psi(\mathcal{S}(w)))$. □

Using Lemma 5.12, we can prove the following result, which is used in justifying Proposition 3.12.

Proposition 5.13. *Let $w \in \widetilde{K((t^{-1}))}$ and let E be an algebraic extension of $K((t^{-1}))$ containing w. Let Ψ be a collection of support-preserving $K((t^{-1}))$-automorphisms of E such that for any $\psi \in \Psi$,*

$$\mathrm{le}(z-\psi(\mathcal{P}(w))) \geq \mathrm{le}(z-\mathcal{P}(w)).$$

Then for all $\psi \in \Psi$,

$$\mathrm{le}(z-\psi(w)) \geq \mathrm{le}(z-\psi(\mathcal{S}(w))).$$

Proof. If we replace w by $\mathcal{P}(w)$ in Lemma 5.12, we obtain

$$\mathrm{le}(z-\psi(\mathcal{P}(w))) \geq \mathrm{le}(z-\psi(\mathcal{S}(\mathcal{P}(w)))).$$

It follows from Definition 5.11 that $\mathcal{S}(w) = \mathcal{S}(\mathcal{P}(w))$, and so

$$\mathrm{le}(z-\psi(\mathcal{P}(w))) \geq \mathrm{le}(z-\psi(\mathcal{S}(w))). \tag{5.15}$$

By Lemma 5.10 with $\psi(w)$ in place of w, we obtain

$$\mathrm{le}(z-\psi(w)) \geq \mathrm{le}(z-\mathcal{P}(\psi(w))).$$

By Lemma 5.9, we have $\mathcal{P}(\psi(w)) = \psi(\mathcal{P}(w))$, and so

$$\mathrm{le}(z-\psi(w)) \geq \mathrm{le}(z-\psi(\mathcal{P}(w))).$$

The proposition follows by combining this inequality with (5.15). □

We will see that in order to generate Λ, we need only consider polynomials whose roots are finite Puiseux series. To demonstrate this, we begin with the following proposition.

Proposition 5.14. *For each nonzero, monic, irreducible element $p(t,y)$ of $K((t^{-1}))[y]$, there exists $f(x,y) \in K[x,y]$ such that the following hold:*

(i) $\deg_y p(t,y) = \deg_y f(x,y)$,
(ii) *$\mathrm{le}(p(t,z)) \geq \mathrm{le}(f(t,z))$, and*
(iii) *$f(t,y)$ is a product of minimal polynomials of series of the form $z^{l(i)-1}$ over $K(t)$.*

Proof. Let E be the splitting field of $p(t,y) \in K(t)[y]$ over $K((t^{-1}))$ and denote by q the degree of inseparability of $E/K((t^{-1}))$. If we define $\Psi = \{\psi_\ell \mid \ell \in \mathbb{Z}\}$ as described in Lemma 3.9, then by Proposition 5.3, each ψ_j sends $\widetilde{K((t^{-1}))}$ to itself, and since E is a normal extension of $K((t^{-1}))$, it follows that each ψ_j sends E to itself. Let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be the orbits of the zeroes of $p(t,y)$ acted on by Ψ, in which case

$$p(t,y) = \left[\prod_{k=1}^{m} \prod_{w \in \mathcal{O}_k} (y-w)\right]^q. \tag{5.16}$$

For each $\mathcal{O}_k$, choose $w_k \in \mathcal{O}_k$ such that for any $w \in \mathcal{O}_k$,

$$\mathrm{le}(z-\mathcal{P}(w)) \geq \mathrm{le}(z-\mathcal{P}(w_k)).$$

Since $\mathcal{O}_k = \{w \mid w \in \mathcal{O}_k\} = \{\psi(w_k) \mid \psi \in \Psi\}$, we have by Lemma 5.9 that

$$\{\mathcal{P}(w) \mid w \in \mathcal{O}_k\} = \{\psi(\mathcal{P}(w_k)) \mid \psi \in \Psi\}.$$

Thus for all $\psi \in \Psi$,

$$\operatorname{le}(z - \psi(\mathcal{P}(w_k))) \geq \operatorname{le}(z - \mathcal{P}(w_k)). \tag{5.17}$$

For each k, if we define $n_k = |\mathcal{O}(w_k)|$, then by Corollary 5.5, we have

$$\mathcal{O}_k = \{\psi_1(w_k), \ldots, \psi_{n_k}(w_k)\}.$$

Then by (5.16),

$$p(t, y) = \left[\prod_{k=1}^{m} \prod_{j=1}^{n_k} (y - \psi_j(w_k))\right]^q. \tag{5.18}$$

Now define

$$f(t, y) = \left[\prod_{k=1}^{m} \prod_{j=1}^{n_k} (y - \psi_j(\mathcal{S}(w_k)))\right]^q. \tag{5.19}$$

We will later show that $f(t, y) \in K[t, y]$, in which case $f(x, y)$ is a polynomial in the variables x and y. Thus, by (5.18) and (5.19), it follows that part (i) of the proposition is justified:

$$\deg_y(p(t, y)) = \deg_y(f(x, y)).$$

If we apply Proposition 5.13 to (5.17), we obtain $\operatorname{le}(z - \psi_j(w_k)) \geq \operatorname{le}(z - \psi_j(\mathcal{S}(w_k)))$ for all j, k. Thus, if we apply (2.1) to (5.18) and (5.19), we obtain part (ii) of the proposition:

$$\operatorname{le}(p(t, z)) \geq \operatorname{le}(f(t, z)).$$

If we denote the ramification index of the Puiseux series $\mathcal{S}(w_k)$ by R_k, then by Proposition 3.10, the minimal polynomial of $\mathcal{S}(w_k)$ over $K(t)$ is

$$f_k(t, y) = \prod_{j=1}^{R_k} (y - \psi_j(\mathcal{S}(w_k)) \in K[t, y].$$

By Corollary 5.8, the ramification index of $\mathcal{P}(w_k)$ divides n_k. Since the ramification index of $\mathcal{S}(w_k)$ divides the ramification index of $\mathcal{P}(w_k)$, we have $R_k \mid n_k$. Since $\mathcal{S}(w_k)$ has ramification index R_k, we have $\psi_j(\mathcal{S}(w_k)) = \psi_{j'}(\mathcal{S}(w_k))$ whenever $j \equiv j' \mod R_k$. Thus,

$$(f_k(t, y))^{(n_k/R_k)} = \prod_{j=1}^{n_k} (y - \psi_j(\mathcal{S}(w_k)) \in K[t, y],$$

and so by (5.19),

$$f(x, y) = \left[\prod_{k=1}^{m} (f_k(x, y))^{(n_k/R_k)}\right]^q.$$

Therefore, $f(x, y)$ is a product of minimal polynomials of finite Puiseux series of the form $\mathcal{S}(w_k)$. If we denote by m_k the largest index to which $\mathcal{P}(w_k)$ and z agree and let L_k be the largest index such that $r_{L_k} = r_{m_k}$, then

$$\mathcal{S}(w_k) = z^{(L_k)}.$$

By Definition 2.6 (iii), we can write each L_k as $l(i_k) - 1$ for some i_k, and so part (iii) of the proposition is justified. □

We now provide a proof of Proposition 3.12, which generalizes Proposition 5.14 to include arbitrary, nonzero polynomials.

Proof of Proposition 3.12. First, factor $p(t, y)$ as a polynomial in y as

$$p(t, y) = q(t) \prod_{i=1}^{m} p_i(t, y),$$

where $q(t) \in K[t]$ and $p_i(t, y)$ is a monic, irreducible element of $K((t^{-1}))[y]$. By Proposition 5.14, for each index i, there exists $f_i(x, y) \in K[x, y]$ such that $\deg_y p_i(t, y) = \deg_y f_i(x, y)$, $\mathrm{le}(p_i(t, z)) = \mathrm{le}(f_i(t, z))$, and the roots of $f_i(t, y)$ are finite Puiseux series of the desired form. It then follows that $f(x, y) = q(x) \prod_{i=1}^{m} f_i(x, y)$ satisfies the conditions of the proposition. □

6. Associated Sequences

In this section, we prove some elementary results about the sequences described in the previous sections. In particular, we will construct recurrence relations and formulas concerning the monoid generating sequence and the sequence $(\lambda_1, \lambda_2, \lambda_3, \dots)$ introduced in Definition 3.17. The following lemma provides a simple recurrence relation for the monoid generating sequence.

Lemma 6.1. *The monoid generating sequence satisfies the following recurrence relation:*

$$\begin{aligned} \rho_1 &= e_{l(1)}, \\ \rho_{i+1} &= s_i \rho_i - e_{l(i)} + e_{l(i+1)}. \end{aligned}$$

Proof. By (2.7), $u_{l(1)-1} = u_{l(1-1)} = u_0 = 0$, and so by Definition 2.6, $\rho_1 = e_{l(1)}$. Moreover,

$$\begin{aligned} u_m + e_{m+1} &= \sum_{j=0}^{m-1} \left(\frac{r_m}{r_j} - \frac{r_m}{r_{j+1}} \right) e_{j+1} + e_{m+1} \\ &= \left(\frac{r_m}{r_{m-1}} \right) \sum_{j=0}^{m-2} \left(\frac{r_{m-1}}{r_j} - \frac{r_{m-1}}{r_{j+1}} \right) e_{j+1} + \left(\frac{r_m}{r_{m-1}} - \frac{r_m}{r_m} \right) e_m + e_{m+1} \\ &= \left(\frac{r_m}{r_{m-1}} \right) \sum_{j=0}^{m-2} \left(\frac{r_{m-1}}{r_j} - \frac{r_{m-1}}{r_{j+1}} \right) e_{j+1} + \left(\frac{r_m}{r_{m-1}} \right) e_m - e_m + e_{m+1} \\ &= \left(\frac{r_m}{r_{m-1}} \right) [u_{m-1} + e_m] - e_m + e_{m+1}. \end{aligned}$$

If we define $\gamma_m := u_{m-1} + e_m$, then

$$\gamma_{m+1} = \left(\frac{r_m}{r_{m-1}} \right) \gamma_m - e_m + e_{m+1}. \tag{6.1}$$

Replacing m by $l(i)$, we obtain

$$\gamma_{l(i)+1} = \left(\frac{r_{l(i)}}{r_{l(i)-1}} \right) \gamma_{l(i)} - e_{l(i)} + e_{l(i)+1} = s_i \gamma_{l(i)} - e_{l(i)} + e_{l(i)+1}. \tag{6.2}$$

If $l(i) < m < l(i+1)$, then $r_m / r_{m-1} = 1$ by (2.6), and so (6.1) yields

$$\gamma_{m+1} = \gamma_m - e_m + e_{m+1}.$$

Multiple applications of this formula yields a telescoping sum, and so

$$\begin{aligned}\gamma_{l(i+1)} &= \gamma_{l(i+1)-1} - e_{l(i+1)-1} + e_{l(i+1)} \\ &= (\gamma_{l(i+1)-2} - e_{l(i+1)-2} + e_{l(i+1)-1}) - e_{l(i+1)-1} + e_{l(i+1)} \\ &= \gamma_{l(i+1)-2} - e_{l(i+1)-2} + e_{l(i+1)} \\ &\vdots \\ &= \gamma_{l(i)+1} - e_{l(i)+1} + e_{l(i+1)}.\end{aligned}$$

This equation in conjunction with (6.2) yields

$$\begin{aligned}\gamma_{l(i+1)} &= \gamma_{l(i)+1} - e_{l(i)+1} + e_{l(i+1)} \\ &= s_i\gamma_{l(i)} - e_{l(i)} + e_{l(i)+1} - e_{l(i)+1} + e_{l(i+1)} \\ &= s_i\gamma_{l(i)} - e_{l(i)} + e_{l(i+1)},\end{aligned}$$

and since $\rho_i = u_{l(i)-1} + e_{l(i)} = \gamma_{l(i)}$ for all i, we have

$$\rho_{i+1} = s_i\rho_i - e_{l(i)} + e_{l(i+1)}.$$

□

We can also construct a recursive formula for the terms of the ramification sequence, as given in the next result.

Lemma 6.2. *For $i \in \mathbb{N}$,*

$$r_{l(i)} = 1 + \sum_{j=1}^{i}(s_j - 1)r_{l(j-1)},$$

where the summation on the right is taken to be zero if $i = 0$.

Proof. For $i = 0$, we have $r_{l(0)} = r_0 = 1$. Otherwise, $i \geq 1$, in which case

$$\begin{aligned}\sum_{j=1}^{i}(s_j - 1)r_{l(j-1)} &= \sum_{j=1}^{i}((r_{l(j)}/r_{l(j-1)}) - 1)r_{l(j-1)} \\ &= \sum_{j=1}^{i}(r_{l(j)} - r_{l(j-1)}) \\ &= r_{l(i)} - r_{l(0)} \\ &= r_{l(i)} - 1.\end{aligned}$$

□

Using Lemma 6.1, we can construct yet another recurrence relation for the terms of the monoid generating sequence.

Lemma 6.3. *For $i \in \mathbb{Z}^+$,*

$$\rho_i = \sum_{j=1}^{i-1}(s_j - 1)\rho_j + e_{l(i)},$$

where the summation on the left is taken to be zero if $i = 1$.

Proof. We proceed by induction. If $i = 1$, then by Lemma 6.1,

$$\rho_1 = e_{l(1)} = 0 + e_{l(1)}.$$

Now suppose the statement holds for the index i. Then by Lemma 6.1 and the induction hypothesis,

$$\begin{aligned} \rho_{i+1} - \sum_{j=1}^{i}(s_j - 1)\rho_j &= \rho_{i+1} - (s_i - 1)\rho_i - \sum_{j=1}^{i-1}(s_j - 1)\rho_j \\ &= \rho_{i+1} - (s_i - 1)\rho_i - (\rho_i - e_{l(i)}) \\ &= s_i\rho_i - e_{l(i)} + e_{l(i+1)} - (s_i - 1)\rho_i - \rho_i + e_{l(i)} \\ &= e_{l(i+1)}. \end{aligned}$$

□

At this point, we prove Lemma 3.16, which states that the terms of the monoid generating sequence are increasing.

Proof of Lemma 3.16. Since $s_j > 1$ for each index j, by Lemma 6.3,

$$\rho_i = \sum_{j=1}^{i-1}(s_j - 1)\rho_j + e_{l(i)} > \sum_{j=1}^{i-1}\rho_j + e_{l(i)} > \rho_{i-1}.$$

□

In addition, we can use Lemma 6.3 to extract information about the denominators of the terms of the monoid generating sequence, as shown in the next result.

Corollary 6.4. *For $i \in \mathbb{Z}^+$,*

$$\rho_i \in (1/r_{l(i)})\mathbb{Z} - (1/r_{l(i-1)})\mathbb{Z}.$$

Proof. The result follows by a simple induction. Indeed,

$$\rho_1 = e_{l(1)} \in (1/r_{l(1)})\mathbb{Z} - \mathbb{Z} = (1/r_{l(1)})\mathbb{Z} - (1/r_{l(0)})\mathbb{Z}.$$

Now, assuming that $\rho_i \in (1/r_{l(i)})\mathbb{Z}$, we see by Lemma 6.3 that

$$\rho_{i+1} = \sum_{j=1}^{i}(s_j - 1)\rho_j + e_{l(i+1)}.$$

Since $\rho_j \in (1/r_{l(j)})\mathbb{Z} \subset (1/r_{l(i)})\mathbb{Z}$ for $1 \leq j \leq i$, we have

$$\sum_{j=1}^{i}(s_j - 1)\rho_j \in (1/r_{l(i)})\mathbb{Z} \subset (1/r_{l(i+1)})\mathbb{Z}.$$

Moreover, $e_{l(i+1)} \in (1/r_{l(i+1)})\mathbb{Z} - (1/r_{l(i)})\mathbb{Z}$, and so

$$\rho_{i+1} = \sum_{j=1}^{i}(s_j - 1)\rho_j + e_{l(i+1)} \in 1/(r_{l(i+1)})\mathbb{Z} - (1/r_{l(i)})\mathbb{Z}.$$

□

The next lemma states that the numerator of any term of the monoid generating sequence is relatively prime to the corresponding term of the partial ramification sequence.

Lemma 6.5. *For $i \in \mathbb{Z}^+$, if we write $\rho_i = c_i/r_{l(i)}$, then gcd$(c_i, s_i) = 1$.*

Proof. First, write each term of the exponent sequence in reduced form:

$$e_i = n_i/d_i,$$

where $n_i, d_i \in \mathbb{Z}^+$ and $\gcd(n_i, d_i) = 1$. By (2.6) and Definition 2.6, we conclude

$$r_{l(i)} = \operatorname{lcm}(r_{l(i-1)}, d_{l(i)}).$$

By Corollary 6.4, for each $j \leq i$, we have

$$\rho_j \in (1/r_{l(j)})\mathbb{Z} \subset (1/r_{l(i)})\mathbb{Z},$$

and so

$$S_i := \sum_{j=1}^{i} (s_j - 1)\rho_j \in (1/r_{l(i)})\mathbb{Z}.$$

Thus, for each i, there exists $f_i \in \mathbb{Z}^+$ such that

$$S_i = \frac{f_i}{r_{l(i)}} = \frac{s_{i+1} f_i}{s_{i+1} r_{l(i)}} = \frac{s_{i+1} f_i}{r_{l(i+1)}}.$$

Then by Lemma 6.3,

$$\rho_i = \frac{c_i}{r_{l(i)}} = S_{i-1} + e_{l(i)} = \frac{s_i f_{i-1}}{r_{l(i)}} + \frac{n_{l(i)}}{d_{l(i)}}. \tag{6.3}$$

Suppose, for contradiction, that $\gcd(s_i, c_i) \neq 1$. Then for some prime $p > 1$, we have $p \mid s_i$ and $p \mid c_i$. Now,

$$s_i = \frac{r_{l(i)}}{r_{l(i-1)}} = \frac{\operatorname{lcm}(r_{l(i)}, d_{l(i)})}{r_{l(i-1)}} = \frac{d_{l(i)}}{\gcd(r_{l(i-1)}, d_{l(i)})}$$

and so

$$p \mid \frac{d_{l(i)}}{\gcd(r_{l(i-1)}, d_{l(i)})}. \tag{6.4}$$

Moreover, by (6.3) we have

$$\frac{c_i}{r_{l(i)}} = \frac{s_i f_{i-1}}{r_{l(i)}} + \frac{n_{l(i)} \operatorname{lcm}(r_{l(i-1)}, d_{l(i)})/d_{l(i)}}{\operatorname{lcm}(r_{l(i-1)}, d_{l(i)})} = \frac{s_i f_{i-1} + n_{l(i)} \operatorname{lcm}(r_{l(i-1)}, d_{l(i)})/d_{l(i)}}{r_{l(i)}}$$

and so

$$c_i = s_i f_{i-1} + n_{l(i)} \operatorname{lcm}(r_{l(i-1)}, d_{l(i)})/d_{l(i)}. \tag{6.5}$$

By (6.4), we have that $p \mid d_{l(i)}$, and since $n_{l(i)}$ and $d_{l(i)}$ are relatively prime, it follows that $p \nmid n_{l(i)}$. Thus, since $p \mid c_i$ and $p \mid s_i$, it follows from (6.5) that

$$p \mid \frac{\operatorname{lcm}(r_{l(i-1)}, d_{l(i)})}{d_{l(i)}}$$

and so

$$p \mid \frac{r_{l(i-1)}}{\gcd(r_{l(i-1)}, d_{l(i)})}. \tag{6.6}$$

Since the expressions in (6.4) and (6.6) are relatively prime, we have a contradiction. □

We are now in a position to prove Lemma 4.3.

Proof of Lemma 4.3. For $i = 1$, we have $\rho_1 = e_{l(1)}$, and so for any $0 < d_1 < s_1$, we have

$$d_1\rho_1 \in (1/r_{l(1)})\mathbb{Z} - \mathbb{Z}.$$

Now, we consider the case $i > 1$. For $j \leq i - 1$, we have by Corollary 6.4,

$$\rho_j \in (1/r_{l(j)})\mathbb{Z} \subset (1/r_{l(i-1)})\mathbb{Z},$$

and so

$$\sum_{j=1}^{i-1} d_j\rho_j \in (1/r_{l(i-1)})\mathbb{Z} \subset (1/r_{l(i)})\mathbb{Z}. \tag{6.7}$$

Write $\rho_i = c_i/r_{l(i)}$. Suppose, for contradiction, $d_i\rho_i = (d_ic_i)/r_{l(i)} \in (1/r_{l(i-1)})\mathbb{Z}$ where $0 < d_i < s_i$. Thus, $r_{l(i)} \mid d_ic_ir_{l(i-1)}$. Now, $s_i = r_{l(i)}/r_{l(i-1)}$, and so $s_i \mid d_ic_i$. By Lemma 6.5, we have $\gcd(c_i, s_i) = 1$ and so $s_i \mid d_i$. Since $0 < d_i < s_i$, we have a contradiction, and so $d_i\rho_i \notin (1/r_{l(i-1)})\mathbb{Z}$. By Corollary 6.4, we know $\rho_i \in (1/r_{l(i)})\mathbb{Z}$ and so

$$d_i\rho_i \in (1/r_{l(i)})\mathbb{Z} - (1/r_{l(i-1)})\mathbb{Z}. \tag{6.8}$$

Combining (6.7) and (6.8), we have

$$\sum_{j=1}^{i} d_j\rho_j \in (1/r_{l(i)})\mathbb{Z} - (1/r_{l(i-1)})\mathbb{Z}.$$

□

The following result gives an explicit formula for λ_d.

Lemma 6.6. *For any $d \in \mathbb{Z}^+$, there exists $J \in \mathbb{Z}^+$ and $d_j \in \mathbb{N}$ for $j \in \{1, \ldots, J\}$ such that $d_J \neq 0$,*

$$\lambda_d = le\left(\prod_{j=1}^{J} f_j(t,z)^{d_j}\right)$$

and

$$d = \sum_{j=1}^{J} d_j \deg_y(f_j(x,y)),$$

where f_j is the minimal polynomial of $z^{l(j)-1}$ over $K(t)$.

Proof. By Definition 3.17, there exists $p(x,y) \in K[x,y]$ such that $\deg_y(p(x,y)) = d$ and $\mathrm{le}(p(t,z)) = \lambda_d$. By Proposition 3.12, there exists $f(x,y) \in K[x,y]$ such that

(i) $\deg_y p(x,y) = \deg_y f(x,y)$,
(ii) $\mathrm{le}(p(t,z)) \geq \mathrm{le}(f(t,z))$, and
(iii) $f(t,y)$ is a product of minimal polynomials of series of the form $z^{(l(j)-1)}$ over $K(t)$.

By the minimality of λ_d in Definition 3.17, we have $\lambda_d = \mathrm{le}(f(t,z))$. Since $f(x,y)$ is a product of minimal polynomials of finite Puiseux series, we can write h as

$$f(t,z) = \prod_{j=1}^{J} f_j(t,z)^{d_j}.$$

□

Using this lemma, we prove Proposition 3.18, which constructs a unique representation for each λ_d in terms of the monoid generating sequence.

Proof of Proposition 3.18. By Lemma 6.6, there exists $f(x,y) \in K[x,y]$ such that $\lambda_d = \mathrm{le}(f(t,z))$, $\deg_y(f(x,y)) = d$, and

$$f(t,z) = \prod_{j=1}^{J} f_j(t,z)^{d_j},$$

where $J \in \mathbb{Z}^+$, and for each $j \in \{1,\ldots,J\}$, we have $d_j \in \mathbb{N}$ (but $d_J \neq 0$) and $f_j(t,z)$ is the minimal polynomial of $z^{(l(j)-1)}$ over $K(t)$. By Lemma 3.15, it follows that $\deg_y f_j(x,y) = r_{l(j)-1}$ and $\mathrm{le}(f_j(t,z)) = \rho_j$, and so

$$\lambda_d = \mathrm{le}\left(\prod_{j=1}^{J} f_j(t,z)^{d_j}\right) = \sum_{i=1}^{J} d_j \mathrm{le}(f_j(t,z)) = \sum_{i=1}^{J} d_j\rho_j$$

and

$$d = \deg_y f(x,y) = \sum_{j=1}^{J} d_j \deg_y f_j(x,y) = \sum_{j=1}^{J} d_j r_{l(j)-1} = \sum_{j=1}^{J} d_j r_{l(j-1)}.$$

Next we show that each d_j satisfies the bound $0 \leq d_j < s_j$. Suppose for contradiction, that $d_k \geq s_k$ for some $k \in \{1,\ldots,J\}$. Define

$$D_j = \begin{cases} d_k - s_k & \text{if } j = k; \\ d_{k+1} + 1 & \text{if } j = k+1; \\ d_j & \text{otherwise.} \end{cases}$$

Using this in conjunction with the recurrence relation given in Lemma 6.1, we obtain

$$\sum_{j=1}^{J} d_j\rho_j - \sum_{j=1}^{J} D_j\rho_j = s_k\rho_k - \rho_{k+1} = e_{l(k)} - e_{l(k+1)},$$

and by Definition 2.6 (vi),

$$\sum_{j=1}^{J} d_j r_{l(j-1)} - \sum_{j=1}^{J} D_j r_{l(j-1)} = s_k r_{l(k-1)} - r_{l(k)} = 0.$$

These equations in conjunction with Lemma 3.15 yield

$$\mathrm{le}\left(\prod_{j=1}^{J} f_j(t,z)^{D_j}\right) = \sum_{j=1}^{J} D_j\rho_j = \sum_{j=1}^{J} d_j\rho_j - e_{l(k)} + e_{l(k+1)} < \sum_{j=1}^{J} d_j\rho_j = \mathrm{le}(f)$$

and

$$\deg\left(\prod_{j=1}^{J} f_j(t,z)^{D_j}\right) = \sum_{j=1}^{J} D_j r_{l(j-1)} = \sum_{j=1}^{J} d_j r_{l(j-1)} = \deg(f).$$

However, since $\mathrm{le}(f) = \lambda_d$, we have contradicted the minimality of λ_d. Thus $0 \leq d_j < s_j$ for each $j \in \{1,\ldots,J\}$.

Finally, we demonstrate that the representation for λ_d given by

$$\lambda_d = \sum_{j=1}^{J} d_j \rho_j$$

is uniquely determined. Suppose, for contradiction, we are given two representations for λ_d:

$$\lambda_d = \sum_{j=1}^{J} d_j \rho_j = \sum_{j=1}^{J} d'_j \rho_j$$

where $0 \leq d_j, d'_j < s_j$. If we define $\Delta_j = d_j - d'_j$, then $\sum_{j=1}^{J} \Delta_j \rho_j = 0$ and $|\Delta_j| < s_j$. Multiplying the expression by $r_{l(J-1)}$, we obtain

$$\sum_{j=1}^{J} r_{l(J-1)} \Delta_j \rho_j = 0.$$

However, $r_{l(J-1)} \Delta_j \rho_j \in \mathbb{Z}$ for $j \in \{1, \ldots, J-1\}$, and so $r_{l(J-1)} \Delta_J \rho_J \in \mathbb{Z}$. Now write ρ_J as $c_J / r_{l(J)}$ where $c_J \in \mathbb{Z}^+$. Then $r_{l(J-1)} \Delta_J c_J / r_{l(J)} \in \mathbb{Z}$, and so $s_J = \frac{r_{l(J)}}{r_{l(J-1)}} \mid \Delta_J c_J$. Since s_J and c_J are relatively prime by Lemma 6.5, it follows that $s_J \mid \Delta_J$. However, $|\Delta_J| < s_J$, and so $\Delta_J = 0$. Thus, $\sum_{j=1}^{J-1} \Delta_j \rho_j = 0$. Repeating this argument, we find $\Delta_{J-1} = \Delta_{J-2} = \cdots = \Delta_1 = 0$, and so $d_j = d'_j$ for all $j \in \{1, \ldots, J\}$. □

The idea that each λ_d has a unique representation can be extended further. In fact, there is a natural bijective correspondence between representations of nonnegative integers and representations of elements of the value monoid of the form λ_d. First, we state the following simple lemma without proof.

Lemma 6.7. *Let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers such that $b_1 = 1, b_{j+1} > b_j$ and $b_j \mid b_{j+1}$ for all j. Then every $d \in \mathbb{Z}^+$ has a unique representation of the form*

$$d = \sum_{j=1}^{J} d_j b_j,$$

where $J \in \mathbb{Z}^+$, $d_J \neq 0$, and $0 \leq d_j < b_{j+1}/b_j$.

For example, if $b_j = 10^{j-1}$, then this says that every positive integer has a unique base 10 representation. Using Lemma 6.7 with $b_j = r_{l(j-1)}$ in conjunction with Proposition 3.18, we produce a method for quickly computing λ_d.

Corollary 6.8. *Given $J \in \mathbb{Z}^+$ and $d_1, \ldots, d_J \in \mathbb{N}$ such that $d_j < s_j$ for each $j \in \{1, \ldots, J\}$, we have*

$$d = \sum_{j=1}^{J} d_j r_{l(j-1)} \;\Leftrightarrow\; \lambda_d = \sum_{j=1}^{J} d_j \rho_j.$$

Lemma 6.9. *The sequence $\lambda_0, \lambda_1, \lambda_2, \ldots$ is increasing.*

Proof. We will show that $\lambda_{d+1} > \lambda_d$ for all d. By Proposition 3.18, we can write $\lambda_d = \sum_{j=1}^{J} d_j \rho_j$ where $0 \leq d_j < s_j$ and

$$d = \sum_{j=1}^{J} d_j r_{l(j-1)}. \tag{6.9}$$

Thus, by Corollary 6.8, it follows that

$$\lambda_d = \sum_{j=1}^{J} d_j \rho_j. \tag{6.10}$$

We now consider different cases, depending on the size of the coefficients d_j.

Case 1: First we consider the case $d_j = s_j - 1$ for all $j \in \{1, \dots, J\}$, in which case

$$d = \sum_{j=1}^{J} (s_j - 1) r_{l(j-1)}, \tag{6.11}$$

and

$$\lambda_d = \sum_{j=1}^{J} (s_j - 1) \rho_j. \tag{6.12}$$

Applying Lemma 6.2 to (6.11), we obtain $d + 1 = r_{l(J)}$, and so by Corollary 6.8,

$$\lambda_{d+1} = \rho_{J+1}. \tag{6.13}$$

If we use Lemma 6.3 in conjunction with (6.12) and (6.13), we obtain

$$\lambda_{d+1} - \lambda_d = \rho_{J+1} - \sum_{j=1}^{J} (s_j - 1) \rho_j = e_{l(J+1)} > 0.$$

Case 2: Consider the case $d_1 < s_1 - 1$. Since $r_{l(0)} = 1$, we can rewrite (6.9) as

$$d + 1 = (d_1 + 1) r_{l(0)} + \sum_{j=2}^{J} d_j r_{l(j-1)},$$

and so by Corollary 6.8,

$$\lambda_{d+1} = (d_1 + 1)\rho_1 + \sum_{j=2}^{J} d_j \rho_j. \tag{6.14}$$

Subtracting (6.10) from (6.14), we obtain

$$\lambda_{d+1} - \lambda_d = \rho_1 > 0.$$

Case 3: Finally, we consider the case where there exists an index $i > 1$ such that $d_i < s_i - 1$, but $d_j = s_j - 1$ for $j \in \{1, \dots, i-1\}$. Write λ_d as

$$\lambda_d = \sum_{j=1}^{i-1} (s_j - 1) \rho_j + \sum_{j=i}^{J} d_j \rho_j. \tag{6.15}$$

By Corollary 6.8,

$$d = \sum_{j=1}^{i-1} (s_j - 1) r_{l(j-1)} + \sum_{j=i}^{J} d_j r_{l(j-1)},$$

and so by Lemma 6.2,

$$\begin{aligned} d+1 &= 1+\sum_{j=1}^{i-1}(s_j-1)r_{l(j-1)}+\sum_{j=i}^{J}d_jr_{l(j-1)} \\ &= r_{l(i-1)}+\sum_{j=i}^{J}d_jr_{l(j-1)} \\ &= (d_i+1)r_{l(i-1)}+\sum_{j=i+1}^{J}d_jr_{l(j-1)}. \end{aligned}$$

Therefore, by Corollary 6.8,

$$\lambda_{d+1}=(d_i+1)\rho_i+\sum_{j=i+1}^{J}d_j\rho_j,$$

and if we subtract (6.15) from this expression and apply Lemma 6.3, we obtain

$$\lambda_{d+1}-\lambda_d=\rho_i-\sum_{j=1}^{i-1}(s_j-1)\rho_j=e_{l(i)}>0.$$

□

Definition 6.10. Given a submonoid M of a commutative monoid N, we define an equivalence relation on N by setting $n_1 \sim_M n_2$ if and only if there exist $m_1, m_2 \in M$ such that $m_1+n_1=m_2+n_2$. Denote by N/M the collection of all equivalence classes under this relation, and define a quotient map π from N to N/M that sends n to the equivalence class containing n. The set N/M has an additive monoid structure where we define $\pi(n_1)+\pi(n_2)=\pi(n_1+n_2)$.

Using this notation, we show that any pair of terms from the sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ are inequivalent modulo $\mathbb{Z}$.

Proposition 6.11. *For all $i \neq k$, we have $\lambda_i \not\sim_{\mathbb{Z}} \lambda_k$.*

Proof. Suppose, toward contradiction, that there exist distinct indices i and k such that $\lambda_i \sim_{\mathbb{Z}} \lambda_k$. Then for some $J \in \mathbb{Z}^+$, by Proposition 3.18 we can write

$$\lambda_i=\sum_{j=1}^{J}d_j\rho_j$$

and

$$\lambda_k=\sum_{j=1}^{J}d'_j\rho_j$$

where $0 \le d_j, d'_j < s_j$ for each $j \in \{1,\ldots,J\}$. For each j, write $\rho_j = c_j/r_{l(j)}$, where c_j and s_j are relatively prime, as promised by Lemma 6.5.

If we define $\Delta_j = d_j - d'_j$, then $|\Delta_j| < s_j$, and since $\lambda_i \sim_{\mathbb{Z}} \lambda_k$, we have

$$\lambda_i-\lambda_k=\sum_{j=1}^{J}\Delta_j\rho_j \sim_{\mathbb{Z}} 0.$$

If we multiply this expression by $r_{l(J-1)}$, we obtain

$$\left(\sum_{j=1}^{J-1} r_{l(J-1)}\Delta_j\rho_j\right) + r_{l(J-1)}\Delta_J\rho_J \sim_{\mathbb{Z}} 0.$$

However, $r_{l(J-1)}\Delta_j\rho_j \in \mathbb{Z}$ for $j \in \{1,\dots,J-1\}$ and so

$$r_{l(J-1)}\Delta_J\rho_J = r_{l(J-1)}\Delta_J c_J/r_{l(J)} \in \mathbb{Z}.$$

That is,

$$\Delta_J c_J/s_J \in \mathbb{Z},$$

and so $s_J \mid \Delta_J c_J$. Since s_J and c_J are relatively prime, $s_J \mid \Delta_J$. However, $|\Delta_J| < s_J$, and so $\Delta_J = 0$. Thus, $\sum_{j=1}^{J-1}\Delta_i\rho_j \sim_{\mathbb{Z}} 0$. Repeating this argument, we find $\Delta_{J-1} = \Delta_{J-2} = \cdots = \Delta_1 = 0$, and so $\lambda_i = \lambda_k$, which contradicts Lemma 6.9. □

Definition 6.12. Given $d \in \mathbb{N}$, define

$$\Lambda_d(z) = \{\mathrm{le}(f(t,z)) \mid f \in K[x,y]^* \text{ and } \deg_y(f(x,y)) \le d\}.$$

We quote the following result from [MoSw2].

Theorem 6.13. *For each $d \in \mathbb{Z}^+$, the quotient Λ_d/Λ_0 has cardinality $d+1$.*

Using this theorem in conjunction with Proposition 6.11, we have the following result.

Corollary 6.14. *The quotient Λ_d/Λ_0 consists precisely of the images of $\lambda_0,\dots,\lambda_d$.*

Finally, we prove Proposition 3.19, which shows how to explicitly write the value monoid as a disjoint union of cosets of $\mathbb{N}$.

Proof of Proposition 3.19. Since $\Lambda_0 = \mathbb{N}$, we have by Definition 6.12 and Corollary 6.14 that $\Lambda_d = \bigcup_{i=0}^{d}(\lambda_i + \mathbb{N})$. Moreover, $\Lambda = \bigcup_{d\ge 0}\Lambda_d$, and so $\Lambda = \bigcup_{d=0}^{\infty}(\lambda_d + \mathbb{N})$. The union is disjoint by Proposition 6.11. □

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